Compressibility Effects on Cavitation in High Speed Liquid Flow

(Second Report-Transonic and Supersonic Liquid Flows)

By Nishiyama Tetsuo** and Omar Faruque Khan***

Some notable features of normal and oblique shock waves of supersonic liquid flow are clarified and also the close relations of cavity streamline to the flow turning by expansion wave are pointed out. And then after defining the Mach number range for transonic and supersonic liquid flow by using the critical Mach number, the governing equations of parabolic and hyperbolic types for velocity potential are analytically solved by local linearization technique. Compressibility effects on the cavity characteristics of symmetrical wedge are clarified through the Mach number range. Comparisons are also made with the results from oblique shock relations.

2. Pressure Waves in Liquid Flows

Though the basic governing principles of the flow field are same for both liquid and gas, some remarkable discrepancy may result from the differences in their equation of state. Therefore the shock relations have to be derived from the conservation laws of mass, momentum and energy in liquid flows.

2.1 Normal Shock Wave

Based on Tait equation of state [1], the conservation laws of mass, momentum and energy in liquid flows yield respectively

\begin{equation}
\rho_1 q_1 = \rho_2 q_2 = \rho \tag{1}
\end{equation}

\begin{equation}
P_1 - P_2 = \rho (q_2 - q_1) = \rho \rho_2^2 - \rho_1 q_2^2 \tag{2}
\end{equation}

\begin{equation}
\frac{\kappa-1}{2} q_1^2 = a_2^2 + \frac{\kappa-1}{2} a_2^2 ; \kappa=7.15 \tag{3}
\end{equation}

From Eqs. (1), (2), and (3) the pressure-density relation is

\begin{equation}
P_2 = \frac{\rho_2}{\rho_1 (\kappa-1)/\kappa+1} \left[ \frac{(P_2+\rho)/(\kappa-1)+\rho}{(P_1+\rho)/(\kappa-1)+\rho} \right]
\end{equation}

From Eq. (4) is shown in Fig. 1. From this we can

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Pressure-density relation for supersonic liquid flows.}
\end{figure}

* Received 21st March, 1980.
** Professor, Dept. of Mechanical Engs., Tohoku University, Sendai.
*** Post Graduate Student, Graduate School of Engs., Tohoku University, Sendai.
see that a little variation of density due to normal shock in liquid produces a violent increment of the pressure compared with gas. For a variation of $p_2/p_1=1.1$, the difference between the pressures by the shock and the isentropic relations is negligibly small. Hence, up to about 10% variation of density in supersonic liquid, the whole flow can be regarded as isentropic. The relation between Mach numbers and pressures ahead of and behind the normal shock respectively is

$$M_2^2 = \frac{2 + (\kappa - 1) M_1^2}{2 \kappa M_1^2 - \kappa + 1}$$  \hspace{1cm} (5)

$$\frac{P_2 + B}{P_1 + B} = \frac{2 \kappa M_1^2}{\kappa + 1} - \frac{\kappa - 1}{\kappa + 1}$$  \hspace{1cm} (6)

2.2 Expansion Fan

Like gases, the vector changes in velocity produced by a pressure wave in a supersonic liquid have a direction normal to the Mach lines. Hence the flow deflection angle through finite expansion past any convex corner can be determined by the above rule. This is given by

$$\alpha + \text{const.} = \frac{1}{2} \arctan \left( \frac{\kappa + 1}{\kappa - 1} \right) \cdot \arctan \left( \frac{M_2^2 - 1}{M_1^2 - 1} \right)$$  \hspace{1cm} (7)

Also, $|\alpha| = |P(M) - P(M_o)| - |\alpha|$  \hspace{1cm} (8)

Eq. (7) is shown in Fig. 2. The maximum turning angle of a liquid flow through expansion past any convex corner is 13.67 degrees which is about 1/10th of the corresponding one of gas.

2.3 Oblique Shock Wave

Using the normal shock relations, Eqs. (5) and (6), we can write the following relations for an oblique shock wave in liquid:

$$\frac{M_2^2 \sin^2(\delta - \theta)}{2 + (\kappa - 1) M_1^2 \sin^2 \delta} = \frac{2 \kappa M_1^2}{(2 \kappa M_1^2 - \kappa + 1)}$$  \hspace{1cm} (9)

$$\frac{P_2 + B}{P_1 + B} = \frac{2 \kappa M_1^2}{\kappa + 1} - \frac{(\kappa - 1)}{(\kappa + 1)}$$  \hspace{1cm} (10)

$$\theta = \arctan \left( \frac{2 \cot \delta}{\kappa + \cos \delta - 1} \right)$$  \hspace{1cm} (11)

The turning angle of flow, $\delta$, for an oblique shock in liquid is shown in Fig. 3.

![Fig. 3 Shock angle versus uniform Mach number and flow turning angle in liquid flows.](image)

2.4 Upper Critical Mach Number

The upper critical Mach number is defined as the supersonic freestream Mach number at which the region of local subsonic flow after shock first disappears. By using the oblique shock relation (9) we can determine this for each turning angle like

$$M_1 = \left[ \frac{2 + (\kappa - 1) \sin^2(\delta - \theta)}{2 \kappa \sin^2 \delta (2 \kappa M_2^2 - \kappa + 1)} \right]^{\frac{1}{2}}$$  \hspace{1cm} (12)

in which $\delta$ is smaller than $\delta_{\theta=\theta_{\text{max}}}$ and also must satisfy Eq. (12).

3. Transonic Liquid Flow

The freestream Mach number for a transonic liquid flow is hereafter defined as being greater than the lower critical and smaller than the upper critical Mach numbers. The flow pattern is shown schematically in Fig. 4. At supersonic speed, the detached shock wave at first will begin normal to the freestream at the point $A$, and then curve progressively downstream such that far from the wedge its strength gradually decreases and its slope tends asymptotically to one of a freestream Mach wave. The liquid first passing through this shock wave will decelerate discontinuously from supersonic to subsonic speed and will confine itself to a limited region bounded by the shock, sonic line and the wedge surface. It will then accel-
erate continuously first to a speed of sound on the sonic line and to a supersonic speed forming an expansion fan at the shoulder of the wedge beyond this sonic line. One of the expansion waves, here known as a separating wave, BFE, will separate this expansion fan into two classes: one which reaches the sonic line and one which does not. Any small disturbances introduced into the expansion fan ahead of the separating wave will travel along the Mach lines to a point on the sonic line. From there they will spread out in the subsonic region thereby influencing the sonic line and the expansion fan itself. As a result, just after crossing the sonic line, the flow will abruptly turn to a direction.

\[
\frac{1}{R} \int_0^{\infty} \phi_1 \left( \frac{2 \phi}{y_1} \right) \frac{1}{4(x-x_1)} \, dx \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \po
For sonic point, which occurs on the wedge surface at the point of minimum pressure i.e. at \(u(x,0)/U_e = (1-M^2)/((c+1)M^2)\), the following condition is to be satisfied:

\[
\frac{d}{dx} \int_0^{x-x_1} \frac{1}{\sqrt{1-x_1^2}} dx_1 = 0 \quad \text{..........................}(24)
\]

For a wedge, Laplace's and its inverse transformations of Eq. (24) yield

\[
\frac{d}{dx} \int_0^{x-x_1} \frac{1}{\sqrt{1-x_1^2}} dx_1 = \frac{\delta}{x} \quad ; 0 < x < C_0 \quad \text{..........................}(25)
\]

Therefore the location of sonic point is at \(x = x_0 = C_0\) i.e. at the shoulder of the wedge. The perturbation velocity on the wedge surface is, by Eq. (23),

\[
u(x,0) = \frac{(1-M^2)}{(c+1)M^2} - \left\{ 1 + \frac{3\beta^2}{(c+1)M^2} \right\} \frac{1}{3}
\]

Hence the pressure and drag coefficients are respectively

\[
C_p(x) = \frac{1}{(2/3)^2} \frac{d}{dx} \left( \frac{1}{\sqrt{1-x}} \right) - \left\{ 1 - \frac{2(1-M^2)}{(c+1)M^2} + \frac{3\beta^2}{(c+1)M^2} \right\} \frac{1}{3}
\]

\[
C_p = \frac{\frac{1}{2} \frac{d}{dx} \nu(x,0)}{U^2}
\]

\[
\text{Here} \Gamma \text{denotes a gamma function. The drag coefficient at transonic flow with infinite trailing cavities is, as a special case of Eq. (28),}
\]

\[
\left\{ \frac{C_p}{M^2} \right\}_{M^2 = 1} = \frac{1.76}{\sigma} \frac{2}{3}
\]

This result differs from the result by Tulin[11] in the factor 1.76 by 1.89.

4. Supersonic Liquid Flow

The freestream Mach number for a supersonic liquid flow is hereafter defined as being greater than the upper critical Mach number. The flow pattern is shown schematically in Fig. 5. With a gradual increase of freestream Mach number, the detached shock will come closer with a weaker strength and will be attached to the wedge nose. Finally, the region of local subsonic flow after the shock will completely disappear at the upper critical Mach number.

4.1 Method of Approximation

The supersonic flows expressed by the equation of hyperbolic type are identified by the sign of total coefficient of \(\partial u/\partial x\) in Eq. (28) in first report. We write this equation in the following form:

\[
\beta \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} = 0 \quad \text{..........................}(30)
\]

where

\[
\beta = \frac{M^4 - 1}{M^2 - 1} \left\{ 1 + \frac{4(c+1)M^2}{(c+1)M^2} \right\} \frac{1}{3}
\]

Let us assume again that \(\beta \leq \) finite and varies so slowly that it can be treated locally as constant [3]. Then Eq. (30) is equivalent to the uniformly linearized supersonic theory by Ackert and the solution on the upper surface of a symmetrical wedge is

\[
u(x,0) = \frac{1}{\sqrt{\beta}} \frac{d}{dx} \frac{u}{x} \quad \text{..........................}(32)
\]

On differentiation, it becomes

\[
\frac{1}{\sqrt{\beta}} \frac{d}{dx} \frac{u(x,0)}{x} = \frac{1}{\sqrt{\beta}} \frac{d}{dx} \frac{d}{dx} \frac{u}{x} \quad \text{..........................}(33)
\]

Now inserting \(\beta \leq \) in Eq. (31) into Eq. (33), the nonlinear ordinary differential equation obtained can be put after integration into the following form:

\[
\nu(x,0) = \frac{(M^4 - 1)}{(c+1)M^2} \left\{ 1 + \frac{3(c+1)M^2}{(2(M^2 - 1)^{3/2})} \right\} \frac{d}{dx} \frac{u}{x} \quad \text{..........................}(34)
\]

In order to insure the hyperbolicity of Eq. (30) and the supersonic condition in the whole flow field, we have initially considered \(\beta \leq 0\), which implies that Eq. (34) is applicable only when the term in the second bracket is positive i.e. only when

\[
\frac{d}{dx} \frac{u}{x} \leq \frac{2(M^2 - 1)^{3/2}}{3(c+1)M^2} \quad \text{..........................}(35)
\]

For a symmetrical wedge, Eq. (34) is further simplified to

\[
u(x,0) = \frac{(M^4 - 1)}{(c+1)M^2} \left\{ 1 + \frac{3(c+1)M^2}{(2(M^2 - 1)^{3/2})} \right\} \frac{d}{dx} \frac{u}{x} \quad \text{..........................}(36)
\]

Hence the pressure coefficient is

\[
C_p(x) = \frac{(P-P_e)}{(1/2)\rho U^2} = \left\{ \frac{1}{(1+\sigma)^2} \left\{ 1 - \frac{3(c+1)M^2}{(2(M^2 - 1)^{3/2})} \right\} \frac{1}{3} \right\} \frac{1}{\beta} \frac{d}{dx} \frac{u}{x} \quad \text{..........................}(37)
\]
both transonic and supersonic flows.

Fig. 7 shows the pressure distribution on the wedge surface at transonic speed. The calculations are made for the freestream Mach numbers ranging from lower critical to upper one. The increment in pressure is mainly due to formation of a local subsonic flow around the wedge formed by a detached shock wave. The pressure distribution on the wedge surface at purely supersonic speed is shown in Fig. 8. The pressure increases with an increase of cavitation number but decreases with an increase of freestream Mach number. Fig. 9 compares the pressure distribution with other theories. In a small supersonic range the results of the present theory are closer to those by oblique shock relation whereas in a supersonic range they tend to be closer to those by uniform linearization.

### 4.2 Upper Critical Mach Number

According to the method of local linearization, Eq. (38) yields the following equation for the upper critical Mach number to be solved for a symmetrical wedge:

$$(M^2 - 1)^{\frac{3}{2}} = \frac{\theta}{3(\alpha+1)M^2}$$

The variation of this upper critical Mach number with wedge half apex angle $\theta$ is shown in Fig. 6 along with that obtained from the oblique shock relation (12). For a wedge of angle 8 degrees, the difference between these two upper critical Mach numbers is negligibly small.

### 5. Cavity Characteristics

On the cavity, Eq. (34) reduces to:

$$\frac{\partial \rho}{\partial z} = \frac{2(M^2 - 1)^{\frac{3}{2}}}{3(\alpha+1)M^2} \left[ \left( 1 + \frac{(\alpha+1)M^2}{2(M^2 - 1)^{\frac{3}{2}}} \right)^{-1} \right] \sigma > 1$$

For a symmetrical wedge, Eq. (40) further reduces to:

$$\theta = \tan \left( \frac{2(M^2 - 1)^{\frac{3}{2}}}{3(\alpha+1)M^2} \left[ \left( 1 + \frac{(\alpha+1)M^2}{2(M^2 - 1)^{\frac{3}{2}}} \right)^{-1} \right] \right)$$

Thus the cavity length as shown in Figs. 4 and 5 can be given by:

$$L = C|\sigma|^{-1}$$

The cavity drag coefficient is:

$$C_D = \frac{D}{(1/2)\rho V^2 L} = \frac{C_D}{(1/2)\rho V^2 L}$$

### 6. Numerical Examples and Discussion

Numerical calculations are made for a supercavitating wedge of angle 4 degrees in
The cavity profile and the corresponding cavity length for various cavitation numbers are plotted in Figs. 10 and 11. From this we can see that the flow deflection through shoulder of the wedge increases with an increase of freestream Mach as well as cavitation numbers. In transonic supercavitating flows the cavity length decreases very fast with an increase of freestream Mach number while in supersonic supercavitating flows it decreases but slowly.

The variation of cavity drag with freestream Mach and cavitation numbers is shown in Fig. 12. It is seen that, for a particular cavitation number, the cavity drag grows sharply with an increase of freestream Mach numbers from high subsonic speed and reaches a maximum at \( (U_c)_{CR, U} \), after which it decreases at first discontinuously at \( (U_c)_{CR, U} \), and then slowly and continuously as should be usual in purely supersonic flows. This increment in cavity drag is mainly due to formation of a local subsonic zone around the wedge formed by a detached shock wave ahead of the wedge nose at \( 1 < (U_c)_{CR, U} \). With a gradual increase in freestream Mach numbers this detached shock wave turns weaker in strength and comes closer to the wedge nose, finally attaching itself to the body nose at \( (U_c)_{CR, U} \), and forming the whole flow field as purely supersonic ones. As a result the cavity drag ceases to increase with a further little increment in freestream Mach numbers at \( (U_c)_{CR, U} \).

7. Conclusions

The flow patterns over a two-dimensional supercavitating symmetrical wedge at transonic and supersonic speeds are clarified by a local linearization concept. The main results may be summarized as follows:

i) Notable features of the pressure waves in supersonic liquid flows are clarified to be minute by conservation laws of mass, momentum and energy. Also the range of freestream Mach numbers is defined for the transonic and supersonic liquid flows.

ii) In transonic supercavitating flows, the cavity drag is expressed in a simple closed analytical form and in purely low supersonic supercavitating flows, the present theory gives results much more precise than the uniform linearization does.

iii) In general, the cavity drag in transonic supercavitating flows increases
while the length decreases sharply with an increase in cavitating and freestream Mach numbers. In purely supersonic supercavitating flows, both cavity drag and length decrease with an increase of freestream Mach numbers.

References