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Elastohydrodynamic Squeeze Problem of Two Rotating Cylinders*

(Part 2, Bingham Solid)

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In this paper, an elastohydrodynamic squeeze problem between two rotating cylinders lubricated by a Bingham solid is analyzed under the conditions of both flooded and starved lubrication to clarify how the yield stress of a Bingham solid affects the properties of film thickness and load capacity. The lubrication equation for a Bingham solid and the elasticity equation are solved simultaneously by the Newton-Raphson method. It is shown that a Bingham solid has a greater load capacity than a Newtonian fluid and its effect is remarkable with a decrease of rolling velocity and an increase of approaching velocity. In the starved lubrication, the effect of inlet coordinate on the load capacity is less in a Bingham solid than in a Newtonian fluid.

1. Introduction

The previous paper⁽¹⁾ presented by the authors, has discussed the elastohydrodynamic lubrication between rotating two cylinders approaching each other and lubricated with a Newtonian fluid, and the boundary conditions of "the starved and the flooded lubrication". It is necessary to study also a similar problem using a grease which is used widely as a lubricant in machine elements such as gears, rolling bearings etc.

The elastohydrodynamic lubrication between the rotating two cylinders lubricated with a grease has been investigated by Kauzlarich⁽²⁾ and Wada et al.^{(3),(4)} and the squeeze lubrication between two elastic cylinders approaching each other has been studied by Wada et al.⁽⁵⁾. However, there are few papers on the elastohydrodynamic lubrication between two cylinders in the rolling motions with the approaching ones. Sasaki et al.⁽⁶⁾ studied similar problems of two rigid cylinders lubricated with a grease. However, studies considering the elastic deformation of the contact surfaces have not been reported.

In this paper, the squeeze problems between rotating two cylinders lubricated with a grease are analyzed by assuming a grease to be a Bingham solid, and the effects of the yield stress of the solid on the film thickness, the pressure distribution and the load capacity are investigated. Furthermore, the effects of the starved and the flooded lubrication conditions on the load capacity are discussed.

2. Nomenclature

- a : width of Hertzian contact
- E : equivalent elastic modulus
- E_1, E_2 : elastic moduli of cylinders 1 and 2
- h : film thickness
- h_u : upper height of core
- h_l : lower height of core
- h_e : film thickness at exit
- p : film pressure
- p_0 : central film pressure
- R : equivalent radius
- R_1, R_2 : radii of cylinders 1 and 2
- u : velocity in the x -direction
- U : surface velocity of cylinder
- V : approaching velocity
- w : load capacity
- x : coordinate
- x_e : exit coordinate (starting point of film pressure at the inlet)
- x_i : inlet coordinate (closing point of film pressure at the exit)
- y : coordinate
- α : pressure coefficient of viscosity
- β : pressure coefficient of yield stress
- λ : velocity ratio (V/U)
- μ : plastic viscosity
- μ_0 : plastic viscosity under atmospheric pressure
- ν_1, ν_2 : Poisson's ratios of cylinders 1 and 2
- τ : shear stress
- τ_0 : yield stress of a Bingham solid
- τ_{∞} : yield stress of a Bingham solid under atmospheric pressure

3. Analysis

It is assumed in this analysis that the length of cylinders are infinitely long and the elastohydrodynamic lubrication between the two cylinders is replaced by that between an elastic cylinder and a flat rigid body by introducing the equivalent elastic modulus E and an equivalent radius R .

$$\frac{1}{E} = \frac{1}{2} \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right), \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

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3.1 Governing equations

The flow equation for a Bingham solid are given by the following ;

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} (\tau \mp \tau_0) \quad (|\tau| \geq \tau_0) \quad \dots\dots\dots(1)$$

$$\frac{\partial u}{\partial y} = 0 \quad (|\tau| < \tau_0) \quad \dots\dots\dots(2)$$

where μ and τ_0 are respectively the plastic viscosity and the yield stress of a Bingham solid which varies with the film pressure exponentially as the following expressions.

$$\mu = \mu_0 e^{\alpha p} \quad \dots\dots\dots(3)$$

$$\tau_0 = \tau_{00} e^{\beta p} \quad \dots\dots\dots(4)$$

From the equilibrium of forces in the film, we have

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \quad \dots\dots\dots(5)$$

The core of a Bingham solid generated in the rolling motion is distributed as shown in Fig.1, and the velocities in each region are obtained from Eqs. (1)~(5) as follows ;

$$u = \frac{1}{2\mu_0 e^{\alpha p}} \frac{\partial p}{\partial x} y(y-2h_s) + U \quad (0 \leq y \leq h_s) \quad \dots\dots\dots(6)$$

$$u = \frac{-1}{2\mu_0 e^{\alpha p}} \frac{dp}{dx} h_s^2 + U \quad (h_s < y < h_s) \quad \dots\dots\dots(7)$$

$$u = \frac{-1}{2\mu_0 e^{\alpha p}} \frac{dp}{dx} (h-y)(h+y-2h_s) + U \quad (h_s \leq y \leq h) \quad \dots\dots\dots(8)$$

where h_s is the lower height of core and h_s the upper one.

The core formed in the rolling motion is symmetric with respect to the center of the film thickness, then the following expression is obtained.

$$h_s = h - h_a \quad \dots\dots\dots(9)$$

From the continuity of flow, we have

$$\frac{\partial}{\partial x} \int_0^h u dy = V \quad \dots\dots\dots(10)$$

Substituting Eqs. (6)~(9) into Eq.(10), the lubrication equation is obtained as follows ;

$$\frac{d}{dx} \left\{ \frac{1}{\mu_0 e^{\alpha p}} \frac{dp}{dx} h_a^2 (3h-2h_a) \right\} = 6U \frac{\partial h}{\partial x} - 6V \quad \dots\dots\dots(11)$$

Fig.1 Schema of film profile and pressure distribution

The boundary conditions are

$$p=0, \quad x=x_i \text{ (inlet)} \quad \dots\dots\dots(12)$$

$$p=dp/dx=0; \quad x=x_e \text{ (exit)} \quad \dots\dots\dots(13)$$

Integrating Eq. (11) with respect to x by applying the boundary conditions to it, the pressure gradient reads

$$\frac{dp}{dx} = \frac{6\mu_0 e^{\alpha p}}{h_a^2 (3h-2h_a)} \{U(h-h_s) - V(x-x_i)\} \quad \dots\dots\dots(14)$$

Considering the linearity of the shear stress distribution in the film, the pressure gradient in the core is expressed by

$$\frac{dp}{dx} = \frac{\mp 2\tau_{00} e^{\beta p}}{h-2h_s} \quad \dots\dots\dots(15)$$

Eliminating h_s from Eqs. (14) and (15), we have

$$h_s^3 \left(\frac{dp}{dx} \right)^3 - 3[(\pm \tau_{00} e^{\beta p})^2 h^2 + 4\mu_0 e^{\alpha p} U(h-h_s - \lambda(x-x_i))] \left(\frac{dp}{dx} \right)^2 + 4(\pm \tau_{00} e^{\beta p})^2 = 0 \quad \dots\dots\dots(16)$$

where the positive signs are taken in $dp/dx > 0$ and the negative one in $dp/dx < 0$, and λ is the ratio of the approaching velocity to the rolling one.

Considering the elastic deformation due to the film pressure, the film thickness is expressed as

$$h = h_c + \frac{x^2 - x_i^2}{2R} - \frac{1}{\pi E} \int_{x_i}^{x_e} p(\xi) \ln \left| \frac{\xi - x}{\xi - x_i} \right| d\xi \quad \dots\dots\dots(17)$$

The core height obtained from Eq. (15) and the load capacity given by integrating the pressure over the contact zone are expressed as follows ;

$$h_s = \frac{1}{2} \left(h - \frac{\pm 2\tau_{00} e^{\beta p}}{dp/dx} \right) \quad \dots\dots\dots(18)$$

$$w = \int_{x_i}^{x_e} p(\xi) d\xi \quad \dots\dots\dots(19)$$

(a) Flooded condition

(b) Starved condition

Fig.2 Lubrication conditions at the inlet

3.2 Flooded and starved lubrication conditions

The flooded and the starved lubrication conditions are shown in Fig.2. In the flooded condition of the lubricant being fully supplied at the inlet as shown in Fig.2, the boundary condition is $p=0$ at $x_i=-\infty$. In the starved condition of poor lubrication, the location of inlet coordinate x_i to be $p=0$ is close to the center of contact. As the pressure zone in the starved condition becomes narrower, it comes to affect the lubrication characteristics.

3.3 Numerical analysis

3.3.1 Nondimensional governing equations The following normalized variables are introduced.

$$\begin{aligned} P &= p/p_0, \quad X = x/x_e, \quad \bar{U} = 12\mu_0 U/ER, \\ X_e &= x_e/a, \quad a/R = 4p_0/E, \quad \bar{\alpha} = \alpha p_0, \quad \bar{\beta} = \beta p_0, \\ H &= h/h_e, \quad H_e = h_e/R, \quad H_a = h_a/h_e, \quad X_i = x_i/x_e, \\ P_{Hz} &= p_0/E, \quad Z = \zeta/x_e, \quad \bar{\tau}_{00} = \tau_{00}/E, \\ W &= w/ER, \quad G = \alpha E, \quad G' = \beta E \end{aligned}$$

Using the variables, Eqs. (16) and (17) are normalized as follows ;

$$\begin{aligned} 2H^3 \left(\frac{dP}{dX} \right)^2 &= \left[\frac{24X_e}{H_e} (\mp \bar{\tau}_{00} e^{\bar{\beta} P})^2 H^2 \right. \\ &+ \left. \frac{8X_e \bar{U}}{H_e^2} \left\{ H-1 - \lambda \frac{4P_{Hz} X_e}{H_e} (X-1) \right\} \right] \left(\frac{dP}{dX} \right)^2 \\ &+ \left\{ \frac{8X_e}{H_e} (\pm \bar{\tau}_{00} e^{\bar{\beta} P}) \right\}^2 = 0 \dots\dots\dots (20) \\ H &= 1 + C_1 \frac{X_e^2}{H_e} (X^2 - 1) \\ &- C_2 \frac{X_e}{H_e} \int_{X_i}^1 P(z) \ln \left| \frac{Z-X}{Z-1} \right| dz \dots\dots\dots (21) \end{aligned}$$

where $C_1 = 8P_{Hz}^2$ and $C_2 = 16P_{Hz}^2/\pi$.

The normalized height of core H_a and the load capacity W are given respectively as,

$$H_a = \frac{1}{2} \left\{ H - \frac{8X_e}{H_e(dP/dX)} (\pm \bar{\tau}_{00} e^{\bar{\beta} P}) \right\} \dots\dots (22)$$

$$W = 4P_{Hz} X_e \int_{X_i}^1 P(X) dX \dots\dots\dots (23)$$

3.3.2 Numerical analysis It is necessary to solve the lubrication equation and the elastic equation in the elastohydrodynamic lubrication. The simultaneous solutions satisfying these equations are obtained from a nonlinear equation derived by substituting Eq. (21) into Eq. (20).

Discretizing Eq. (20) and (21), we have

$$\begin{aligned} 2H_K^3 \Phi_K^2 &= \left[\frac{24X_e}{H_e} (\pm \bar{\tau}_{00} e^{\bar{\beta} P_K})^2 H_K^2 \right. \\ &+ \left. \frac{8X_e \bar{U}}{H_e^2} \left\{ H_K - 1 - \lambda \frac{4P_{Hz} X_e}{H_e} (X_K - 1) \right\} \right] \Phi_K^2 \\ &+ \left\{ \frac{8X_e}{H_e} (\pm \bar{\tau}_{00} e^{\bar{\beta} P_K}) \right\}^2 = 0 \dots\dots\dots (24) \end{aligned}$$

$$H_K = 1 + C_1 \frac{X_e^2}{H_e} (X_K^2 - 1) - C_2 \frac{X_e}{H_e} \sum_{i=1}^{K-1} L_{K,i} \Phi_i \dots\dots\dots (25)$$

where $\Phi_K = (dP/dX)_K$, and the coefficient $L_{K,i}$ is a function of coordinate only which is derived when the integral in the elastic equation (21) is rewritten by the summation expression as shown in Eq. (25). By substituting Eq. (25) into Eq. (24), we have a nonlinear equation with respect to the pressure gradient Φ_K at each grid point. Applying the Newton-Raphson method to it, the solution is obtained.

The numerical procedure is as follows,

(a) After setting the yield stress $\bar{\tau}_{00}$, the exit film thickness H_e , the central film pressure P_{Hz} and the velocity ratio λ , the initial values of the rolling velocity $\bar{U}$ and the exit coordinate X_e are specified.

(b) Using an initial pressure gradient estimated from a pressure distribution, the pressure gradient Φ_K satisfying Eq. (24) is obtained by the Newton-Raphson method, and integrating it, the pressure distribution is calculated again.

(c) If the central pressure P_{Hz} differs from the specified value, the exit coordinate X_e is modified.

(d) After repeating the processes (b) and (c) until the expected central pressure is obtained, we confirm that the inlet pressure satisfies $P_i = 0$.

(e) If it is not so, the rolling velocity $\bar{U}$ is modified and the processes (2)~(4) are repeated.

(f) If a pressure satisfying the boundary conditions is obtained, the height of core H_a , the load capacity W and the film thickness H are calculated from Eqs. (22), (23) and (25).

The above procedures are conducted for various yield stresses, central pressures, exit film thicknesses and velocity ratios.

4. Analytical results and discussions

4.1 Flooded lubrication condition

For the flooded condition of the lubricant being fully supplied at the inlet, the inlet coordinate X_i is set at X_m which is such a point where the pressure gradient is very small. In the domain of this analysis, the flooded condition is satisfied if the inlet coordinate is $X_m = -4.0 \sim -5.0$ as described in the previous paper⁽⁴⁾. Figure 3 shows the relation between the film thickness and the rolling velocity without a squeeze motion. In this figure, the pressure coefficient of viscosity of a grease is similar to that of a Newtonian fluid, that is, $\alpha = 1.5 \times 10^{-1} \text{ mm}^2/\text{kg}$, and the elastic modulus is $E = 2 \times 10^4 \text{ kg/mm}^2$ when the cylinders are made of steel. Then $G(\alpha E)$ is 3000. As $G'(\beta E)$ affects less the elastohydrodynamic lubrication, it seems to be zero ($\beta = 0$). Also the central pressure p_0 to deform the cylinder surfaces is about 8 kg/mm^2 . From this pressure and $E = 2 \times 10^4 \text{ kg/mm}^2$, the nondimensional central pressure $P_{Hz} (p_0/E)$ becomes equal to 0.0004. The results obtained under such a condition are shown in Fig.3.

Fig.3 Effects of rolling velocity on film thickness at the exit

Fig.4 Effects of velocity ratio on film thickness at the exit

The film thickness decreases with a decrease of the rolling velocity, and the film thickness and the load capacity for a Bingham solid are larger than those for a Newtonian fluid. Figure 4 shows the relation between the film thickness and the rolling velocity with a squeeze motion. In such a case, the rolling velocity under a constant film thickness decreases with an increase of the approaching velocity, and the rate of decrease becomes larger with a decrease of film thickness. Neglecting the squeeze motion, the behavior of a Bingham solid is similar to that of a Newtonian fluid. Under a constant film thickness, the rolling velocity of a Bingham solid is smaller than that of a Newtonian fluid according as the approaching velocity decreases. Figure 5 shows the difference between the film thickness and the pressure distribution with a squeeze motion ($\lambda = 0.2 \times 10^{-4}$) and those without a squeeze motion ($\lambda = 0$). The pressure at the inlet region decreases and it comes to be distributed at the central region, and also the film thickness at the inlet region decreases and the film profile at the central region becomes flat as shown in Fig.5. According as the approaching velocity becomes larger, an entrapment in the film begins to appear at the central region, and the minimum film thickness comes to be located closer to the center.

It is known that the characteristics of the elastohydrodynamic lubrication are given by three parameters of the film thickness,

Fig.5 Effects of approaching velocity on film thickness and pressure distribution

Fig.6 Effects of approaching velocity on film thickness and load capacity

the rolling velocity and the load capacity. Under a constant film thickness and rolling velocity, the effects of the approaching velocity on the load capacity are shown in Fig. 6 in which the load capacity becomes larger due to the squeeze motion rather than the rolling one. As mentioned above, the pressure at the inlet region decreases and comes to be distributed at the central region, and the film profile becomes flat and an entrapment begins to appear at the central region.

Under a constant load capacity, the behavior of the squeeze motion is discussed in the following. We have analyzed the film thickness and the film pressure in the squeeze motion only. Then we discuss the relation between the film thickness and the velocity ratio in the squeeze motion with the rolling one as shown in Fig.7. The approaching velocity decreases with a decrease of the film thickness as in the case of the squeeze motion without the rolling one. Under a constant rolling velocity, the load capacity becomes larger according as the approaching velocity increases, that is, the rolling velocity under a constant load capacity becomes larger with a decrease of the approaching velocity. Figure 8 shows the film thickness and the film pressure generated by the squeeze motion under a constant load capacity. According as the film thickness decreases, the pressure distribution becomes narrow, and also the entrapment in the film increases.

Fig. 7 Effects of velocity ratio on film thickness at the exit

Fig. 8 Film thickness and pressure distribution in a constant load capacity

4.2 Starved lubrication condition

The starved condition of poor lubrication is discussed to show the effects of the lubricant flow on the inlet by setting $X_i = -3.0, -2.0, -1.8$, and -1.6 of the inlet coordinate which are closer to the center than that of the flooded condition.

Under a constant load capacity, rolling velocity and velocity ratio, the inlet coordinate affects the film thickness and the film pressure as shown in Fig. 9 in which $\bar{P} = p/E$ and $\bar{H} = h/R$ are used instead of $P = p/p_0$ and $H = h/h_e$ in the previous article to make the effects of the inlet coordinate evident. For instance, the film thickness with $X_i = -1.6$ becomes thinner than the one under the flooded condition ($X_i = X_m$), and the film profile at the central region becomes flat and the entrainment is more liable to appear.

Under a constant rolling velocity and film thickness, the effects of the inlet coordinate on the load capacity are shown in Fig. 10. The load capacities with or without the squeeze motion decrease according as the inlet coordinate becomes closer to the center, and they decrease abruptly in the region below $X_i = -2.0$. The load capacity of a Bingham solid is larger than that of a Newtonian fluid, and the load capacity under the starved condition varies less in a Bingham than in a Newtonian fluid.

Fig. 9 Effects of lubrication condition on film thickness and load capacity

Fig. 10 Effects of lubrication condition on load capacity

5. Conclusions

Analyzing the elastohydrodynamic lubrication between rotating two cylinders with a squeeze motion under the flooded and the starved conditions in a Bingham solid, the following conclusions are obtained.

In the flooded condition of full lubrication,

(1) The film thickness under a constant rolling velocity becomes larger in a Bingham solid than in a Newtonian fluid. The difference between them increases according as the rolling velocity becomes small and the approaching velocity becomes large.

(2) When the approaching velocity is large under a constant rolling velocity and film thickness, the pressure distribution is narrow and the film thickness is flat and an entrainment appears.

(3) Under a constant rolling velocity and load capacity, the more the approaching velocity increases, the more the film thickness decreases, and it becomes larger with an increase of the load capacity under a constant rolling velocity and with a decrease of the rolling velocity under a constant load capacity.

In the starved condition of poor lubrication,

(4) When the inlet coordinate is close to the center under a constant rolling velocity and load capacity, the film pressure comes to be distributed at the central region and the maximum pressure becomes large, and the film becomes thin and flat.

(5) The load capacity is affected less by the inlet coordinate when the approaching velocity is large, and the inlet coordinate affects less the lubricating behavior of a Bingham solid than that of a Newtonian fluid.

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