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A Method for Analysis of Elasto-dynamic Contact Problems

by Finite Element Method*

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The basic equations of motion with contact-impact behavior for two bodies are formulated by the finite element method based on the principle of virtual work, in which the unknown contact-impact forces acting on contact surfaces are treated as the forces acting on surfaces of each body. Regarding two bodies as being interconnected through contact surfaces makes it possible to eliminate the unknown force components from the basic equations. Then, the equations of contact motion for two bodies can be derived for various contact states.

The equations of contact motion are applied to a two-dimensional analysis of longitudinal impact of two prismatic rods with an equal cross-sectional area. Although the calculated results fluctuate periodically, the mean value agrees well with that obtained by the theory of propagation of one-dimensional elastic stress wave. This method is practically applicable to elasto-dynamic contact problems.

1. Introduction

Contact and impact problems arise in many machine parts such as gears, roller and so on. Exact solutions of contact and impact stresses are the product of highly sophisticated mathematical analysis for idealized model configurations, (1), (2) whereas impact with friction is an open problem. (3)

In recent years, large-scale computational capabilities have been developed in many areas of structural analysis. The primary technique used in these development is the finite element method (FEM) especially available for complicated structural configurations. Ohte, (4) Tsuta and Yamaji (5) proposed an FEM for elasto-static problems involving contact effect. On the other hand, the authors formulated a contact-impact theory with friction, (6), (8) based on both the conventional FEM for small strain and a new non-incremental FEM for shearing deformation. (7)

In this paper we summarize some aspects of our work in developing a general finite element formulation (6) for two-dimensional elastic contact and impact analysis of structures. In section 2, the basic equations of motion with contact-impact behaviors for two bodies are formulated by the FEM based on the principle of virtual work, in which the unknown contact-impact forces acting on contact surfaces are treated as the forces acting on surfaces of each body. In section 3, regarding two bodies as being interconnected through contact surfaces makes it possible to eliminate the unknown force components from the basic equations expressed in section 2. Then the equations of contact motion for two bodies can be derived from the basic equations for various contact

states. A procedure for calculation is described in section 4. In section 5, the equations of contact motion for two bodies are applied to a two-dimensional analysis of longitudinal impact of two prismatic rods with an equal cross-sectional area and flat ends. In order to investigate that this method can be applied to two-dimensional elasto-dynamic contact problems, the results calculated by this method are compared to those obtained by the theory of propagation of one-dimensional elastic stress wave.

2. Formulation by Finite Element Method for Two Bodies Based on Principle of Virtual Work

For simplicity, let us consider two-dimensional elastic problems of two bodies (1) and (2) in a contact state, as shown in Fig. 1. Let b , A and L represent the thickness, area and boundary length of a body, respectively. And let the subscripts on A and L indicate the body number. An arbitrary point in the body (ξ), ($\xi = 1, 2$), moves with a displacement vector $[U_\xi]$, velocity vector $[\dot{U}_\xi]$ and acceleration vector $[\ddot{U}_\xi]$ by an action of body force vector $[F_{B\xi}]$ and surface force vector $[P_{L\xi}]$. A relative contact force vector $[R]$ acts upon the contact part L_c of the boundary length for each body, and is decomposed into the vector $[R^n]$ in n -direction perpendicular to the contact plane and the vector $[R^t]$ in t -direction tangential

Fig. 1 Two-dimensional contact structures for two bodies.

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to the plane. The vector $[R^n]$ and $[R^t]$ act upon two bodies in the opposite directions. As L_c and $[R]$ change with configurations and deformations of the contact plane, their values are usually unknown and are determined by a trial calculation on the contact process.

Figure 2 shows an element division near the contact area between two bodies divided by constant strain elements. Let n_ξ denotes the number of element divisions in the body (ξ). Nodal points discretized on the contact line L_c are arranged with proper separation l_c which makes it possible to investigate the variation of configurations and stress distributions in contact states. When the element number n_c with the contact nodes of each body is identical, the contact nodal points of two bodies are put in position with pairs of those in a contact state, as shown in Fig. 2. When the discrepancy of position on each pair of the contact nodes is not negligible, alignment of candidate contact nodes must be made, for example, by remeshing two bodies or interpolating the results obtained.

On two bodies in the fixed coordinate system, the displacement vector $[U_\xi]$, ($\xi=1,2$), in an element for each body is,

$$\begin{aligned} [U_i] &= [S][C][d_i], \quad (\xi=1,2), \quad [U_i] = [u_i, v_i]', \\ [C] &= [S', S_j', S_k']^{-1}, \quad [S] = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \\ [S] &= \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \end{bmatrix}, \quad (i=j, k) \\ [d_i] &= [u_{i\xi}, v_{i\xi}, u_{j\xi}, v_{j\xi}, u_{k\xi}, v_{k\xi}]', \quad \dots (1) \end{aligned}$$

where $[d_{e\xi}]$ denotes a nodal displacement vector of a divided element in the body (ξ), and $[]^{-1}$, $[]'$ represent transposed matrix and inverse one, respectively. Also an acceleration vector $[\ddot{U}_\xi]$ is,

$$[\ddot{U}_i] = [S][C][\ddot{d}_i], \quad (\xi=1,2) \quad (2)$$

where $[\ddot{d}_{e\xi}]$ is a nodal acceleration vector of an element.

Strain vector $[\epsilon_\xi]$ and stress vector $[\sigma_\xi]$ in the body (ξ) can be expressed as follows:

$$\begin{aligned} [\epsilon_i] &= [B][C][d_i], \quad [\sigma_i] = [D_i][\epsilon_i], \quad [\epsilon_i] = [\epsilon_{x\xi}, \epsilon_{y\xi}, \gamma_{xy\xi}]' \\ [B] &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad [\sigma_i] = [\sigma_{x\xi}, \sigma_{y\xi}, \tau_{xy\xi}]' \quad \dots (3) \end{aligned}$$

where $[D_\xi]$ denotes stress-strain matrix and depends on assumption such as plane stress, plane strain and so on.

When a relative vector $[R]$ of contact forces is regarded as surface force vector $[P_{L\xi}]$, the principle of virtual work except viscous effect is

$$\int_{t_1}^{t_2} (\delta E_i + \delta T_i - \delta W_i - \delta Q_i) dt = 0, \quad (\xi=1,2) \quad (4)$$

Fig. 2 Two-dimensional element divisions near the contact area.

where t_1 and t_2 denote time instants, δE_ξ , δT_ξ and δW_ξ are virtual kinetic, internal and external works, respectively, and δQ_ξ represents virtual work dependent on the vector $[R]$.

In Fig. 1, the vector $[R]$ is expressed by

$$[R] = [R^n, R^t]' \quad (5)$$

As $[R]$ acts on both bodies in the opposite directions, δQ_1 and δQ_2 are

$$\begin{aligned} \delta Q_1 &= \sum_{e=1}^{n_c} b \int_{l_c} [\delta U_1]' (-[R]) dl \\ \delta Q_2 &= \sum_{e=1}^{n_c} b \int_{l_c} [\delta U_2]' (+[R]) dl \quad \dots (6) \end{aligned}$$

On the other hand, the virtual works are expressed by

$$\begin{aligned} \delta E_i &= \sum_{e=1}^{n_i} b \int_{\Delta} [\delta \epsilon_i]' [D_i] [\epsilon_i] d\Delta, \quad \delta T_i = \sum_{e=1}^{n_i} b \int_{\Delta} [\delta U_i]' \rho_i [\dot{U}_i] d\Delta \\ \delta W_i &= \sum_{e=1}^{n_i} b \left(\int_{\Delta} [\delta U_i]' [F_{B_i}] d\Delta + \int_l [\delta U_i]' [P_{L_i}] dl \right) \quad \dots (7) \end{aligned}$$

where ρ_ξ denotes the density of the body (ξ), and $[P_{L\xi}]$ is the circumference force vector of an element being subjected to the surface force vector $[P_{L\xi}]$.

Rewriting Eqs. (6) and (7) by Eqs. (1) to (3), and substituting them into Eq. (4), we obtain,

$$\left. \begin{aligned} \int_{t_1}^{t_2} [\delta X_1]' ([M_1] [\dot{X}_1] + [K_1] [X_1] - [F_1] + [G]) dt &= 0 \\ \int_{t_1}^{t_2} [\delta X_2]' ([M_2] [\dot{X}_2] + [K_2] [X_2] - [F_2] - [G]) dt &= 0 \end{aligned} \right\} \quad \dots (8)$$

The matrices for the body in Eq. (8) are

$$[X_i] = \sum_{e=1}^{n_i} [d_i], \quad [M_i] = \sum_{e=1}^{n_i} [m_i], \quad [K_i] = \sum_{e=1}^{n_i} [k_i],$$

$$[F_i] = \sum_{e=1}^{n_i} [F_{e_i}], \quad [m_i] = b \int_{\Delta} [C]' [S]' \rho_i [C] d\Delta$$

$$[k_i] = b \int_{\Delta} [C]' [B]' [D_i] [B] [C] d\Delta, \quad [G] = \sum_{e=1}^{n_i} b \int_{l_c} [C]' [S]' [R] dl,$$

$$[F_{e_i}] = b \left(\int_{\Delta} [C]' [S]' [F_{B_i}] d\Delta + \int_l [C]' [S]' [P_{L_i}] dl \right) \quad \dots (9)$$

where Σ means matrix representation.

Eliminating components of displacements, accelerations and loads in locking directions at supported nodes from Eq. (9), we obtain

$$\begin{bmatrix} M_{e1} & 0 \\ 0 & M_{e2} \end{bmatrix} \begin{bmatrix} \ddot{X}_{e1} \\ \ddot{X}_{e2} \end{bmatrix} + \begin{bmatrix} K_{e1} & 0 \\ 0 & K_{e2} \end{bmatrix} \begin{bmatrix} X_{e1} \\ X_{e2} \end{bmatrix} = \begin{bmatrix} F_{e1} \\ F_{e2} \end{bmatrix} + \begin{bmatrix} -G_e \\ G_e \end{bmatrix} \quad \dots (10)$$

where $[M_{C\xi}]$, $[K_{C\xi}]$, $[\ddot{X}_{C\xi}]$, $[X_{C\xi}]$, $[F_{C\xi}]$ and $[G_C]$ denote the matrices condensed in matrices of Eq. (9), respectively. The condensed matrices are classified into three groups (n, t, r) with respect to kinds of the nodes:

- (1) contact nodes in n-direction perpendicular to contact surface,
- (2) contact nodes in t-direction tangential to contact surface, and
- (3) other nodes called r except all contact nodes.

Using group name as superscript, the matrices for the body (ξ) in Eq. (10) are expressed by

$$\left. \begin{aligned} [X_{e\xi}] &= \begin{bmatrix} Y_{e\xi}^n \\ Y_{e\xi}^t \\ Y_{e\xi}^r \end{bmatrix}, \quad [F_{e\xi}] = \begin{bmatrix} H_{e\xi}^n \\ H_{e\xi}^t \\ H_{e\xi}^r \end{bmatrix} \\ [G_e] &= \begin{bmatrix} 0 \\ G_e^n \\ G_e^t \end{bmatrix}, \quad [M_{e\xi}] = \begin{bmatrix} m_{e\xi}^{nn} & m_{e\xi}^{nt} & m_{e\xi}^{nr} \\ m_{e\xi}^{tn} & m_{e\xi}^{tt} & m_{e\xi}^{tr} \\ \text{SYM.} & & \end{bmatrix} \\ [K_{e\xi}] &= \begin{bmatrix} k_{e\xi}^{nn} & k_{e\xi}^{nt} & k_{e\xi}^{nr} \\ k_{e\xi}^{tn} & k_{e\xi}^{tt} & k_{e\xi}^{tr} \\ \text{SYM.} & & \end{bmatrix} \end{aligned} \right\} \quad \dots (11)$$

where double superscripts in $[M_{C\xi}]$ and $[K_{C\xi}]$ are shown in Fig. 3.

Fig.3 Coupled relation for double superscripts in $[M_C\xi]$ and $[K_C\xi]$

Fig.4 State variables for two-dimensional contact nodes.

When Eq.(10) is rearranged by Eq.(11), state variables at contact nodes can be simply represented in Fig.4.

As the vector $[R]$ is absent in a separate state of two bodies, the vector $[G]$ of Eq.(9) becomes zero. Then the motions for two bodies are independent of each other, from Eq.(10).

3. Constitution of Equations of Contact Motion for Two Bodies

The vector $[G_C]$ of Eq.(10) changes with configurations and deformations on contact surfaces, and with contact states between two bodies. The values are always unknown. Hence, for eliminating $[G_C]$ from Eq.(10), the procedures used in ref.(4) are applied. Therefore, two bodies are regarded as a connective body through contact nodes. With an assumption of coincidence in position of all pairs at contact nodes, we develop the equations of motion in both sticking and slipping states for two bodies.

3.1 Sticking State of Two Bodies

For simplicity, let us consider the condition of motion for two bodies regarded as being interconnected without slipping at every pair of contact nodes. The contact force vector $[G_C]$ at a contact node can be decomposed into the vector $[G_C^n]$ in n-direction and the vector $[G_C^t]$ in t-direction. The following relation exists between their vectors:

$$|[G_C^t]| \leq \mu |[G_C^n]| \quad (12)$$

where μ indicates a frictional coefficient at a pair of contact points. When eliminating the vector $[G_C]$ from Eq.(10), two bodies are considered to be continuous. From Eqs.(10) and (11), we obtain

$$\left. \begin{aligned} [G_d] &= [I^{t1}] \begin{bmatrix} -G_c^n \\ G_c^t \end{bmatrix} = [0] \\ [G_d] &= \begin{bmatrix} 0 \\ -G_c^n + G_c^t \\ -G_c^t + G_c^n \\ 0 \end{bmatrix} \\ [I^{t1}] &= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad (13)$$

where $[I^{t1}]$ is the matrix eliminating the vector $[G_C]$ and converting two bodies into one body.

Let us consider the relative displacement vector which results from a difference in the size of every contact point, such as interference in a fit mechanism. Since the relative vector is composed of the vector $[V^n]$ in n-direction and the vector $[V^t]$ in t-direction, we obtain

$$\left. \begin{aligned} [Y_1^n] &= [Y_2^n] + [V^n] \\ [Y_1^t] &= [Y_2^t] + [V^t] \end{aligned} \right\} \quad (14)$$

where $[V^n]$ and $[V^t]$ have constant values provided by an initial condition. Matrices $[I^{u1}]$ and $[I^{v1}]$ related to displacements can be obtained by substituting Eq.(14) into the vector $[X_{C1}, X_{C2}]'$ of Eq.(10).

As acceleration of the vector $[V^n]$ and $[V^t]$ does not exist, acceleration vectors at contact nodes between both bodies are expressed by

$$[\ddot{Y}_1^n] = [\ddot{Y}_2^n], \quad [\ddot{Y}_1^t] = [\ddot{Y}_2^t] \quad (15)$$

Rearranging the vector $[\ddot{X}_{C1}, \ddot{X}_{C2}]'$ of Eq.(10) by Eq.(15), we obtain the matrix $[I^{a1}]$ associated with accelerations.

When Eq.(10) is rewritten by use of matrices $[I^{u1}]$, $[I^{v1}]$, $[I^{a1}]$ and $[I^{f1}]$, we obtain the equation of contact motion for the sticking state:

$$[I^{f1}] [M_c] [I^{a1}] [\ddot{Z}] + [I^{f1}] [K_c] [I^{u1}] [Z] = [I^{f1}] [F_c] + [I^{f1}] [K_c] [I^{v1}] [V]$$

$$[Z] = [Y_1^t, Y_1^n, Y_2^t, Y_2^n]', \quad [V] = [0, V^n, V^t, 0]'$$

$$[I^{u1}] = [I^{v1}] = [I^{f1}]', \quad [\ddot{Z}] = \frac{d^2}{dt^2} [Z]$$

$$[I^{u1}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \quad [M_c] = \begin{bmatrix} M_{e1} & 0 \\ 0 & M_{e2} \end{bmatrix}, \quad [K_c] = \begin{bmatrix} K_{e1} & 0 \\ 0 & K_{e2} \end{bmatrix},$$

$$[F_c] = \begin{bmatrix} F_{e1} \\ F_{e2} \end{bmatrix} \quad (16)$$

where $[0]$ and $[I]$ are zero and unit matrices, respectively.

3.2 Slipping State of Two Bodies

Let us consider the condition of motion for two bodies with slipping at every pair of contact nodes. Slipping condition in the t-direction is expressed by

$$|[G_c^t]| = |\mu [G_c^n]| \quad (17)$$

On the other hand, two bodies are regarded as being interconnected in the n-direction of contact plane. Hence, the two bodies can be considered to be continuous in the n-direction. From Eq.(10), we obtain

$$[G_d] = [I^{f2}] \begin{bmatrix} -G_c^n \\ G_c^t \end{bmatrix}, \quad [G_c] = \begin{bmatrix} 0 \\ G_c^n \\ G_c^t \end{bmatrix}, \quad [G_d] = \begin{bmatrix} 0 \\ -G_c^n + G_c^t \\ -G_c^t \\ 0 \\ G_c^t \end{bmatrix}$$

$$[I^{f2}] = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (18)$$

If Eq.(17) is expressed by $[G_c^t] = \pm \mu [G_c^n]$, the vector $[G_C]$ is

$$[G_c] = \begin{bmatrix} 0 \\ G_c^n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_c^t \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \pm \mu I & I \end{bmatrix} \begin{bmatrix} 0 \\ G_c^n \\ 0 \end{bmatrix} \quad (19)$$

Then the term $[G_C]$ in Eq.(10) is given by

$$\begin{aligned} \begin{bmatrix} -G_e \\ G_e \end{bmatrix} &= [I^n] \begin{bmatrix} -G_e^n \\ G_e^n \end{bmatrix}, & [G_e^n] &= \begin{bmatrix} 0 \\ G_e^n \\ 0 \end{bmatrix} \\ [I^n] &= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & \mp \mu I & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & \pm \mu I & 0 & 0 & 0 & I \end{bmatrix}, \end{aligned} \quad (20)$$

From Eqs. (18) and (20), we obtain

$$[G_e] = [I^n] \begin{bmatrix} -G_e^n \\ G_e^n \end{bmatrix} = [0], \quad [I^n] = [I^{n2}] [I^n] \quad (21)$$

where $[I^n]$ is the matrix of slipping forces.

As the displacement vectors $[Y_1^t]$ and $[Y_2^t]$ in slipping direction between both bodies are independent of each other, these vectors can be treated in the same manner as the vectors $[Y_1^t]$ and $[Y_2^t]$. Then, the following relation holds in the n-direction:

$$[Y_1^n] = [Y_2^n] + [V^n] \quad (22)$$

Substituting Eq. (22) into the vector $[X_{C1}, X_{C2}]'$ of Eq. (10), we obtain the matrices $[I^{u2}]$ and $[I^{v2}]$ related to contact nodes. By referring to Eq. (15), the relation between acceleration vectors in the n-direction is provided by

$$[\ddot{Y}_1^n] = [\ddot{Y}_2^n] \quad (23)$$

Rearranging the vector $[X_{C1}, X_{C2}]'$ of Eq. (10) by Eq. (23), we obtain the acceleration matrix $[I^{a2}]$. When Eq. (10) is rewritten by matrices $[I^{u2}]$, $[I^{v2}]$, $[I^{a2}]$ and $[I^{f2}]$, we obtain the equation of contact motion for a slipping state of two bodies:

$$[I^n] [M_c] [I^{a2}] [\ddot{Z}] + [I^n] [K_c] [I^{u2}] [Z] = [I^n] [F_c] + [I^n] [K_c] [I^{v2}] [V]$$

$$[Z] = [Y_1^n, Y_2^n, Y_1^n, Y_2^n, Y_1^n, Y_2^n]', \quad [V] = [0, V^n, 0, 0, 0, 0]'$$

$$\begin{aligned} [M_c] &= \begin{bmatrix} M_{c1} & 0 \\ 0 & M_{c2} \end{bmatrix}, & [I^{u2}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ [K_c] &= \begin{bmatrix} K_{c1} & 0 \\ 0 & K_{c2} \end{bmatrix}, & [I^{v2}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ [F_c] &= \begin{bmatrix} F_{c1} \\ F_{c2} \end{bmatrix}, \end{aligned}$$

$$[I^{a2}] = [I^{u2}] = [I^{v2}]', \quad [\ddot{Z}] = \frac{d^2}{dt^2} [Z] \quad (24)$$

3.3 Other State of Two Bodies

Let us consider the mixed condition of two-dimensional motion for two bodies with sticking, slipping contacts and separation at multitudinous pairs of contact nodes. The contact force vectors $[G_e^n]$ and $[G_e^t]$ in a separate state become zero. Equations (16) and (24) are applicable to each pair of contact nodes. Then, contact components for the body (ξ) in Eq. (11) are divided as follows:

$$\begin{aligned} Y_{\xi}^n, Y_{\xi}^t &\rightarrow Y_{\xi}^{n1}, Y_{\xi}^{t1}, Y_{\xi}^{n2}, Y_{\xi}^{t2}, \dots \\ \dot{Y}_{\xi}^n, \dot{Y}_{\xi}^t &\rightarrow \dot{Y}_{\xi}^{n1}, \dot{Y}_{\xi}^{t1}, \dot{Y}_{\xi}^{n2}, \dot{Y}_{\xi}^{t2}, \dots \\ H_{\xi}^n, H_{\xi}^t &\rightarrow H_{\xi}^{n1}, H_{\xi}^{t1}, H_{\xi}^{n2}, H_{\xi}^{t2}, \dots \\ G_{\xi}^n, G_{\xi}^t &\rightarrow G_{\xi}^{n1}, G_{\xi}^{t1}, G_{\xi}^{n2}, G_{\xi}^{t2}, \dots \\ V^n, V^t &\rightarrow V^{n1}, V^{t1}, V^{n2}, V^{t2}, \dots \\ \mu &\rightarrow \mu_1, \mu_2, \dots \end{aligned}$$

Also, matrices $[M_{C\xi}]$ and $[K_{C\xi}]$ change according to above vectors. Global matrices related to contact nodes are each formed based on procedures of various contact states investigated at all pairs of the nodes. By substituting these matrices into Eq. (10), we obtain the equation of the mixed contact state.

Procedures in this chapter are for two-dimensional contact problems and their methods can be readily expanded into three-dimensional fields.

4. Method for Analysis

4.1 Procedures for Calculation

In order to calculate numerically the equations of the contact motion for two bodies, we determine the time interval ΔT for integration less than the time required for the stress wave to propagate through the element with the smallest size. Table 1 shows the procedures for computation. At the time T_{i-1} in [3] of the table, we first assume the contact states at all pairs of contact nodes after ΔT . Then we calculate the equations of the contact motion in [5] after composing various matrices and determining initial value vectors in [4].

Results obtained in [5] represent various quantities at the time T_i after ΔT . Contact states at each pair of contact nodes are investigated based on the results in [5]. If there is a discrepancy between the states obtained in [6] and those assumed in [3] for all pairs of contact nodes in [7], we return to the original time T_{i-1} in [4] and calculate again the equations of the contact motion by assuming the contact states obtained in [6] instead of those in [3]. The contact states at the time T_i are determined by repeating the above described procedures, until the states obtained agree with those assumed at the original time.

At the next time T_i instead of T_{i-1} , we calculate the equations of the contact motion by application of the same procedures described above and determine the states at the time T_{i+1} . At the time T_{i+1} , a procedure for calculation goes back to [3] by assuming first the sticking states at the nodes which are in the slipping state at the time T_i , in order to investigate duration of the slipping state at the nodes. When no pairs of contact nodes are in the slipping state, a procedure is returned to [5] for purpose of efficient calculation.

4.2 Derivation of Lumped Mass Matrix

Various methods are used for numerical solutions of ordinary differential equations. Simizu, et al. (9) reported that there exist PC-12, Newmark- β (N- β), Runge-Kutta-Gill (RKG) methods etc., to obtain a solution with good

Table 1. Procedures for computation

- [1] Input and output of data for two bodies.
- [2] Composition and storage of the mass and stiffness matrices for each body.
- [3] Assumption of contact states.
- [4] Composition of matrices of the mass, stiffness, load and initial conditions for the contact states.
- [5] Numerical calculation for equations of contact motion.
- [6] Computation for contact states.
- [7] Comparison of the results obtained in [6] with those assumed in [3] $\rightarrow$ [3]
- [8] Calculation for element stresses.
- [9] Print of results.
- [10] Check of required number for integration $\rightarrow$ [3] or [5]
- [11] End.

phase and gain characteristics, and that PC-12 is the best method of them. When we use PC-12 and N- β method, the coefficient matrix is composed of stiffness and mass matrices. We must repeatedly produce and decompose the coefficient matrix whenever the contact state changes at each node.

On the contrary, in RKG method which gives a solution with a good phase characteristic, the coefficient matrix is composed only of the mass matrix. By use of the lumped mass matrix (LM), we are able to explain the physical properties, to compose the coefficient matrix of a high order for various contact states, and to economize computer memory and executing time for computation as compared with use of the consistent mass matrix (CM). Also the use of the LM makes it easy to obtain the inverse matrix even in the slipping state including frictional effect. Then the RKG method of fourth order is used to calculate the equations of the contact motion such as Eqs. (16) and (24).

For simplicity, let us consider a two-dimensional triangular element with constant strain. If the mass of an element at the center of gravity (x_g, y_g) is represented by m_g , the equivalent lumped masses (m_i, m_j, m_k) at nodal points (i, j, k) are,

$$\begin{Bmatrix} m_i \\ m_j \\ m_k \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x_g \\ y_g \end{bmatrix} m_g \quad \left. \vphantom{\begin{Bmatrix} m_i \\ m_j \\ m_k \end{Bmatrix}} \right\} \dots (25)$$

$$m_g = b \int_{\Delta} \rho d\Delta$$

where b, ρ and Δ are the thickness, the density and the area of an element, respectively. Also the body (ξ) is easily treatable with the LM.

5. Application of the Method to Analysis of Elasto-dynamic Contact Stress between Two Rods subjected to Impact

In order to verify the applicability of this method, we investigate a longitudinal impact problem of two prismatic rods with equal cross-sectional area and flat ends.

5.1 Condition for Analysis

A model for two-dimensional analysis is shown in Fig. 5. Two elastic rods different in length are subjected to impact on their flat ends. As the contact area between both rods remains constant, calculation is carried out on impact velocity $V=1$ [m/s] of the rod (2). Unchangeability of the contact area makes it unnecessary to perform the n - t coordinate transformation at each contact node. The frictional coefficient $\mu=0.1$ at all pairs of contact nodes is assumed for calculation. To investigate the contact stress distribution subjected to impact, contact parts of the models consist of finer meshing elements. Also finely divided elements at a free end of the rod (2) are necessary to improve the response of reflected

Fig. 5 A model for two-dimensional analysis

wave. Although the time interval for integration becomes 0.976 μ s. according to the size of the smallest element in the structure, the calculation at $\Delta t=0.5$ μ s. is convenient.

5.2 Stress Conditions Based on Theory of Propagation of One-dimensional Stress Wave

As the impact stress based on the theory of propagation of one-dimensional elastic stress wave (1D theory) distributes uniformly in any cross-sectional area of rods (10) the stress of the 1D theory is used to verify the element stress values of the model shown in Fig. 5. When a free end of elastic rod with semi-infinite length is struck at the velocity v , the stress σ^* in any cross-sectional area is expressed by

$$\sigma^* = v \sqrt{\gamma E / g} \dots (26)$$

where E, γ and g are the modulus of elasticity, the specific gravity and the acceleration of gravity, respectively.

Using the velocity of propagation of sound wave along the rod $c = \sqrt{gE / \gamma}$, Eq. (26) is rewritten as

$$\sigma^* = E v / c \dots (27)$$

On the other hand, let A_1 and A_2 be the cross-sectional areas of the rods (1) and (2), respectively. The stresses σ_1 and σ_2 produced in two rods subjected to impact are expressed by

$$\sigma_1 + \sigma_2 = \sigma^*, \quad P = A_1 \sigma_1 = A_2 \sigma_2 \dots (28)$$

where P denotes the concentrated load applied to the cross-sectional area of a rod. Then the stresses σ_1 and σ_2 are formulated by

$$\sigma_1 = \frac{A_2}{A_1 + A_2} \sigma^*, \quad \sigma_2 = \frac{A_1}{A_1 + A_2} \sigma^* \dots (29)$$

As A_1 is equal to A_2 for the model in Fig. 5, the stresses are rewritten as

$$\sigma_1 = \sigma_2 = \sigma^* / 2 \dots (30)$$

For example, the compression waves are reflected from the free end as tension waves, and from the fixed end as compression waves, respectively, and vice versa. Then the stress conditions at any point along the rods in Fig. 5 can be obtained by the superposition of the progressive waves and reflected ones, as shown in Fig. 6.

5.3 Behavior for Internal Stress

Figure 7 represents the stress behavior at the distance 10 mm. apart from the contact end on the axis of the rod (1). Although the results by the FEM fluctuate periodically depending on the deformability of elements due to the lateral effect of inertia (11) on two-

Fig. 6 Stress conditions by the theory of propagation of one-dimensional elastic stress wave.

Fig.7 Behavior for internal stress of the rod (1).

dimensional problems, the mean value of the results agrees well with that of the 1D theory. A slow response is also recognized in the FEM results, as compared with that of the 1D theory.

The fluctuation of the results by the CM is relatively smaller than that by the LM. The LM becomes a diagonal matrix without coupled relation, and its diagonal components are large compared to those of the CM. The use of the LM in the equation of motion is equivalent to neglecting the coupling effect of inertia motion, and coupled relation between neighboring nodes and between directions depends on the stiffness matrix only. Hence, the inertia force and the stress fluctuation of the LM become larger than those of the CM, and also, the period of the stress fluctuation becomes longer.

From the discussion described above, this method can be verified and its applicability can be defined.

5.4 Behavior for Elasto-dynamic Contact Stress

At present it is difficult to measure and calculate the elasto-dynamic contact stress and its distribution. The equations of the contact motion in section 3 make it possible to investigate the contact behaviors.

Figure 8 shows the elasto-dynamic contact stress and its distribution of the rod (1), using the CM. Although the stress distribution becomes relatively uniform, the value of the stress fluctuates near that of the 1D theory. The separation between both rods occurs at 31.3 μs. in the 1D theory and at 36.5 μs. in this method. Small difference at the separated time between both methods comes from the lateral effect of inertia, and from the response of elements divided in this method, as described in section 5.3.

From the facts described above, this method is found practically applicable to elasto-dynamic problems.

6. Conclusions

The following results are obtained:

(1) The basic equations of motion with contact-impact behavior for two bodies are formulated by the finite element method based on the principle of virtual work, in which the unknown contact-impact forces acting on contact surfaces are treated as the forces acting on surfaces of each body. Regarding two bodies as being interconnected through contact surfaces makes it possible to eliminate the unknown contact-impact forces from the basic equations of motion for two bodies. Then the equations of contact motion for two bodies can be derived

Fig.8 Behavior for elasto-dynamic contact stress.

from the basic equations for various contact states.

(2) The equations of contact motion for two bodies are applied to a two-dimensional analysis of longitudinal impact of two prismatic rods with an equal cross-sectional area and flat ends. Although the calculated results fluctuate periodically, the mean value agrees well with that obtained by the theory of propagation of one-dimensional elastic stress wave. This method is practically applicable to elasto-dynamic contact problems.

(3) A method for analysis of elasto-static contact problems in reference (4) can be introduced by eliminating the inertia term from the equations of contact motion for two bodies.

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