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Particle Collision on the Wall in a Stagnation Flow*

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The behavior of a single particle which moves in the stagnation flow of an ideal fluid is studied theoretically, assuming that the drag force is proportional to both the first and second powers of the velocity difference between a particle and a fluid. In the case that the interaction force is assumed to follow only Stokes law, an analytic solution is sought and the critical condition for collision is obtained. The drag force based on the second power of the relative velocity tends to prevent the particle collision on the wall. These characteristics are clarified through the effective collision radius, and so on.

1. Introduction

In driving a car we are sometimes troubled with a collision of the front glass with a small particle, for instance, insects, droppings of birds, dust, etc. In such cases, whether floating particles in air collide against the car or not depends on the relative velocity, the size and the weight of the particle, and so on. It is, therefore, important to know under what condition a particle in air collides against the body. This condition is useful for solving various problems in the powder industry and the painting engineering.

In a steady flow past a large body, the phenomenon of the collision between a small particle and the body is theoretically studied in this paper. The collision phenomenon is observed in case that the relative velocity between the particle and the body is large and the collision usually takes place on the forward surface of the body. Therefore, the fluid is assumed to be incompressible and ideal. Since the size of the particle floating in air is usually very small compared with that of the body, the flow is regarded as a forward stagnation flow. In studying the motion of a particle-fluid mixture, particles as well as fluid are usually considered as continuous medium⁽¹⁾. But we have an interest in the collision between a small particle and a large body. Then small particles are assumed to be isolated and the influence of particles on the flow of the fluid is neglected. It is assumed in our analysis that the interaction between a particle and fluid is approximated by Oseen's drag formula, which contains a term proportional to the second power of the relative velocity. Under the assumptions

mentioned above we examine the collision between a particle and the wall in the stagnation flow.

2. Basic equations

As mentioned in the previous chapter, we consider the stagnation flow of a particle-fluid mixture as shown in Fig.1. We take x-axis on the wall and z-axis perpendicular to the wall. The x and z components of fluid velocity (u_f, w_f) are represented by

$$u_f = c\alpha x \dots\dots\dots(1)$$

$$w_f = -\alpha z \dots\dots\dots(2)$$

Where α denotes the velocity gradient of the stagnation flow and the constant c is equal to unity in the case of a two dimensional flow and $1/2$ in the case of a three dimensional flow. We apply the drag law⁽²⁾ derived from Oseen's approximation to the interaction force between a particle and fluid. Then the force F exerted on a spherical particle with radius a in a fluid is given as

$$F = 6\pi\mu a \Delta V \left(1 + \frac{3a\rho}{8\mu} |\Delta V|\right) \dots\dots\dots(3)$$

Where μ and ρ denote viscosity and density of the fluid, ΔV is a vector denoting the velocity difference between fluid and a par-

Fig. 1 Coordinate system

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title. Denoting the velocity components of particle by (u, w) and the density of particle by ρ_p , the equations of motion of a particle in x and z directions are written in the following forms:

$$\frac{4}{3}\pi a^3 \rho_p \frac{du}{dt} = 6\pi \mu a (u_f - u) \left\{ 1 + \frac{3}{8} \frac{a \rho}{\mu} |u_f - u| \right\} \quad \dots (4)$$

$$\frac{4}{3}\pi a^3 \rho_p \frac{dw}{dt} = 6\pi \mu a (w_f - w) \left\{ 1 + \frac{3}{8} \frac{a \rho}{\mu} |w_f - w| \right\} \quad \dots (5)$$

where, t denotes time and is an independent variable in our problem. On the other hand, denoting the coordinates of particle position by (x, z) , the particle trajectory is defined by

$$\frac{dx}{dt} = u \quad \dots (6)$$

$$\frac{dz}{dt} = w \quad \dots (7)$$

Substituting Eqs. (6), (7), (1) and (2) into Eqs. (4) and (5), the equations governing the particle trajectory are obtained. It is assumed for the initial conditions of these equations that a particle moves with the same velocity as that of the fluid at a place apart from the origin by a distance l in the x direction and L in the z direction:

$$t=0: x=l, \quad \frac{dx}{dt} = c\alpha l \quad \dots (8)$$

$$z=L, \quad \frac{dz}{dt} = -\alpha L \quad \dots (9)$$

In order to express the governing equations (1) - (9) in a dimensionless form, the coordinates of the initial position, l and L , are used as the reference length for x and z coordinates, respectively. The reciprocal of the velocity gradient $1/\alpha$ is used as the reference time. Then the equations governing the particle trajectory are written in a dimensionless form as

$$\frac{d^2 x}{dt^2} = \frac{1}{\sigma} \left(cx - \frac{dx}{dt} \right) \left\{ 1 + \epsilon_x \left| cx - \frac{dx}{dt} \right| \right\} \quad \dots (10)$$

$$\frac{d^2 z}{dt^2} = \frac{1}{\sigma} \left(-z - \frac{dz}{dt} \right) \left\{ 1 + \epsilon_z \left| -z - \frac{dz}{dt} \right| \right\} \quad \dots (11)$$

$$t=0: x=1, \quad \frac{dx}{dt} = c \quad \dots (12)$$

$$z=1, \quad \frac{dz}{dt} = -1 \quad \dots (13)$$

Here, we use the same symbol to denote the corresponding dimensional and dimensionless quantities. And

$$\sigma = \frac{2a^2 \rho_p \alpha}{9\mu}, \quad \epsilon_x = \frac{3}{8} \frac{a \alpha l \rho}{\mu}, \quad \epsilon_z = \frac{3}{8} \frac{a \alpha L \rho}{\mu} \quad \dots (14)$$

The equations of particle trajectory in a dimensionless form (11) and (12) are solved under the initial conditions (12) and (13) and the properties of the solution for these equations are examined in the following chapter 5.

3. Transformation of three dimensional motion into two dimensional motion

In the basic equations (10) - (13) we consider a three dimensional case with $c=1/2$. Equations in the x direction (10) and (12) can be rewritten in the following form

$$\frac{d^2 x}{d(t/2)^2} = \frac{1}{\sigma/2} \left(x - \frac{dx}{d(t/2)} \right) \times \left\{ 1 + \left(\frac{\epsilon_x}{2} \right) \left| x - \frac{dx}{d(t/2)} \right| \right\} \quad \dots (10)'$$

$$t=0: x=1, \quad dx/d(t/2)=1 \quad \dots (12)'$$

If we introduce the following transformation:

$$t/2 \rightarrow t, \quad \sigma/2 \rightarrow \sigma, \quad \epsilon_x/2 \rightarrow \epsilon_x \quad \dots (15)$$

these equations of three dimensional motion completely coincide with those of two dimensional motion with $c=1$ in Eqs. (10) and (12). Hence, we shall solve only the two dimensional case. The solutions of three dimensional motion are obtained with the aid of the transformation equation (15). It should be noted that the solution of the motion in the z direction is unchanged through that transformation.

4. The case with the interaction given by Stokes drag law

We consider first that the flow is two dimensional. In case of Stokes drag law ϵ_x and ϵ_z are equal to zero. Then, Eqs. (10)-(13) become

$$\frac{d^2 x}{dt^2} + \frac{1}{\sigma} \left(\frac{dx}{dt} - x \right) = 0 \quad \dots (10)''$$

$$\frac{d^2 z}{dt^2} + \frac{1}{\sigma} \left(\frac{dz}{dt} + z \right) = 0 \quad \dots (11)'$$

$$t=0: x=1, \quad \frac{dx}{dt}=1 \quad \dots (12)''$$

$$z=1, \quad \frac{dz}{dt}=-1 \quad \dots (13)''$$

In order to obtain a solution to these equations we assume that x and z are proportional to $e^{\lambda t}$, where λ is a characteristic root. Substituting the solutions in this form into Eqs. (10)'' and (11)' then we obtain the following characteristic equations:

$$\lambda_x^2 + (\lambda_x - 1)/\sigma = 0 \quad \lambda_z^2 + (\lambda_z + 1)/\sigma = 0$$

From these equations, λ_x and λ_z can be obtained as follows:

$$\lambda_x = \frac{-1 \pm \sqrt{1+4\sigma}}{2\sigma} \quad \dots (16)$$

$$\lambda_z = \frac{-1 \pm \sqrt{1-4\sigma}}{2\sigma} \quad \dots (17)$$

σ is always positive from Eq. (14). Therefore the solution to Eqs. (10)'' and (12)'', x , is found as

$$x = \frac{1}{2\sqrt{1+4\sigma}} \exp\left(-\frac{1}{2\sigma}t\right) \left[(\sqrt{1+4\sigma}+1+2\sigma) \exp\left(\frac{\sqrt{1+4\sigma}}{2\sigma}t\right) + (\sqrt{1+4\sigma}-1-2\sigma) \exp\left(\frac{-\sqrt{1+4\sigma}}{2\sigma}t\right) \right] \quad \dots (18)$$

On the other hand, the solution to Eqs. (11)' and (13)', z , differs as follows according to the value of σ

(i) $\sigma > 1/4$:

$$z = \frac{2\sigma}{\sqrt{4\sigma-1}} \exp\left(-\frac{1}{2\sigma}t\right) \cos\left(\frac{\sqrt{4\sigma-1}}{2\sigma}t + \tan^{-1} \frac{2\sigma-1}{\sqrt{4\sigma-1}}\right) \quad \dots (19)$$

(ii) $\sigma = 1/4$: $z = (1+t)e^{-2t}$ (19)'

(iii) $\sigma < 1/4$:

$$z = \frac{1}{2\sqrt{1-4\sigma}} \exp\left(-\frac{1}{2\sigma}t\right) \left[(\sqrt{1-4\sigma}+1-2\sigma) \exp\left(\frac{\sqrt{1-4\sigma}}{2\sigma}t\right) + (\sqrt{1-4\sigma}-1+2\sigma) \exp\left(\frac{-\sqrt{1-4\sigma}}{2\sigma}t\right) \right] \quad \dots (19)''$$

4.1 Collision Condition

From the solutions, Eqs.(19)-(19)" , the value of z can not be negative at any finite time except for case (1) of $\sigma > 1/4$ in order that the particle collides against the body, while it is necessary that z becomes zero in the course of a finite time. Hence, the nondimensional quantity σ must be greater than $1/4$ in order that a particle collides against the wall under the action of Stokes drag force. This value coincides with that obtained by Healy in the case of the flow of a particle-fluid mixture past a plate with finite length, which is placed perpendicular to the ideal fluid flow⁽⁴⁾.

He concluded, assuming σ to be small, that only particles lying on the stagnation line can collide against the body under the condition described above. However, the collision condition, $\sigma > 1/4$, obtained here does not depend on the initial position of the particle. The result obtained here is different from Healy's one. Typical examples of the solution $z(t)$ are shown in Fig.2 (a) for the cases with a collision and without one. Since the curves in Fig.2(a) depend only on z , they are the solutions of the three dimensional case. Examples of particle trajectory are shown in Fig.2(b).

4.2 Collision time

The collision time T_c is defined by the required total time for a particle to

Fig. 2 (a) Solution for the particle (in case of Stokes drag)

Fig. 2 (b) Particle trajectory (in case of Stokes drag)

be moved from an arbitrary initial point to the wall. T_c is obtained as a function of σ by equating the left hand side of Eq. (19) to zero. Thus, T_c is given by

$$\cos\left(\frac{\sqrt{4\sigma-1}}{2\sigma} T_c + \tan^{-1} \frac{2\sigma-1}{\sqrt{4\sigma-1}}\right) = 0 \quad \dots\dots\dots(20)$$

This relation is shown in Fig.3. From Fig.3, the collision time T_c tends to unity as σ increases. This means that as the size of particle becomes large a particle moves with initial velocity. That is, the motion of the particle is independent of the flow. As σ approaches the critical value for collision, i.e. $1/4$, the collision time becomes infinite. Since Fig.3 is obtained from the equation of z , it is also applicable to three dimensional case. It should be noted that the collision time does not depend on whether the flow is two dimensional or three dimensional.

4.3 The effective collision radius

Let L_c be the distance from the origin to a position at which a particle collides against the wall. It is non-dimensionalized by the x coordinate of the initial position of the particle. We introduce the concept of an effective collision cross-section⁽⁴⁾ used widely in the problem of scattering. Then in case of a two dimensional flow considered, the quantity $1/L_c$ corresponds to it. We call it the effective collision radius. When $1/L_c$ is small, particles scarcely collide against the wall. Conversely, when $1/L_c$ is large, particles easily collide.

Fig. 3 Collision time (in case of Stokes drag)

Fig. 4 Effective collision radius (in case of Stokes drag)

In order to make a comparison between two dimensional and three dimensional cases, that quantity is also applied to the three dimensional case as it is. Now, the effective collision radius $1/L$, can be obtained by substituting the collision time T , into Eq.(18). It is shown in Fig.4 as a function of σ . It is found from Fig.4 that in the region where σ is greater than $1/4$ the effective collision radius rapidly increases as σ increases, that is, the particle is easy to collide against the wall. As σ becomes large the effective collision radius approaches a constant value, $1/2$. This fact means that, since the increase in σ corresponds to the increase in particle radius, particles move with the initial velocity independently of fluid.

Because the relation shown in Fig. 4 depends on x , the effective collision radius of a three dimensional flow differs from that of a two dimensional flow. Applying the transformation (15) to σ and t on the right hand side of Eq.(18) and substituting T , given by Eq.(20) into it, then we obtain the effective collision radius of the three dimensional case. It is shown in Fig. 4 with a dashed line. It is found that particles collide more easily against a three dimensional body than against a two dimensional one. This is based on the fact that fluid particles coming from the same position at upstream reach closer to the wall in a three dimensional flow than in a two dimensional one.

5. The case of the interaction given by Oseen's drag law

In the equations governing particle motion, Eqs.(10) and (11), the terms involving ϵ_x and ϵ_z are the correction terms to make Stokes drag law more accurate and they remain small in Reynolds number of the order of unity. Hence, ϵ_x and ϵ_z can be regarded as small quantities. The solution to Eqs.(10)-(13) are sought in an expanded form as follows:

$$x = x_0 + \epsilon_x x_1 + \dots \quad (21)$$

$$z = z_0 + \epsilon_z z_1 + \dots \quad (22)$$

Substituting this expanded form into Eqs. (10)-(13), and taking the following relations into account:

$$\begin{aligned} |x - dx/dt| &= x - dx/dt \\ |z - dz/dt| &= -z - dz/dt \end{aligned}$$

Fig. 5 Correction term of the solution in the x direction

z_1 is found as

$$\begin{aligned} z_1 = & \frac{\sigma^2}{1+4\sigma} \left[\sigma \exp\left(-\frac{1}{\sigma}t\right) \left\{ \frac{1}{1+3\sigma-\sqrt{1+4\sigma}} \exp\left(\frac{\sqrt{1+4\sigma}}{\sigma}t\right) \right. \right. \\ & + \frac{1}{1+3\sigma+\sqrt{1+4\sigma}} \exp\left(-\frac{\sqrt{1+4\sigma}}{\sigma}t\right) \left. \right\} + 2 \exp\left(-\frac{1}{\sigma}t\right) \\ & - 4\sqrt{1+4\sigma} \exp\left(-\frac{1}{2\sigma}t\right) \left\{ \frac{1}{3\sqrt{1+4\sigma}+1} \exp\left(\frac{\sqrt{1+4\sigma}}{2\sigma}t\right) \right. \\ & \left. \left. + \frac{1}{3\sqrt{1+4\sigma}-1} \exp\left(-\frac{\sqrt{1+4\sigma}}{2\sigma}t\right) \right\} \right] \quad (23) \end{aligned}$$

On the other hand, z_1 , is given in three forms corresponding to the value of σ

(i) $\sigma > 1/4$:

$$\begin{aligned} z_1 = & \frac{2\sigma^2}{4\sigma-1} \left[\exp\left(-\frac{1}{\sigma}t\right) - \sqrt{\frac{\sigma}{9\sigma-2}} \exp\left(-\frac{1}{\sigma}t\right) \cos\left\{\frac{\sqrt{4\sigma-1}}{\sigma}t\right\} \right. \\ & + \tan^{-1}\left\{\frac{\sqrt{4\sigma-1}}{1-3\sigma}\right\} - 2\sqrt{\frac{4\sigma-1}{9\sigma-1}} \exp\left(-\frac{1}{2\sigma}t\right) \cos\left\{\frac{\sqrt{4\sigma-1}}{2\sigma}t\right\} \\ & \left. + \tan^{-1}\left\{\frac{1}{3\sqrt{3\sigma-1}}\right\} \right] \quad (24) \end{aligned}$$

(ii) $\sigma = 1/4$:

$$z_1 = (t^2 - 4t + 6)e^{-t} - (2t + 6)e^{-2t} \quad (24)'$$

(iii) $\sigma < 1/4$:

$$\begin{aligned} z_1 = & \frac{\sigma^2}{1-4\sigma} \left[\sigma \exp\left(-\frac{1}{\sigma}t\right) \left\{ \frac{1}{1-3\sigma-\sqrt{1-4\sigma}} \exp\left(\frac{\sqrt{1-4\sigma}}{\sigma}t\right) \right. \right. \\ & + \frac{1}{1-3\sigma+\sqrt{1-4\sigma}} \exp\left(-\frac{\sqrt{1-4\sigma}}{\sigma}t\right) \left. \right\} - 2 \exp\left(-\frac{1}{\sigma}t\right) \\ & + 4\sqrt{1-4\sigma} \exp\left(-\frac{1}{2\sigma}t\right) \left\{ \frac{1}{3\sqrt{1-4\sigma}+1} \exp\left(\frac{\sqrt{1-4\sigma}}{2\sigma}t\right) \right. \\ & \left. \left. + \frac{1}{3\sqrt{1-4\sigma}-1} \exp\left(-\frac{\sqrt{1-4\sigma}}{2\sigma}t\right) \right\} \right] \quad (24)'' \end{aligned}$$

Representative examples of z_1 and z_1 are shown in Fig.5 and Fig.6. As shown in these figures z_1 and z_1 are positive in the range $\sigma > 1/4$. But, the increment in z_1 with time depends on the collision time.

Equating the right hand side of Eq.(22) to zero, the collision time T is obtained as

$$T = T_1 + \epsilon_z T_1 \quad (25)$$

$$T_1 = -z_1 / \left. \frac{dz_1}{dt} \right|_{T_1} \quad (26)$$

$dz_1/dt|_{T_1}$ is the velocity at that time when a particle collide against the wall. Therefore, its sign is negative. T_1 is positive. Thus, the correction term proportional to the second power of the velocity difference in Oseen's drag law makes the collision time long and suppresses the collision. The relation of T_1 versus σ is shown in Fig.7. An increase in σ , that is, an increase in the size of the particle, causes a decrease in the correction term in Oseen's drag law.

Fig. 6 Correction term of the solution in the z direction

As the effective collision radius $1/L$ is $1/(x_0 + \epsilon_s x_1)$, we have

$$\frac{1}{L} = \frac{1}{L_s} \left\{ 1 - \frac{1}{L_s} \left(\epsilon_s \frac{\partial x_0}{\partial t} \Big|_{T_s} T_1 + \epsilon_s x_1 \Big|_{T_s} \right) \right\} \dots (27)$$

$\partial x_0 / \partial t|_{T_s}$ is the velocity along the wall and, of course, it is positive, then the effective collision radius is decreased by the correction term in Oseen's drag law. This result is derived from the increase in the collision time T . Now, we examine the particle motion assuming the nondimensional quantities ϵ_s and ϵ_1 to be small. The ratio of these parameters to σ is

$$\frac{\epsilon_1}{\sigma} = \frac{27}{16} \frac{L}{a} \frac{\rho}{\rho_p}, \quad \frac{\epsilon_s}{\sigma} = \frac{l}{L} \dots (28)$$

Let us consider the motion of a particle which comes from a point 100 times the particle radius far from the wall. In the case of particles floating in air then ρ/ρ_p is of the order of 10^{-3} . Thus ϵ_s and ϵ_1 become sufficiently small quantities. Hence, the results of this chapter are useful to examine the motion of a solid or liquid particle floating in air. However, there is, of course, a practical case where the ratio of Eq.(28) has a large value. Then the case of the interaction given by Newton drag, which is proportional to the second power of velocity difference, is examined in the following.

Fig. 7 Correction term of collision time

Fig. 8 Particle trajectory (in case of Newton drag)

6. The case of the interaction given by Newton's drag law

We consider the case that the interaction force consists of only the term in the middle parenthesis on the right hand side of Eqs.(10) and (11). That is,

$$\frac{d^2 x}{dt^2} = \frac{1}{N_s} \left(x - \frac{dx}{dt} \right)^2 \dots (10)'''$$

$$\frac{d^2 z}{dt^2} = \frac{1}{N} \left(-z - \frac{dz}{dt} \right)^2 \dots (11)'''$$

$$t=0 \quad x=1, \quad \frac{dx}{dt} = C \dots (12)'''$$

$$t=0 \quad z=1, \quad \frac{dz}{dt} = -1 \dots (13)'''$$

where

$$\frac{1}{N_s} = \frac{3}{8} C_D \frac{\rho}{\rho_p} \frac{l}{a}, \quad \frac{1}{N} = \frac{3}{8} C_D \frac{\rho}{\rho_p} \frac{L}{a}$$

C_D is the drag coefficient. These equations are nonlinear, hence are solved numerically. The representative particle trajectories are shown in Fig.8 in terms of $1/L$ instead of N_s . The relation of collision time T versus N and the relation of the effective collision radius $1/L$ versus N are shown in Fig.9 and Fig.10, respectively. There occur always collision phenomena for all positive values of N . In the case of large N there is no difference between a two dimensional flow and a three dimensional flow. However, in the case that N is of the order of unity, the effective collision radius for the three dimensional flow is greater than that for the two dimensional one.

Fig. 9 Collision time (in case of Newton drag)

Fig. 10 Effective collision radius (in case of Newton drag)

7. The case that Stokes drag and Newton drag are of the same order of the magnitude

x and z are found from Eqs. (10)-(13) numerically with $\epsilon_z=5$ and $1/L=0.1$. The particle trajectory in the case of the drag force proportional to the first and the second power of the velocity difference is shown in Fig.11. The relations of the collision time T and the effective collision radius $1/L$ versus σ are shown in Fig.12 and Fig.13, respectively. In these figures the collision time and the effective collision radius for the case of Stokes drag law, which is obtained in Chap.4, are described for comparison. It is clear from these figures that the drag force derived from the second power of the velocity difference tends to prevent the collision against the wall. This fact agrees with the results stated in Chap.5.

Fig. 11 Particle trajectory

Fig. 12 Collision time

Fig. 13 Effective collision radius

8. Conclusions

We examine the phenomena of a collision between the wall and small floating particles in an incompressible and inviscid stagnation flow. The trajectory of a single particle is found theoretically under the assumption that the drag force is proportional to both the first and the second power of the velocity difference. In the case that the interaction force is given by only Stokes law, an analytic solution is found. When the nondimensional quantity σ which means the size of particles is greater than $1/4$, all particles always collide against the wall. This result is independent of the initial position of the particle. Newton's drag proportional to the second power of the relative velocity tends to prevent the particle collision against the wall. These properties are made clear through the collision time, the effective collision radius, the particle trajectory, and so on.

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