Analysis of Vibration of Bevel Gears

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Equation of motion for free vibrations of a pair of gears is derived in a general form. A vibration system of two-degrees of freedom which represents the fundamental characteristics of bevel gears is constructed mathematically. It is found from the analysis of the system that the fundamental characteristics of vibrations which distinguish bevel gears from spur and helical gears are caused by a change in the direction of the tooth-normal in tooth-meshing. The influences of parameters such as the contact ratio, stiffness of gear-carrying shafts and damping ratios on the band width of unstable regions of the vibration are also shown.

1. Introduction

The dynamic behaviours of spur and helical gears are known better than those of bevel gears. Especially the vibration of spur gears has been investigated theoretically as well as experimentally. In order to clarify the dynamic behaviours of bevel gears it is useful, therefore, to introduce a vibratory model which can represent bevel gears as well as spur gears. In the present paper an equation of motion for a pair of gears is derived in a general form, which can make evident the fundamental difference of vibrations between bevel gears and spur and helical gears. Then it is clear from the equation that the difference between both types of gears can be typically represented by a coupled Hill-type vibration model of two degrees of freedom.

It is shown that the coupled Hill-type equations of the model can be analyzed strictly under some simplifications of the model. The dynamic behaviours of bevel gears are compared with those of spur and helical gears in terms of vibration instability. The model and the method of analysis can be used also to estimate the accuracy of approximate methods of analysis for coupled Hill-type equations, which will be applied for a more complicated model of gear vibrations.

2. Equation of motion

Vibrations of a pair of gears are described by the coordinate systems in Fig. 1. The subscripts I and II show the driving and driven sides respectively. The origin 0\(j\) (j=I,II) coincides with the centroid of the corresponding gear when the gears are matched without displacements. The Zj-axis (j=I,II) is taken in the axial direction of each gear. The unit vectors specifying the Xj, Yj and Zj directions are designated by \(i_j\), \(j_j\) and \(k_j\) respectively. Each gear has six degrees of freedom, which are displacements in the direction of Xj, Yj and Zj axes and angular displacements about these axes. The vibrational displacements are represented by a vector which has the components of (Xj, Yj, Zj, \(\theta_{j1}\), \(\psi_{j1}\), \(\Theta_{j1}\)). Suppose that \(l\)-pairs of teeth are in contact and the point \(P_l\) (l=1 to \(L\)) is the centre of distributed load of the \(l\)-th pair of meshing teeth. The vector \(x_{lj}\) extends from the origin 0j to the point \(P_l\). The unit vector of the tooth-normal at the point \(P_l\) is designated by \(n_{lj}\). Then the deflection of the \(l\)-th meshing pair of teeth due to the vibration displacements is expressed as follows:

\[
\delta_{lj} = (p_{l1}n_{lj}) + (p_{l2}n_{lj})
\]

where \(p_{lj} = X_l i_j + Y_l j_j + Z_l k_j\) \[[1]\]

\[
+ \phi_{l1}[i_j r_{lj}] + \psi_{l1}[j_j r_{lj}] + \Theta_{l1}[k_j r_{lj}]
\]

The unit vectors of the tangential and binormal directions at the point \(P_l\) are designated by \(b_{lj}\) and \(b_{lj}\) respectively. Then the distortion of the \(l\)-th meshing teeth due to the vibration displacements is written as follows:

\[
\alpha_{lj} = (b_{l1}Q_j) + (b_{l2}Q_j)
\]

where \(Q_j = \phi_{lj} i_j + \psi_{lj} j_j + \Theta_{lj} k_j\).  

Fig. 1 Coordinate systems for a pair of gears

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Let $K_{n1}$ and $\psi_1$ be the stiffness of a pair of teeth against the deflection and the distortion respectively. Then the tooth load $W_{n1}$ and the tooth moment $M_{n1}$ are represented as follows:

$$W_{n1} = K_{n1}\delta_{1n}, \quad M_{n1} = J_{1n}\delta_{1n} \tag{3}$$

In most of bevel gears, the load $W_{n1}$ is distributed over a small region on the contact surface. The moments about three axes due to $M_{n1}$ are, therefore, negligibly small in comparison with those due to $W_{n1}$. For this reason the term $M_{n1}$ is neglected in the following discussion.

Equation (1) can be re-written as follows:

$$\bar{\mathbf{F}}_{n1} = \left( \mathbf{q}_{11} \mathbf{X}_1 \right) + \left( \mathbf{q}_{12} \mathbf{X}_2 \right)$$

where

$$\mathbf{q}_{ij} = \left( (n_{i1},n_{i2}), (n_{i3},n_{i4}), (r_{i1},r_{i2}), (r_{i3},r_{i4}), \right) \quad (j = 1, 2) \tag{4}$$

The vector $\mathbf{q}_{ij}$ represents the direction of the tooth stiffness in the six degrees of freedom system. This direction is that of the tooth normal when only the rectilinear displacements are considered.

Supposing $\mathbf{q}_{ij}$ and $\mathbf{q}_{ij}$ are the column and the row matrices, the stiffness matrix of meshing teeth is expressed as follows:

$$K_{ij} = \sum_{k} K_{ik} \delta_{ik} \delta_{jk} \quad (j, k = 1, 2) \tag{5}$$

The damping matrix with respect to the damping force between meshing teeth is represented as follows:

$$C_{ij} = \sum_{k} C_{ik} \delta_{ik} \delta_{jk} \quad (j, k = 1, 2) \tag{6}$$

where $c_{n1}$ is the damping coefficient of the damper of the $i$-th pair of meshing teeth. Using Eqs. (5) and (6), the stiffness and damping matrix with respect to the vector $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ are written as:

$$K_{\mathbf{x}} = \begin{bmatrix} K_{111} & K_{112} \\ K_{121} & K_{122} \end{bmatrix}, \quad C_{\mathbf{x}} = \begin{bmatrix} C_{111} & C_{112} \\ C_{121} & C_{122} \end{bmatrix}$$

The equation of motion for a pair of gears is expressed as follows:

$$(m_{11} + m_{12}) \ddot{\mathbf{X}} + (C_{11} + C_{12}) \dot{\mathbf{X}} + K_{11} \mathbf{X} = 0 \tag{7}$$

where

$$\mathbf{K}_{11} = \sum_{i=1}^{l} K_{11} \mathbf{q}_{1i}, \quad \mathbf{C}_{11} = \sum_{i=1}^{l} C_{11} \mathbf{q}_{1i}, \quad \mathbf{C}_{12} = \sum_{i=1}^{l} C_{12} \mathbf{q}_{1i} \tag{8}$$

The terms $m_{11}$ and $m_{12}$ in Eq. (7) are mass matrices of gears and shafts respectively and $C_{11}$ and $K_{11}$ are damping and stiffness matrices with respect to the gear carrying shafts.

Using Eqs. (4), (5) and (6), Eq. (7) can be re-written as:

$$(m_{11} + m_{12}) \ddot{\mathbf{X}} + \sum_{i=1}^{l} C_{1i} \dot{\mathbf{X}} + \sum_{i=1}^{l} K_{1i} \mathbf{X} = 0 \tag{8}$$

where

$$\mathbf{q}_{i} = (\mathbf{q}_{1i}, \mathbf{q}_{2i}) \tag{9}$$

Neglecting the terms $m_{11}$, $\mathbf{c}_{11}$ and $\mathbf{k}_{11}$, Eq. (8) is written as follows:

$$(\mathbf{X} \dot{\mathbf{q}}_{1} + \sum_{i=1}^{l} C_{1i} \eta_{1i}) + \sum_{i=1}^{l} K_{1i} \eta_{1i} = 0 \tag{9}$$

where

$$\eta_{1i} = (\mathbf{q}_{1i}^{m} - \mathbf{q}_{1i}^{r}), \quad \delta_{1i} = (\mathbf{X} \dot{\mathbf{q}}_{i})$$

In spur gears the vector $\mathbf{q}_{1i}$ is not dependent on time nor on subscript $i$'. Equation (9) is, therefore, reduced to an equation for the variable $\delta_{1i}$. On the other hand, in bevel gears, the vector $\mathbf{q}_{1i}$ is a function of time and depends on $i$. Therefore in bevel gears Eq. (9) represents a coupled vibration of $l$-degrees of freedom. It is of course necessary to note that in any cases the vibration represented by Eq. (9) is a Hill-type vibration, because the stiffness $K_{1}$ is a function of time.

3. A vibration model of two degrees of freedom

Figure 2 shows one of the simplest vibration systems which can represent the above introduced difference between spur and bevel gears. The springs $k_{3x}$ and $k_{3y}$ represent the stiffness of gear carrying shafts. The damping force acting through the shafts is represented by the dampers $c_{3x}$ and $c_{3y}$. The directions of forces acting on these elements are fixed in the space. The springs $k_{n1}, k_{n2}$ and the dampers $c_{n1}, c_{n2}$ represent the meshing pairs of teeth. These elements rotate with an angular velocity $\omega$ which corresponds to the term $\mathbf{q}_{1i}$ in Eq. (9). A pair of teeth continue to contact in the interval of $\theta_{1} - \theta_{2}$, and $\theta_{1}$, $\theta_{2}$, $\omega_{1}$, $\omega_{2}$ are obtained by an angle of one meshing period. A double contact occurs in the interval of $(\theta_{1} - \theta_{2})$.

For simplicity, the case of $k_{3x} = k_{3y}$ and $c_{3x} = c_{3y} = c_{3}$ is investigated in the following discussion. The essential difference between spur and bevel gears will not be lost by this simplification. The coefficients are nondimensionalized as follows:

$$k_{3} = k_{0}, \quad c_{3} = c_{0}, \quad \omega_{0} = \omega_{0}/\omega_{f}, \quad k_{n1} = k_{0}/k_{0} = k_{0}/k_{0}, \quad \omega_{f} = \omega_{f}/\omega_{f} = 2\pi \omega_{0}/\omega_{f}$$

where $e$ and $t_{z}$ are the contact ratio and the tooth meshing period respectively. Defining the vector $\mathbf{q}_{s}(x, y)$, the vector $\mathbf{q}_{1}$ is given as:

$$\mathbf{q}_{1} = \begin{bmatrix} \cos \omega_{f}t_{z} \\ \sin \omega_{f}t_{z} \end{bmatrix}$$

$$\mathbf{q}_{s} = \begin{bmatrix} \cos (\omega_{f}t_{z} + \theta_{z}) \\ \sin (\omega_{f}t_{z} + \theta_{z}) \end{bmatrix}$$

Instead of $\mathbf{q}_{2}$, the vector $\mathbf{q}_{3}$ which is perpendicular to the vector $\mathbf{q}_{1}$ can be used:

$$\mathbf{q}_{3} = \begin{bmatrix} -\sin \omega_{f}t_{z} \\ \cos \omega_{f}t_{z} \end{bmatrix}$$

![Fig. 2 A simplified vibration model of bevel gears](image-url)
Defining the vector $\mathbf{x}$ as

$$x = (x_1, x_2), \quad x_1 = (q, X), \quad x_2 = (q, X). \quad \quad (12)$$

the equation of motion for the system in Fig. 2 is expressed with respect to the coordinates $x_1$ and $x_2$ as follows:

$$I\ddot{r} + \Gamma \dddot{r} + \kappa \dot{r} = 0. \quad \quad (13)$$

where the elements of the matrix $\Gamma'$ and $\kappa$ are

$$\Gamma_{11} = 2\zeta_0 \sqrt{\kappa_1} + 2\zeta_0 \sqrt{\kappa_2} \sin \theta_1 \cos \phi_1 + 2 \omega, \quad \Gamma_{12} = \Gamma_{21} = -4 \omega, \quad \Gamma_{22} = 2\zeta_0 \sqrt{\kappa_1} \sin \theta_1 \sin \phi_1 + 2 \omega\sqrt{\kappa_2} \sin \phi_1, \quad \Gamma_{33} = \kappa_1 \sin^2 \theta_1 + \kappa_2 \sin^2 \phi_1 + 2 \omega \sqrt{\kappa_1} \sin \theta_1 + 2 \omega \sqrt{\kappa_2} \sin \phi_1. \quad \quad (14)$$

In the single contact region the term $\kappa_{12}$ should be eliminated. In the spur and helical gears $\kappa_{12} = 0$.

Strictly speaking the stiffness $k_{11}$ ($i = 1, 2$) is dependent on time. But the differences between spur and helical gears and bevel gears are not lost by regarding the term $\kappa_{12}$ as a constant. In the following discussion $\kappa_{11}$ is regarded as a constant and necessarily $\kappa_{12}$ is equal to $k_{12}$, which will be denoted by $k_{12}$.

The normalized form of Eq. (13) can be expressed as:

$$\ddot{y} = Ay, \quad y = (x_1, x_2, \dot{x}_1, \dot{x}_2). \quad \quad (15)$$

The elements of matrix $A$ are constant in the single and double contact regions respectively. The eigenvalues and the eigenvectors of $A$ can be, therefore, calculated for each region. The general solutions of Eq. (13) can be expressed by using these eigenvalues and eigenvectors. Denoting the double and the single contact regions by subscripts 'a' and 'b' respectively, and expressing the general solutions by $y_a(t)$ and $y_b(t)$, the continuity conditions of the solution at the time $t = t_a$ are:

$$y_a(t_a) = y_b(t_a), \quad \tau_a = (t_a - t_b). \quad \quad (16)$$

The conditions of normal solutions are expressed as follows:

$$\sigma y_i(0) = \left( \begin{array}{c} R \cos \theta_1 \\ R \sin \theta_1 \end{array} \right) \sin \phi_1, \quad \tau_s = \pi - \tau_a. \quad \quad (17)$$

From Eqs. (16) and (17) a set of equations for eight unknown constants which are included in the expressions of $y_a(t)$ and $y_b(t)$ are obtained. When the coefficients matrix is expressed by $B$, the condition in which a nontrivial solution of the set of equations is possible is expressed:

$$|B| = 0. \quad \quad (18)$$

Equation (18) is a fourth order equation in $\tau_a$. When absolute values of the roots $\tau_a$ of Eq. (18) are less than unity, the system will be stable. In the following section the results of numerical analysis of Eq. (18) for various conditions of the system will be shown.

The unstable region of the system can be simply estimated by making an approximation in the damping matrix. Eliminating the terms of damping in Eq. (13), the natural frequencies and modes of the system can be expressed as

$$\omega_{11} = \kappa_1 - \omega^2 + \omega_1(1 - \cos \theta_1), \quad \omega_{12} = -\omega_2 \cos \theta_1, \quad \omega_{21} = \kappa_2 - \omega^2 + \omega_2(1 - \cos \theta_1), \quad \omega_{22} = \kappa_1 - \omega^2. \quad \quad (19)$$

For simplicity the damping ratio of the vibration in each natural mode is estimated as

$$2 \zeta_1 = \left(2 \zeta_0 \sqrt{\kappa_1} + 2 \omega_1(1 - \cos \theta_1) / \omega_1 \right), \quad 2 \zeta_2 = \left(2 \zeta_0 \sqrt{\kappa_1} + 2 \omega_2(1 - \cos \theta_1) / \omega_1 \right). \quad \quad (20)$$

The off-diagonal elements of damping are neglected. The solutions of the system are

$$x_i = \sum_{i=1}^{\infty} e^{-\zeta_i \omega_1 t_i^i} \left( \begin{array}{c} \cos \sqrt{-\zeta_i^2 \omega_1} t_i^i \\ \sin \sqrt{-\zeta_i^2 \omega_1} t_i^i \end{array} \right), \quad \tau_i = \tau - \tau_a. \quad \quad (21)$$

The expression by Eqs. (16) and (17) can be reduced to Eq. (22) and Eq. (23) respectively:

$$x_a(t_a) = x_b(t_a), \quad \dot{x}_a(t_a) = \dot{x}_b(t_a), \quad \alpha x_a(0) = R x_b(0), \quad \alpha \dot{x}_a(0) = R \dot{x}_b(0). \quad \quad (22)$$

The relationship among the modal vectors is written as follows:

$$\tau = \tau_a, \quad (x_a(x) = (x_1 x_2) = \cos \theta_1 / 2 = p, \quad \dot{x}_a(x) = (x_1 \dot{x}_2) = \sin \theta_1 / 2 = q), \quad \tau = \tau_a, \quad x_a(x) = x_2 x_1 = p, \quad \dot{x}_a(x) = -x_1 x_2 = q. \quad \quad (24)$$

Then the characteristic equation corresponding to Eq. (18) is written:

$$|\mathbf{B}| = f_i(p) + f_i(p) q + f_i(p) a^2 + f_i(p) a + f_i(p) a^2 + f_i(a) d^2 + f_i(a) d^4, \quad \alpha = q / p, \quad \sigma = \sigma / p^2. \quad \quad (25)$$

where $f_i$ is a polynomial of order $i$ in $\mathbf{G}$. It is easily shown that the $f_i$ in $\mathbf{G}$ is resolved as a multiplication of two second order factors in $\mathbf{G}$. They are correspondent to independent characteristic equations of Melrinder-type vibration of one-degree of freedom. It can also be shown approximately that the zeros of $f_i$ in $\mathbf{G}$ approach maximum value under the following condition:

$$\sqrt{1 - \zeta_1^2 \omega_1^2 \omega_2^2} + \sqrt{1 - \zeta_2^2 \omega_1^2 \omega_2^2} = 2 \eta \quad \quad (26)$$

$$(i = 1, 2, m = 1, 2, 3, 2, \ldots)$$

When the $i$-th natural frequency $\omega_i$ which is measured over one tooth-mesh period is defined by

$$\omega_i = \sqrt{1 - \zeta_1^2 \omega_i^2 (t-1)} + \sqrt{1 - \zeta_2^2 \omega_i^2 (t-2)} \omega_i. \quad \quad (27)$$

Eq. (26) can be re-written as:

$$\omega_i / \omega_i = \eta_i \quad (i = 1, 2, m = 1, 2, 3, 2, \ldots) \quad \quad (28)$$
It should be noted that the frequencies $\omega_n$ in Eq. (27) are calculated by choosing pairs of modal vectors $\mathbf{x}_4$ and $\mathbf{x}_5$ whose direction cosine is largest. In the corresponding pairs of natural modes the ratios of stiffness variation are given as follows:

$$\Delta_1 = \frac{\varepsilon_4 (1 - \cos \theta_1)}{1 - \varepsilon_4 (1 - \cos \theta_1)}, \quad \Delta_2 = \frac{\varepsilon_4 (1 + \cos \theta_1)}{1 - \varepsilon_4 (1 - \cos \theta_1)}$$  \hspace{1cm} (29)

Since $\varepsilon \neq 0$, Eq. (25) may have zeros of large absolute value even if the condition given in Eq. (26) is not satisfied. The unstable regions with respect to these zeros occur at frequencies in the neighbourhood of

$$\omega_n = \frac{(\omega_1 + \omega_2)}{2} \omega_n = \nu_n, \quad (n = 1/2, 1, 3/2, \ldots),$$

where $\nu_n$ is a function of the natural frequencies $\omega_1$ and $\omega_2$. The values $n$ given in Eqs. (28) and (30) are called the order of resonance with natural frequency $\omega_n$.

4. Results of numerical analysis

The tooth stiffness $K_m$, the contact ratio $\varepsilon$, the angle of tooth $\theta$, the rotation angle of the tooth-normal vector in a tooth-mesh period and damping ratios $\gamma_1$, $\gamma_2$ are essential parameters of the model under consideration. The parameters $K_m$ and $\varepsilon$ respectively determine the ratio of stiffness variation $\Delta_1$ and the spectrum of the stiffness function. Figures 3 to 6 show the results of a numerical analysis of Eqs. (13) by the procedure of Eqs. (15) to Eq. (18).

Figure 3 shows the dependence of unstable regions on the angle of tooth $\theta$ with respect to the tooth mesh frequency $\omega_1/\omega_2$. The resonant order $n_1$ which is obtained from Eqs. (28) and (30) is indicated by chained lines. The natural frequencies $\omega_n$, $\omega_{n+1}$ and $\omega_{n+2}$ for Eqs. (28) and (30) are calculated from Eq. (19) by neglecting the term $\omega_1$. The curve $\omega_1 = 0$ in Fig. 3 shows the condition that the angular velocity $\omega_1$ is equal to $\omega_2$ obtained in the fixed coordinate system. Unstable regions with respect to $\omega_2$ are not found in the model under consideration. The curves of $n_1 (i = 1, 2, 3)$ are almost coincident with the center lines of corresponding unstable regions. This fact means that the three kinds of resonant frequencies which cause instability can be approximately estimated by Eqs. (28) and (30).

The unstable regions due to resonance with $\omega_2$, which are typically seen in the regions of $\gamma_2 = 1/2, 1$, tend to be narrower with an increase of $\theta$. On the other hand the unstable regions due to resonance with $\omega_1$ tend to be wider with $\theta$. These tendencies are correspondent to those of $\Delta_2$ and $\Delta_3$ given in Eq. (29). The unstable regions of $\gamma_3$ due to resonance with $\omega_3$ show an intermediate tendency between the others.

An unstable region which is caused by the influence of the angular velocity $\omega$ is seen under the condition of $\omega_2/\omega_1 > 2$ and $\theta > 50^\circ$. No evident influence of $\omega$ is found except under this condition. It should be noted that the value $\theta_2$ may not exceed 30$^\circ$ and $\omega_1/\omega_2$ may not exceed 2 in usual bevel gears. The importance of $\theta_2$ is therefore recognized in the fact that the direction of modal vectors and the stiffness variation of the natural modes are dependent on the angle $\theta_2$.

Figure 4 shows the dependence of the unstable regions on the damping ratios $\gamma_1$ and $\gamma_2$ with respect to $\omega_2/\omega_1$. In the case where $\gamma_1$ is increased from 0 to 0.15 the value of $\gamma_1$ is fixed at zero. In the case where $\gamma_2$ is varied the value of $\gamma_2$ is fixed at 0.01. It is apparent that the unstable regions of $n_1$ do not diminish rapidly with an increase of $\gamma_2$. On the other hand the unstable regions of $n_2$ disappear rapidly as $\gamma_2$ increases. The tendencies of the effect of $\gamma_2$ on these two kinds of unstable region are contrary to those of that of $\gamma_1$. The dependence of the unstable regions of $n_3$ on $\gamma_1$ and $\gamma_2$ shows an intermediate tendency between the others. These influences of $\gamma_1$ and $\gamma_2$ on instability found in Fig. 4 are

![Fig. 3 Influence of the rotation angle of the tooth-normal vector on the unstable regions](image-url)
correspondent to the effects of $\zeta_1$ and $\zeta_2$ on
the damping ratios of natural modes given
in Eq. (20).

It is known that the damping ratio $\zeta_1$
is about 0.07 in spur gears. Although the
value of $\zeta_3$ may change in cases, it may not
be expected to be larger than $\zeta_1$. For example
the value of $\zeta_3$ is estimated to be 0.02
for bearings of shafts.

Figure 5 shows the dependence of the unstable regions on the contact ratio with
respect to the tooth-mesh frequency $\omega_k/\omega_k$. The dependence of instability on the contact
ratio may be explained by the relationship of it with the Fourier coefficients of the
stiffness function. The $k$-th Fourier coefficient of the stiffness function can be
expressed by the stiffness variation ratio
$\Delta_k$ and the contact ratio $\xi$ as follows:

$$C_k = (\Delta_k/\pi k) \sin k(\xi-1)\pi.$$  \hspace{1cm} (31)

It is found in Fig. 5 that the band width of
unstable regions of $n_2 = 1/2$, 1 and 3/2 corresponds to the magnitude of the coefficient
$C_{2k}$, $k=1,2$ and 3 respectively. There is seen
a similar tendency between the band width of
the unstable region of $n_1$ and the correspond-
ing coefficients $C_{1k}$. Strictly speaking, the
value of $\Delta_k$ in Eq. (31) increases with the
contact ratio, of which the influence can be
also found on the unstable regions of $n_1$. It
should be noted that the possibility of $\kappa n$
decreasing with an increase of $\xi$ is not con-
sidered in the calculation of Fig. 5.

For the unstable regions of $n_3$ as seen
typically in the case of $n_3 = 1/2$ can not be
found any simple relationship with the Fourier
coefficients $C_{1k}$. But in most cases the un-
stable regions of $n_1$ and $n_3$ which distinguish
the bevel gears from spur and helical gears,
tends to be wider with an increase of $\xi$.

Figure 6 shows the dependence of the unstable regions on the magnitude $\kappa_n$ with
respect to the tooth-mesh frequency $\omega_k/\omega_k$. The case of $\kappa_n = 1/\xi$ is correspondent to the
case where the stiffness of shafts is reduced
to zero. In the case where $\kappa_n = \zeta_3 = 0$, the unstable regions may appear even in the
neighbourhood of the axis of $\kappa_n = 0$. The
unstable regions seen in small $\kappa_n$ disappear
when a small damping of $\zeta_1 = \zeta_3 = 0.025$ is

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**Fig. 4 Influence of damping ratios on unstable regions**

**Fig. 5 Influence of the contact ratio on unstable regions**
taken into consideration. Especially the unstable regions of $n_1$ become narrower rapidly with a decrease of $K_m$ under existence of small damping $3\beta$. This is because the reduction of $K_m$ makes strong the influence of $3\beta$ on the damping ratio of the natural mode corresponding to $\omega_m$, while it reduces the stiffness variation $\Delta_k$.

It is also seen in Fig. 6 that the unstable region of $n_1$ and $n_3$ shifts to a higher frequency range with reduction of $K_m$. The increasing of the stiffness of gear carrying shafts $k_s$, also for this reason, useful for stabilization of the vibration system.

5. Conclusions

A mathematical model of two degrees of freedom which can represent the essential difference of vibrational characteristics between bevel gears and spur and helical gears has been constructed and investigated. From the analysis of the coupled Hill-type vibration of the model, some fundamental characteristics of the vibrations of bevel gears have been clarified:

1. The vibrational characteristics of bevel gears are due to the fact that the tooth-normal varies its direction with rotation of gears. Number of natural modes which are represented by parametric vibrations of single-degree of freedom tends to increase because of this rotation of the tooth-normal direction.

2. In the equation of motion the angular velocity of the tooth-normal vector may be regarded as a characteristic of bevel gears. But this term can not have any important influence on the vibration of bevel gears under usual running conditions.

3. The unstable regions due to the higher order natural mode show a similar tendency to those of the spur gears which can be regarded as a single degree of freedom system. The lower order natural mode can not be affected by the damping force due to tooth-meshing. The unstable region caused by this natural mode may become noticeably wide when the damping force acting on gear carrying shafts is not large.

4. The variation of contact ratio between $1 \sim 2$ may cause a change of the frequency spectra of the stiffness function. The width of unstable regions has close relation to the frequency spectra. But as a whole, the variation of the contact ratio does not have such an important influence in the problem of instability of gear vibrations as the other parameters.

5. By increasing the stiffness of gear carrying shafts the system can be stabilized through effects of reducing the stiffness variation ratio as well as effect of raising the lower natural frequency of the system.

6. The coupled Hill-type vibration of bevel gears can be approximately resolved as uncoupled Hill-type vibrations by applying the modal analysis in the single and the double contact regions separately. The uncoupled Hill-type equations can be estimated by applying the results which have been obtained for spur gears of Meißner-type vibration.

References

3. reference (2), p. 258