The Torsion of a Hollow Cylinder with Many Internal Circumferential Cracks*

By Toshikazu SHIBUYA**, Takashi KOIZUMI*** and Kazuyoshi SUZUKI****

In a solid with many cracks, the stress field is affected by the mutual interference of cracks. The effect of the interference of many cracks comes between those of two and infinite internal cracks. Therefore, in the present paper, the torsional stress state of a hollow cylinder with two or infinite parallel internal circumferential cracks is considered on the basis of the theory of elasticity. Numerical results are illustrated for the distributions of displacements and shear stresses and for the relations between the stress intensity factors and the crack dimensions. The stress intensity factors in the case of infinite cracks are smaller than those in the case of two cracks. The stress intensity factors in both cases increase with an increasing distance between cracks, and tend to that in the case of one crack.

1. Introduction

In the problems involving a solid with many cracks, the authors are here concerned with the interference of crack stress fields. The effect of the interference of many cracks comes between those of two and an infinite row of cracks. The purpose of the present paper is to consider the problems of axisymmetric torsion of a hollow cylinder with two parallel internal circumferential cracks or an infinite row of them on the basis of the theory of elasticity. This paper is a sequel to the previous ones**-***. The boundary value problems in this study are reduced to solving a pair of dual series equations. The equations are reduced to an infinite system of simultaneous equations by the method of the previous paper***. Numerical results are obtained for stress fields and stress intensity factors.

2. Stress functions

We use the cylindrical coordinates \((r, \theta, z)\). If \((u_r, u_\theta, u_z)\) denotes the displacement vector and \((\sigma_r, \sigma_\theta, \sigma_z, \tau_r\theta, \tau_rz, \tau_\theta z)\) the stress tensor, the solution of an axisymmetric equation with torsion can be written by using Boussinesq's stress function \(\psi(\tau, z)\) as follows:

\[
\begin{align*}
\tau_{r\theta} &= 2 \frac{\partial \psi}{\partial \theta}, \\
\tau_{rz} &= -\frac{\partial^2 \psi}{\partial r \partial z}, \\
\tau_{\theta z} &= 2 \frac{\partial \psi}{\partial r}, \\
\sigma_r &= \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = \tau_{\theta z} = 0
\end{align*}
\]

where \(G\) is the shear modulus, and \(\psi\) is an axisymmetric harmonic function.

3. Two parallel internal circumferential cracks

In this section, we consider an axisymmetric torsional problem of an infinite hollow cylinder of radii \(r_c\) and \(r_i\) with two parallel internal cracks of equal radius \(r_0\) which are spaced at distance \(2k\), as shown in Fig.1. In the cylindrical coordinates \((r, \theta, z)\), the \(\theta\)-axis coincides with the axis of the hollow cylinder, and the cracks occupy the regions \(r=\pm k\) and \(r_0 < r < r_c\). Due to the symmetry of the stress field with respect to \(r=0\), the problem is equivalent to that of a semi-infinite hollow cylinder \((\theta \geq 0)\) of which the plane \(r=0\) is bonded to a rigid body. Therefore, the boundary conditions are

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\[ (1) \: (\tau_\theta)^\alpha, r_\alpha = (\tau_\theta)^\alpha = (\tau_\theta)^\alpha = (\tau_\theta)^\alpha = 0 \]  
\[ (2) \]

\[ (2) \: (\tau_\phi)^\alpha, r_\phi = (\tau_\phi)^\alpha (r \leq r_s) \]  
\[ (3) \]

\[ (3) \: (\tau_\alpha)^\alpha = (\tau_\alpha)^\alpha (r \geq r_s) \]  
\[ (4) \]

\[ (4) \: (\tau_\phi)^\phi = (\tau_\phi)^\phi = 0 (r \geq r_s, r_s < r) \]  
\[ (5) \]

\[ (5) \: (\tau_\theta)^\theta = 0 (r \geq r_s) \]  
\[ (6) \]

\[ (6) \: Torque = T \text{ at } r \to \infty \]  
\[ (7) \]

where unstarae quantities refer to region \(0 \leq r \leq h\) and starred ones to region \(h \leq r \leq \infty\).

The stress functions \(\lambda\) \((0 \leq r \leq h)\) and \(\lambda^*\) \((h \leq r < \infty)\) are given in the following forms:

\[
\begin{align*}
\lambda &= Bz(2z^2 - 3p^2) + \sum_{n=1}^{\infty} A_n C_n(\xi_s) \sinh(\xi_s) \quad (0 \leq r \leq h) \\
\lambda^* &= A^*(r^2 - 2p^2) + B^*(2z^2 - 3p^2) + \sum_{n=1}^{\infty} A^*_n C_n(\xi_s) \exp(-\xi_s(x - h)) \quad (h \leq r < \infty)
\end{align*}
\]

where

\[
C_n(\xi_s) = J_0(\xi_s) - J_1(\xi_s) Y_0(\xi_s) / Y_1(\xi_s)
\]

and \(J_0(\xi_s), Y_0(\xi_s)\) are Bessel functions of the first and second kinds of order \(n\). Moreover, \(B, A^*, B^*, A_n, A^*_n\) are arbitrary constants. The displacements and stresses due to Eqs.(8) are

\[
\begin{align*}
r_s &= 6Br + \sum_{n=1}^{\infty} \xi_s A_n C_n(\xi_s) \sinh(\xi_s) \\
t_s &= GBr + G \sum_{n=1}^{\infty} \xi_s^2 A_n C_n(\xi_s) \cosh(\xi_s) \\
t_r &= -G \sum_{n=1}^{\infty} \xi_s^2 A_n C_n(\xi_s) \sinh(\xi_s) \\
r_s^* &= 2A^*r + 6Br + \sum_{n=1}^{\infty} \xi_s A_n C_n(\xi_s) \exp(-\xi_s(x - h)) \\
t_s^* &= 6GB^* - G \sum_{n=1}^{\infty} \xi_s^2 A_n C_n(\xi_s) \exp(-\xi_s(x - h)) \\
t_r^* &= -G \sum_{n=1}^{\infty} \xi_s^2 A_n C_n(\xi_s) \exp(-\xi_s(x - h))
\end{align*}
\]

The conditions of Eq.(6) and \((\tau_\theta)^\alpha, r_\alpha = (\tau_\theta)^\alpha, r_\alpha = 0\) in Eq.(2) are always satisfied. By using the condition of \((\tau_\theta)^\alpha, r_\alpha = (\tau_\theta)^\alpha, r_\alpha = 0\) in Eq.(11), we obtain \(\xi_s\) as the positive zeros of the equation

\[
C_0(\xi_s) = 0
\]

where \(0 < \xi_s < \xi_s < \ldots\).

The continuity condition of the stress \((\tau_\theta)^\alpha, r_\alpha = (\tau_\theta)^\alpha, r_\alpha = 0\) and the condition of Eq.(7) require that

\[
A^*_n = -A_n \cosh(\xi_s), \quad B^* = B = \frac{T}{2\pi r_s^2 - r_t^2}
\]

The remaining two boundary conditions (3) and (5) may be written in the forms:

\[
\begin{align*}
2A^*r + \sum_{n=1}^{\infty} \xi_s A_n C_n(\xi_s) \exp(-\xi_s(x - h)) &= 0 \quad (r \leq r_s, r) \\
\frac{2T}{\pi(r^2 - r_t^2)} + G \sum_{n=1}^{\infty} \xi_s^2 A_n C_n(\xi_s) \cosh(\xi_s) &= 0 \quad (r \geq r_s, r_s < r)
\end{align*}
\]

To solve the dual series equations (13) and (14), the technique described in [1] is used. We introduce the following transformations:

\[
\begin{align*}
h &= (r_s + r_t)/2, \quad \omega = (r_s - r_t)/2, \quad \rho = B + B - 2B \cos(\phi - \pi) \quad (0 \leq \phi \leq \pi)
\end{align*}
\]

Using Eq.(15), the variable \(r\) in \(r \leq r_s, r\) can be changed to a new one \(\phi\) in \(0 \leq \phi \leq \pi\), when \(r = r_s\) corresponds to \(\phi = 0\) and \(r = r_t\) to \(\phi = \pi\).

There exists a stress field which is singular in the vicinity of the crack tip. We assume that shear stress \((\tau_\theta)^\alpha, r_\alpha\) has a singularity as \(r^{-\alpha} r_\alpha\) at \(r = r_s\) and takes a continuous finite value in \(r < r_s, r\). Then, the stress \((\tau_\theta)^\alpha, r_\alpha\) can be expressed by Fourier series with respect to \(\phi\) as follows:

\[
(\tau_\theta)^\alpha, r_\alpha = \frac{r}{r_s^2 - r_t^2} \sum_{n=1}^{\infty} a_n \cos n\phi r(r_s - r) \quad (r \geq r_s, r)
\]
where $H(x)$ denotes Heaviside unit function and $\alpha_n$ (n = 0, 1, 2,...) are unknown coefficients. Eq. (16) may be rewritten as

$$ (r_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} T_n \frac{\partial}{\partial r} \left[ \cos \left( n + \frac{1}{2} \right) \phi \right] H(r-r_n) $$

(17)

where

$$ T_n = \sqrt{2} \delta_n \left[ (1+\delta_n) \alpha_n - \alpha_{n+1} \right] / (2n+1) \quad (n = 0, 1, 2, ...) $$

and $\delta_n$ denotes the Kronecker delta.

Eq. (17) is equivalent to the equation

$$ \frac{2T}{\pi (r^3 - r_n^3)} = - \sum_{n=1}^{\infty} A_n \sum_{k=1}^{\infty} C_{mk}^2 \left[ (2n+1) \right] \frac{\partial}{\partial r} \left[ \cos \left( n + \frac{1}{2} \right) \phi \right] H(r-r_n) $$

(18)

and we get, after some manipulation,

$$ T = 2 \pi \sum_{n=1}^{\infty} T_n \frac{r_n^3}{r^3} + \frac{\delta_n}{(2n+3)(2n-1)} $$

(19)

$$ A_n = \frac{1}{G \cosh (2 \omega h)} \sum_{n=1}^{\infty} \delta_n \omega T_n P(m,n) \quad (m = 1, 2, 3, ...) $$

(20)

where

$$ P(m,n) = \frac{1}{\cosh (2 \omega h)} \frac{1}{(2n+3)(2n-1)} \\
= \left[ -2C_m C_{m+k} - 2 \sum_{k=1}^{\infty} \left[ F_m C_{m+k} \right] \right] $$

$$ F(m,n) = 2 \sum_{k=1}^{\infty} \frac{2n+3}{x_m^2 - x^2} - \frac{2n+1}{(2n+1)^2 - 4 \omega^2} $$

If the coefficients $A_n$ are given by Eq. (20), the boundary condition of $(r_n)_{n=1}^{\infty}$ is satisfied for arbitrary values of $T$. Therefore, to determine the coefficients $T_n$, we use the condition of $(\phi)_{n=1}^{\infty}$. Using the following form of Gegenbauer's addition theorem

$$ C_{mk}^2(\xi r) = 2 \xi r \sum_{j=1}^{\infty} \sin (p+1) \phi \sum_{j=1}^{\infty} \sin (p+1) \phi \left[ F_{mk} C_{m+j} C_{m+k} \right] $$

(21)

and substituting the above equation into Eq. (13), we obtain

$$ A_n + \sum_{n=1}^{\infty} A_n \sum_{j=1}^{\infty} \sin (p+1) \phi \sum_{j=1}^{\infty} \sin (p+1) \phi \left[ F_{mk} C_{m+j} C_{m+k} \right] = 0 $$

(22)

Since the above equation must hold for arbitrary values of $\phi$, we get the following equation

$$ \sum_{n=1}^{\infty} A_n \left[ F_{mk} C_{m+j} C_{m+k} \right] = - A_n \delta_n \quad (p = 0, 1, 2, ...) $$

(23)

Substituting Eq. (20) into Eq. (23) and interchanging the order of summations, we obtain an infinite system of simultaneous equations

$$ \sum_{n=1}^{\infty} T_n Q(n,p) = A_n \delta_n \quad (p = 0, 1, 2, ...) $$

(24)

where

$$ Q(n,p) = \sum_{n=1}^{\infty} \left[ 2F_{mk} C_{m+j} C_{m+k} \right] P(m,n) $$

A physical quantity of interest is the stress intensity factor $K_v$ given by

$$ K_v = \lim_{r \to r_n} \sqrt{2} \int_{r_n}^{\infty} \left( \eta \right)_{n=1}^{\infty} \left( \frac{r_n}{r} \right)^{n-1} \left( \frac{r_n}{r} \right)^{n+1} T_n $$

(25)
4. An infinite row of parallel external circular cracks

We now investigate the torsion of a hollow cylinder of radii \( r_1 \) and \( r_0 \) with an infinite row of parallel internal circumferential cracks which are spaced at equal distance \( 2a \) (Fig.2). The cracks are assumed to lie in planes perpendicular to the hollow cylinder axis and occupy the region \( z = \pm 2nh (n=0,1,2,\ldots) \), \( r_0 \leq r \leq r_1 \). Since the array of the cracks is periodic with spacing \( 2a \), it follows that the described problem can be replaced by one of a finite hollow cylinder subjected to a rotation \( \theta \). The state of stress can be obtained from an analysis of the finite hollow cylinder with length \( L \), which is partially bonded to a rigid surface in an annular region \( r_0 \leq r \leq r_1 \), as shown in Fig.3. The boundary conditions of the problem are

\[
(1) \quad (z_0)_{n+1} = (z_0)_n = 0 \quad (0 \leq z \leq h) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (26)
\]

\[
(2) \quad (n_1)_{n+1} = \theta r \quad (r_0 \leq r \leq r_1) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (27)
\]

\[
(3) \quad (n_0)_{n+1} = 0 \quad (r_0 \leq r \leq r_1) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (28)
\]

\[
(n_2)_{n+1} = 0 \quad (r_0 \leq r < r_1) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (29)
\]

It is convenient to represent \( \lambda \) in the following form:

\[
(30)
\]

Displacement and stresses can now be written as

\[
(31)
\]

If \( \xi_n \) are the positive zeros of Eq.(11), the boundary condition (26) is satisfied. The boundary condition (27) requires that

\[
(32)
\]

The torque \( T \) required to produce the twist \( \theta \) is given by the equation

\[
(33)
\]

Substituting Eq.(31) into this expression and integrating, we find that

\[
(34)
\]

Similarly from Eqs.(31), we see that the remaining boundary conditions (28) and (29) are equivalent to a pair of dual series equations

\[
(35)
\]

To solve the dual series equations (34) and (35), the same method as discussed in the previous section is used. Then, from Eq.(17), the following equation is obtained

\[
(36)
\]

and then the coefficients satisfy the relations

\[
(37)
\]

\[
(38)
\]

To determine the coefficients \( \tau_n \), we use Eqs.(34). Using Gegenbauer's addition theorem which is shown in the previous section, the following equation is obtained

\[
(39)
\]
Substituting Eq. (38) into the above equation and interchanging the order of summations, we obtain an infinite system of simultaneous equations

\[ \sum_{n=0}^{N} Q'(n, p) T_n = A' G b_{np} \quad (p=0, 1, 2, \ldots) \]

where

\[ Q'(n, p) = \sum_{n=0}^{N} \frac{1 - \exp(-2 \pi k)}{1 + \exp(-2 \pi k)} \times J_0(|\omega|) C_{np} (\omega k) \Phi(m, n) \]

The stress intensity factor \( K_b \) is presented in Eq. (25). As \( 2k/r_s \to \infty \), all the results are reduced to those obtained in [1] for an external circular crack.

5. Numerical results and discussion

In the numerical examples the main question is the behavior in convergence of the coefficients \( \tau_n \) and the series giving the stress intensity factors. Since the infinite system of simultaneous equations is to be truncated at a finite number of terms, Eqs. (24) and (40) are approximated by

\[ \sum_{n=0}^{N} Q'(n, p) T_n = A' G b_{np} \quad (p=0, 1, 2, \ldots, N-1) \]

### Table 1

<table>
<thead>
<tr>
<th>( N/2h/r_s )</th>
<th>Two cracks</th>
<th>Infinite cracks</th>
<th>One crack</th>
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Table 1 shows the convergence behavior of the nondimensional stress intensity factor \( K_b/\sigma \sqrt{r_s} \) with the distance between the cracks, \( 2h/r_s \), when \( r_s/r_b = 0.5 \), \( r_s/r_b = 0.75 \); where \( r_s = 2T/r_s(\pi r_s - r_c) \) is the net-section stress. The table shows a rapid convergence of the results even for relatively small values of \( N \).

Figs. 3 and 4 show the distributions of \( \tau_n \) and \( \tau_n \) in the case of two cracks for \( r_s/r_b = 0.5 \), \( r_s/r_b = 0.75 \), \( r_s/r_b = 0.25 \), where the displacement \( \sigma_n^h = 2T(r_s/\pi r_s(\pi r_s - r_c)) \) and the stress \( \tau_n = 2T/r_s(\pi r_s - r_c) \) are the solution of an uncracked hollow cylinder under torsion. The displacements in \( x-h \) (solid lines) are similar to those in one crack \( x-h \); and they rapidly approach straight line \( -2A(\pi h) \) as \( 2h/r_s \) increases. The displacements on the upper and lower surfaces of the crack have different values. The displacements in \( 0 \leq h \leq h \) (dashed lines) are always smaller than those in \( x-h \). The stresses in \( x-h \) are similar to those in the case of one crack. The stress on \( x-h \) is infinite at \( r-r_s \) and it decreases monotonically as \( r/r_s \) increases.

Fig. 5 shows the distributions of \( \tau_n \) in the case of infinite cracks for \( r_s/r_b = 0.5 \), \( r_s/r_b = 0.75 \), \( r_s/r_b = 0.25 \), and \( r_s/r_b = 0.25 \). The stresses in \( x-h \) are smaller than that in \( x-h \). Both results approach that in the case of one crack as \( 2h/r_s \to 1.0 \). \( K_b/\sigma \sqrt{r_s} \) in the case of several cracks is expected to fit in between these results.
Fig. 3 The distribution of $\tau_{xx}$ (two cracks)

Fig. 4 The distribution of $\tau_{yy}$ (two cracks)

Fig. 5 The distribution of $\tau_{zz}$ (infinite cracks)

Fig. 6 The variation of $K_{f}r_{o}\sqrt{\gamma_{f}}$ with $c/t_0$ (two cracks)

Fig. 7 The variation of $K_{f}r_{o}\sqrt{\gamma_{f}}$ with $c/t_0$ (infinite cracks)

Fig. 8 The variation of $K_{f}r_{o}\sqrt{\gamma_{f}}$ with $2h/r_o$

6. Conclusions

In the present paper, the torsion stress states of a hollow cylinder with two or infinite internal circumferential cracks have been analyzed on the basis of the theory of elasticity. The conclusions are summarized as follows:

1. The interference between the stress fields around infinite cracks is more remarkable than that around two cracks.
2. The stress intensity factors for mode I in both cases are always smaller than that in the case of one crack, and they tend to come closer to the result in the case of one crack as the interval between cracks increases.

References


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