532.517:532.526:532.582

Paper No. 192-6

Linear Stability Theory of Boundary Layer along a Cone Rotating in Axial Flow*

By Ryōji KOBAYASHI**

A linear stability theory is treated for an incompressible laminar boundary layer along a cone rotating about its axis of symmetry with a constant angular speed in an external forced flow field. Small perturbations are assumed to be of spiral vortices. This leads to one set of perturbation equations including an order of magnitude (δ_1/R_0) , where δ_1 denotes a displacement thickness of the boundary layer, calculated from a velocity component in the meridional section, and R_0 is a local radius of the cone at instability. A numerical procedure for solving the eigenvalue problem is shown. As an example, calculations are carried out for a cone having a total included angle of 30° at a rotating speed ratio 3, where the rotating speed ratio is defined as the ratio of the circumferential velocity at the cone surface to the external flow velocity at the outer edge of the boundary layer. From stability diagrams obtained herein, critical Reynolds number, spiral angle and the associated eigenvalues are determined. The critical Reynolds number is compared with an experiment.

1. Introduction

Detailed studies on transition phenomena of boundary layers along a cone rotating in an axial flow are important to promote understanding of basic mechanism in transition regions of complex three-dimensional boundary layers in internal flows of turbomachines or on rotating axisymmetric bodies, because one can investigate characteristics caused by the rotational motion using a small number of geometrical and mechanical parameters. Previous investigations on rotating cones were carried out for inviscid flows,⁽¹⁾⁽²⁾ for boundary layers with heat and mass transfer (theories⁽³⁾⁽⁵⁾ and experiment⁽⁶⁾⁽⁷⁾) and for those in still fluid⁽⁸⁾⁽¹³⁾. For a circular disk rotating in still fluid, instability and transition of the boundary layer were studied in detail,⁽¹⁴⁾⁽¹⁷⁾ while, for rotating cones in an axial flow, mechanism of transition has not been clarified.

The purpose of the present paper is to consider theoretically the instability criterion of the three-dimensional laminar boundary layer along a cone rotating in an axial flow. This instability problem includes Tollmien-Schlichting instability and Taylor instability. Perturbation equations for the present phenomena have not yet been obtained. Although perturbation equations for boundary layers on rotating disk in still fluid⁽¹⁴⁾⁽¹⁷⁾ and for Couette-Poiseuille flow between rotating coaxial cylinders⁽¹⁸⁾⁽¹⁹⁾ have been known, they can not be applied to the case of a rotating

cone. We therefore analyze at first the perturbation equations, and then show a numerical example of a cone having a total included angle of 30° .

2. Nomenclature

- $c = c_r + ic_i$
 c_r : phase velocity = $\lambda_r h_1 / (a u_e)$, dimensionless
 c_i : amplification factor = $\lambda_i h_1 / (a u_e)$, dimensionless
 R : radius of a circular cross section
 R_0 : radius R of the origin O'
 Re : Reynolds number = $U_e \delta_1 / \nu$
 S : rotational speed ratio = $R_0 \omega / U_e$
 U, V, W : velocity components of x' -, y' - and z' -axes in laminar boundary layer, dimensionless
 U_e : meridional velocity at outer edge of boundary layer
 u, v, w : amplitude functions of velocity disturbances in x' -, y' -, z' -directions, dimensionless
 a : wave number, Eq. (23)
 δ_1 : displacement thickness obtained from meridional velocity distribution in laminar boundary layer
 δ' : ratio of boundary layer thickness to rotational radius = δ_1 / R_0
 ω : angular velocity of rotating cone
 ϵ : spiral angle
 θ : half angle of cone
 σ : wave number = $a \delta_1 / h_1$, dimensionless
 ν : kinematic viscosity of fluid

3. Perturbation equations

3.1 Orthogonal curvilinear co-ordinate system

A rectangular co-ordinate system (X, Y, Z) is taken as shown in Fig. 1, where the origin O is on the apex of a cone. The cone is rotating around the X -axis with

* Received 4th June, 1979.

** Professor, Institute of High Speed Mechanics, Tohoku University, Katahira 2-1-1, Sendai.

Fig. 1 Rotating cone in axial flow and co-ordinate system

constant angular velocity ω . An approaching flow is directed toward the X-axis.

We adopt here an orthogonal curvilinear co-ordinate system (x', y', z') , where the origin o' is fixed on the surface of the rotating cone. It is known in the case of a rotating disk in still fluid,⁽¹⁴⁾⁽¹⁷⁾ which is a special case of the present problem, that logarithmic spiral vortices appear in the transition region of boundary layer. Spiral vortices are also observed in an annular flow between two co-axial cylinders, where the outer cylinder is at rest, and the inner cylinder rotates around its axis and moves in the direction of the center axis with constant speed.⁽²⁰⁾ We expect that a similar phenomenon appears in a transition region of the boundary layer on the rotating cone. The two axes x' and y' are, therefore, set along a pair of logarithmic spirals making a right angle with each other, as shown in Fig. 1, where the direction of the y' -axis is chosen to coincide with the axis of spiral vortices, so that the x' -axis is in the direction of propagation of the spiral vortices. The z' -axis is normal to the surface of the cone. The angle ϵ between the x' -axis and the meridional direction is unknown. It should be determined by stability theories or by experiments. We treat the angle ϵ as a variable parameter at the start of the present stability theory, and discuss later an expected value of ϵ from numerical results.

The x' -axis is written as follows:

$$\frac{dR/\sin\theta}{Rd\varphi} = \cot\epsilon \quad (1)$$

where θ denotes the half angle of the cone,

R_0 the distance of the origin o' measured from the X-axis, R the distance of a point $(x', y', 0)$ at the cone surface from the X-axis, and φ a rotating angle measured from the meridional line including the origin o' . Integration of Eq. (1) gives

$$\ln \frac{R}{R_0} = \varphi \sin\theta \cot\epsilon \quad (2)$$

Similarly, for the y' -axis,

$$-\frac{dR/\sin\theta}{Rd\varphi} = \tan\epsilon \quad (3)$$

Integration gives

$$\ln \frac{R}{R_0} = -\varphi \sin\theta \tan\epsilon \quad (4)$$

Using Eqs. (2) and (4), we write

$$M = \ln \frac{R}{R_0} - \varphi \sin\theta \cot\epsilon \quad (5)$$

$$N = \ln \frac{R}{R_0} + \varphi \sin\theta \tan\epsilon \quad (6)$$

where M and N denote parameters. Replacing M and N with x' and y' as follows:

$$\frac{x'}{R_0} = N \cos\epsilon, \quad \frac{y'}{R_0} = M \sin\epsilon \quad (7)$$

we then obtain the following relations between (R, φ) and (x', y') :

$$x' = R_0 \left(\cos\epsilon \ln \frac{R}{R_0} + \varphi \sin\theta \sin\epsilon \right) \quad (8)$$

$$y' = R_0 \left(\sin\epsilon \ln \frac{R}{R_0} - \varphi \sin\theta \cos\epsilon \right) \quad (9)$$

and

$$R = R_0 \exp \left\{ \frac{1}{R_0} (x' \cos\epsilon + y' \sin\epsilon) \right\} \quad (10)$$

$$\varphi = \frac{1}{R_0 \sin\theta} (x' \sin\epsilon - y' \cos\epsilon) \quad (11)$$

The relations between the rectangular co-ordinate system (X, Y, Z) and the orthogonal curvilinear co-ordinate system (x', y', z') are, therefore, expressed as follows:

$$X = R \cot\theta - z' \sin\theta \quad (12)$$

$$Y = (R + z' \cos\theta) \cos(\varphi + \omega t) \quad (13)$$

$$Z = (R + z' \cos\theta) \sin(\varphi + \omega t) \quad (14)$$

We assume that the thickness of the boundary layer is small compared with the radius R . The parameters (h_1, h_2, h_3) of the orthogonal curvilinear co-ordinate system (x', y', z') are hence written as

$$h_1 = \left\{ \left(\frac{\partial X}{\partial x'} \right)^2 + \left(\frac{\partial Y}{\partial x'} \right)^2 + \left(\frac{\partial Z}{\partial x'} \right)^2 \right\}^{1/2} = \frac{R}{R_0 \sin\theta} \left(1 + 2 \frac{z'}{R} \cos\theta \sin^2\epsilon + \frac{z'^2}{R^2} \cos^2\theta \sin^2\epsilon \right)^{1/2} \\ \doteq \frac{R}{R_0 \sin\theta} \left(1 + \frac{z'}{R} \cos\theta \sin^2\epsilon \right) \quad (15)$$

$$h_2 = \left\{ \left(\frac{\partial X}{\partial y'} \right)^2 + \left(\frac{\partial Y}{\partial y'} \right)^2 + \left(\frac{\partial Z}{\partial y'} \right)^2 \right\}^{1/2} = \frac{R}{R_0 \sin\theta} \left(1 + \frac{z'}{R} \cos\theta \cos^2\epsilon \right) \quad (16)$$

$$h_3 = \left\{ \left(\frac{\partial X}{\partial z'} \right)^2 + \left(\frac{\partial Y}{\partial z'} \right)^2 + \left(\frac{\partial Z}{\partial z'} \right)^2 \right\}^{1/2} = 1 \quad (17)$$

3.2 Perturbation equations

The equations governing the motion of an incompressible viscous fluid are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \times \boldsymbol{\omega}) \times \mathbf{v} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\nabla \left(\frac{p}{\rho} + \frac{1}{2} \mathbf{v}^2 \right) + \nu \nabla^2 \mathbf{v} \quad (18)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (19)$$

where $\mathbf{v}$ denotes the velocity vector measured in the co-ordinate system (x', y', z') , $\mathbf{r}$ the position vector for the origin O , $\boldsymbol{\omega}$ the rotation vector of the co-ordinate system, p the pressure, ρ the density of fluid, ν the kinematic viscosity and t the time.

We shall now suppose that a laminar boundary layer on the rotating cone (the velocity $\bar{\mathbf{v}}$ and the pressure $\bar{p}$) is slightly perturbed. The velocity $\mathbf{v}$ and the pressure p are then written as

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}', \quad p = \bar{p} + p' \quad (20)$$

where $\mathbf{v}'$ and p' are perturbations, which are small enough compared with $\bar{\mathbf{v}}$ and $\bar{p}$. Substituting Eq. (20) into Eqs. (18) and (19), and linearizing for the small perturbations, we obtain the following equations for the perturbations:

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}' \times \bar{\mathbf{v}}) \times \bar{\mathbf{v}} + (\mathbf{v}' \times \bar{\mathbf{v}}) \times \mathbf{v}' + 2\bar{\boldsymbol{\omega}} \times \mathbf{v}' = -\nabla \left(\frac{p'}{\rho} + \bar{\mathbf{v}} \cdot \mathbf{v}' \right) + \nu \nabla^2 \mathbf{v}' \quad (21)$$

$$\nabla \cdot \mathbf{v}' = 0 \quad (22)$$

The vortex-like perturbations are expressed as

$$\mathbf{v}' = \hat{\mathbf{v}}(z') e^{i(a x' - \lambda t)} \quad (23)$$

$$p' = \hat{p}(z') e^{i(a x' - \lambda t)} \quad (24)$$

where a is the wave number, the real part (λ_r) of λ is related with the phase velocity (λ_r/a) of the perturbations in the x' -direction, and the imaginary part (λ_i) of λ is their amplification rate. The amplitude function $\hat{\mathbf{v}}(z')$ of velocity perturbation has three components ($\hat{u}(z')$, $\hat{v}(z')$, $\hat{w}(z')$) for the x' -, y' - and z' -directions, respectively.

Now the expressions (23) and (24) are substituted into Eqs. (21) and (22). For the velocity vector $\bar{\mathbf{v}}(\bar{u}, \bar{v}, \bar{w})$, we neglect the z' -component $\bar{w}$ because of $\bar{w} \ll (\bar{u}, \bar{v})$, and assume that $\bar{u}$ and $\bar{v}$ are functions of z' only. We further neglect the quantities of the order of magnitude $(\delta_1/R_0)^2$ and variations of the perturbations in the y' -direction, but consider $\partial h_1/\partial y$, because it is not small. The two vector equations (21) and (22) are then reduced to the following perturbation equations:

$$-i\lambda \hat{u} + 2\omega(\hat{w} \cos \theta \sin \varepsilon + \hat{v} \sin \theta) - \bar{V} \left(\frac{h_{2x}}{h_1 h_2} \hat{v} - \frac{h_{1y}}{h_1 h_2} \hat{u} \right) - \hat{v} \left(\frac{h_{2x}}{h_1 h_2} \bar{V} - \frac{h_{1y}}{h_1 h_2} \bar{U} \right) + \hat{w} \left(\frac{h_{1z}}{h_1} \bar{U} + \bar{U}_z \right) = -\frac{i\alpha}{\rho h_1} \hat{p} - \frac{i\alpha}{h_1} \bar{U} \hat{u} + \nu \left(\hat{u}_{zz} - \frac{i\alpha}{h_1} \hat{w}_z \right) \quad (25)$$

$$-i\lambda \hat{v} - 2\omega(\hat{u} \sin \theta + \hat{w} \cos \theta \cos \varepsilon) + \bar{U} \left\{ \left(\frac{h_{2x}}{h_1 h_2} + \frac{i\alpha}{h_1} \right) \hat{v} - \frac{h_{1y}}{h_1 h_2} \hat{u} \right\} + \left(\frac{h_{2x}}{h_2} \bar{V} + \bar{V}_z \right) \hat{w} + \left(\frac{h_{2x}}{h_1 h_2} \bar{V} - \frac{h_{1y}}{h_1 h_2} \bar{U} \right) \hat{u} = \nu \left(\hat{v}_{zz} - \frac{\alpha^2}{h_1^2} \hat{v} \right) \quad (26)$$

$$-i\lambda \hat{w} + 2\omega(\hat{v} \cos \theta \cos \varepsilon - \hat{u} \cos \theta \sin \varepsilon) - \bar{U} \left(\frac{h_{1z}}{h_1} \hat{u} - \frac{i\alpha}{h_1} \hat{w} \right) - \frac{2h_{2z}}{h_2} \bar{V} \hat{v} - \frac{h_{1z}}{h_1} \bar{U} \hat{u} = -\frac{1}{\rho} \hat{p}_z - \nu \left(\frac{i\alpha}{h_1} \hat{u}_z + \frac{\alpha^2}{h_1^2} \hat{w} \right) \quad (27)$$

$$\left(\frac{h_{2x}}{h_1 h_2} + \frac{i\alpha}{h_1} \right) \hat{u} + \frac{h_{1y}}{h_1 h_2} \hat{v} + \left(\frac{h_{1z}}{h_1} + \frac{h_{2z}}{h_2} \right) \hat{w} + \hat{w}_z = 0 \quad (28)$$

where the subscripts x , y , z denote differentiations with respect to x' , y' , z' , respectively. Eliminating the pressure $\hat{p}$ from Eqs. (25) and (27), substituting h_1 and h_2 of Eqs. (15) and (16), and introducing the following dimensionless expressions:

$$\left. \begin{aligned} z &= \frac{z'}{\delta_1}, & U &= \frac{\bar{U}}{U_e}, & V &= \frac{\bar{V}}{U_e}, & u &= \frac{\hat{u}}{U_e}, & v &= \frac{\hat{v}}{U_e}, & w &= \frac{\hat{w}}{U_e} \\ R_e &= \frac{U_e \delta_1}{\nu}, & \delta' &= \frac{\delta_1}{R_0}, & \sigma &= \frac{\alpha \delta_1}{h_1}, & c &= \frac{\lambda \delta_1}{\sigma U_e}, & S &= \frac{R_0 \omega}{U_e} \end{aligned} \right\} \quad (29)$$

U_e : meridional velocity at the outer edge of the boundary layer,

δ_1 : displacement thickness obtained from the meridional velocity distribution in the laminar boundary layer

we reach finally a system of ordinary differential equations:

$$(U-c)(w_{zz} - \sigma^2 w) - U_{zz} w + \frac{i}{\sigma R_e} (w_{zzzz} - 2\sigma^2 w_{zz} + \sigma^4 w) = \delta' \left[\frac{i \sin \theta}{\sigma} \{ (c-U) \cos \varepsilon + V \sin \varepsilon \} w_{zz} + \{ (c-U) \cos \theta - U \cos \theta \sin^2 \varepsilon \right.$$

$$\begin{aligned}
& + \frac{i}{\sigma} (V_z \sin \varepsilon - U_z \cos \varepsilon) \sin \theta \} w_z - U_z w \cos \theta \cos^2 \varepsilon \\
& + 2 \cos \varepsilon \{ i \sigma (V \cos \varepsilon - S) \cos \theta - V_z \sin \theta \} v + \sin \theta (c \sin \varepsilon - 2V \cos \varepsilon + 2S) v_z \} \dots \dots \dots (30)
\end{aligned}$$

$$\begin{aligned}
& \frac{i}{\sigma R_e} (v_{zz} - \sigma^2 v) + (U - c) v - \frac{i}{\sigma} V_z w = \delta' \left\{ \frac{i}{\sigma} U v \sin \theta \cos \varepsilon + \frac{\sin \theta}{\sigma^2} (2S - V \cos \varepsilon + 2U \sin \varepsilon) w_z \right. \\
& \left. + \frac{i}{\sigma} \cos \theta \cos \varepsilon (V \cos \varepsilon - 2S) w \right\} \dots \dots \dots (31)
\end{aligned}$$

$$i \sigma u + w_z + \delta' \{ (u \cos \varepsilon + v \sin \varepsilon) \sin \theta + w \cos \theta \} = 0 \dots \dots \dots (32)$$

in which the subscript z denotes differentiation with respect to z . It is noted that Eqs. (30) and (31) govern the amplitude functions $v(z)$ and $w(z)$ and do not include $u(z)$, so that $u(z)$ can be calculated using Eq. (32), after $v(z)$ and $w(z)$ are determined from Eqs. (30) and (31).

In Eq. (30), the left-hand side is in the same form as the Orr-Sommerfeld equation. The right-hand side is additional terms in the order of δ' in magnitude, of which the fourth term including the velocity component V is related with centrifugal force and the terms including the rotational speed ratio S come from Coriolis force. It should be noted that the local rotational speed ratio $S (= R_0 \omega / U_e)$ is a significant parameter governing the effect of rotation on the present instability problem.

The boundary conditions which follow from the requirement of no slip at the wall surface ($z = 0$) are $u = v = w = 0$. As the other three boundary conditions it is reasonable to take $u, v, w \rightarrow 0$ not at the outer edge of the boundary layer but at a point ($z \rightarrow \infty$) far from the wall, because the spiral vortices of small wave number grow through the outer edge of the boundary layer. In view of the continuity equation, these boundary conditions are written as follows:

$$\begin{aligned}
& z = 0 : v = w = w_z = 0 \} \dots \dots \dots (33) \\
& z \rightarrow \infty : v, w, w_z \rightarrow 0 \}
\end{aligned}$$

4. Procedure of solving perturbation equations

The present linear stability theory is now reduced to an eigenvalue problem to solve the perturbation equations (30) to (32) under the boundary conditions mentioned above. There exists a nontrivial solution for a suitable combination between the parameters $(\theta, S, Re, \varepsilon, \sigma, \text{ and } c)$. For temporal stability, $c (= \lambda \delta_1 / (\sigma U_e))$ is complex. The real part of c , namely c_r , is the phase velocity, and the imaginary part of c , namely c_i , represents the amplification or damping of the perturbations with the passage of time. If c_i is negative in the solution, the flow is stable, while for $c_i > 0$ the flow becomes unstable. A shooting technique^{(2,3) (2,4)} was employed for solving the present eigenvalue problem.

The procedure of numerical calculations is as follows. For given numerical values of the parameters $\theta, S, Re, \varepsilon$ and σ , the other parameter c is determined as

eigenvalue. Two predicted values of the parameter c are first given, and corrections for c are made in the course of the calculation. The outer boundary conditions were set at infinity ($z \rightarrow \infty$), but in numerical work, it is desirable to replace the interval $0 \leq z < \infty$ by the interval $0 \leq z \leq z_0$. The position z_0 is set where the basic velocity profiles $U(z)$ and $V(z)$ become nearly equal to their limiting values.

For the outer range ($z \geq z_0$), the coefficients of the perturbation equations (30) to (32) become constant because U and V are constant. Solutions of the equations can be then expressed in the form

$$v = \sum_{i=1}^6 a_i \exp(\gamma_i z) \dots \dots \dots (34)$$

where a_i ($i = 1 \sim 6$) are constant and the exponents γ_i ($i = 1 \sim 6$) are roots of the characteristic equation of the sixth order, which is obtained by substituting Eq. (34) into the perturbation equations (30) and (31). The characteristic equation can be solved by using Lin's technique. Among the six roots γ_i ($i = 1 \sim 6$), three roots having a positive real part should be excluded because such terms in Eq. (34) do not satisfy the boundary conditions for $z \rightarrow \infty$. The other three roots γ_i (say, $i = 1, 2, 3$) having a negative real part are adopted.

Next, the perturbation equations (30) and (31) are numerically integrated inward from $z = z_0$ to the wall surface $z = 0$ using Runge-Kutta-Gill's technique and Adams-Moulton's technique. A starting value of the first integral at $z = z_0$ is Eq. (34) with a combination ($a_1 = 1, a_2 = 0, a_3 = 0$). The solutions of $v(z)$, $w(z)$ and $w_z(z)$ are written as $v_1(z)$, $w_1(z)$ and $w_{z1}(z)$. Similarly, the second and the third integrals are carried out for ($a_1 = 0, a_2 = 1, a_3 = 0$) and for ($a_1 = 0, a_2 = 0, a_3 = 1$), respectively. The solutions are written as (v_2, w_2, w_{z2}) and (v_3, w_3, w_{z3}). Because Eqs. (30) and (31) are linear, the final solutions can be expressed with linear combinations of the three integrals as follows:

$$v(z) = \sum_{i=1}^3 b_i v_i(z) \dots \dots \dots (35)$$

$$w(z) = \sum_{i=1}^3 b_i w_i(z) \dots \dots \dots (36)$$

$$w_z(z) = \sum_{i=1}^3 b_i w_{zi}(z) \dots \dots \dots (37)$$

where b_i ($i = 1, 2, 3$) are constant. Putting $b_1 = 1$ does not cause the generality

of the problem to be lost. Applying the boundary conditions $v = 0$ and $w = 0$ at the solid surface ($z = 0$) to Eqs. (35) and (36), we have

$$v_1(0) + b_2 v_2(0) + b_3 v_3(0) = 0 \quad \dots\dots\dots(38)$$

$$w_1(0) + b_2 w_2(0) + b_3 w_3(0) = 0 \quad \dots\dots\dots(39)$$

which give

$$b_2 = \frac{w_1(0)v_3(0) - v_1(0)w_3(0)}{\Delta} \quad \dots\dots\dots(40)$$

$$b_3 = \frac{v_1(0)w_2(0) - w_1(0)v_2(0)}{\Delta} \quad \dots\dots\dots(41)$$

where

$$\Delta = v_2(0)w_3(0) - w_2(0)v_3(0) \quad \dots\dots\dots(42)$$

The value of $w_z(0)$ at $z = 0$ can be calculated from the following equation:

$$w_z(0) = w_{z1}(0) + b_2 w_{z2}(0) + b_3 w_{z3}(0) \quad (\equiv K) \quad \dots\dots\dots(43)$$

If the value of c predicted initially is correct, the K -value in Eq. (43) should be zero. Because of $K \neq 0$ in general, it is necessary that the numerical integrations be made for the two predicted values of c , and then interpolation or extrapolation of c be done by using the two predicted c in order to make the value of K zero. After the successive approximation procedure, one obtains finally a correct c as eigenvalue

Fig. 2 Velocity distributions in laminar boundary layer
(x' -component $U(z)$)
 $2\theta = 30^\circ$, $S = 3$

Fig. 3 Velocity distributions in laminar boundary layer
(y' -component $V(z)$)

and the amplitude functions $v(z)$ and $w(z)$ from Eqs. (35) and (36) as eigenfunctions. The other amplitude function $u(z)$ is calculated by using Eq. (32).

5. Numerical results and discussions

5.1 Laminar boundary layer on a rotating cone

A numerical example for the present theory was carried out for a case of $2\theta = 30^\circ$ and $S (= R_0\omega/U_\infty) = 3$, which is expected to cause a remarkable effect of rotation. Velocity profiles of the laminar boundary layer for that case are shown in Figs. 2 and 3, which are calculated by using Koh-Price technique⁽⁴⁾. The range of ϵ in these figures is selected to include those for the minimum of critical Reynolds number and for the maximum amplification of perturbations, as stated later. The displacement thickness δ_1 and the momentum thickness δ_2 of the boundary layer, calculated from the meridional component of the resultant velocity, are

$$\delta_1 \sqrt{\frac{U_\infty}{\nu R}} = 0.819, \quad \delta_2 \sqrt{\frac{U_\infty}{\nu R}} = 0.453 \quad \dots\dots(44)$$

Fig. 4 Eigenvalues σ_{c1} and σ_{c2}
 $2\theta = 30^\circ$, $S = 3$, $\epsilon = 15^\circ$

Fig. 5 Stability diagram
 $2\theta = 30^\circ$, $S = 3$, $\epsilon = 15^\circ$
Curve a: σ -value for the maximum of σ_{c1} at each Reynolds number Re

The shape parameter $H(=\delta_1/\delta_2)$ is hence $H = 1.806$. The parameter δ' is related to the Reynolds number R_e as

$$\delta' = \frac{0.6698}{R_e} \dots \dots \dots (45)$$

5.2 Instability of boundary layer

Figure 4 shows eigenvalues (phase velocity c_r and amplification factor $\sigma_{ci}(=\lambda_1\delta_1/U_e)$) obtained for $\epsilon = 15^\circ$ in relation to the wave number σ for various Reynolds numbers R_e . The phase velocity c_r varies little for wide ranges of R_e and σ . The amplification factor σ_{ci} remains negative for all values of σ , when R_e is less than about 11, which is in stable range. It is also clear that, when R_e is increased, σ_{ci} increases at first, reaches a maximum, and then decreases.

Figure 5, which is obtained from Fig. 4, is a stability diagram in a relation between R_e and σ for various values of σ_{ci} . A curve for $\sigma_{ci} = 0$ gives a state of neutral stability, and the range for $\sigma_{ci} > 0$ is unstable. Figure 5 indicates that the critical Reynolds number R_{ec} is 10.6 and the associated wave number σ_c is 0.65. A curve *a* is drawn for the wave number, at which the amplification factor σ_{ci} becomes the maximum $(\sigma_{ci})_{\max}$ for each value of the Reynolds number. There exists further the maximum value of $(\sigma_{ci})_{\max}$ along the curve *a* at a certain Reynolds number, namely R_{em} . Figure 5 for $\epsilon = 15^\circ$ gives the Reynolds

number $R_{em} = 41$ and the associated wave number $\sigma_m = 1.10$.

Similar calculations were carried out for various values of the spiral angle ϵ . Figure 6 shows changes in the critical Reynolds number R_{ec} and the Reynolds number for the maximum amplification R_{em} as a function of the spiral angle ϵ . It is clear that the critical Reynolds number R_{ec} has the minimum value 10.4 at $\epsilon = 12^\circ \sim 13^\circ$ and is insensible between $\epsilon = 5^\circ$ and 20° ; and that R_{em} has the maximum value 42.5 at $\epsilon = 14^\circ \sim 15^\circ$.

Figure 7 gives the associated wave numbers σ_c and σ_m for the Reynolds numbers R_{ec} and R_{em} , respectively. The critical wave number σ_c has the maximum value 0.70 at $\epsilon = 12^\circ \sim 13^\circ$, while the maximum of σ_m is 1.11 at $\epsilon = 14^\circ \sim 15^\circ$.

Figure 8 indicates relations between the maximum amplification factor $(\sigma_{ci})_{\max}$ and the spiral angle ϵ for various Reynolds numbers R_e . The curve *b* represents the values of $(\sigma_{ci})_{\max}$ for $R_e = R_{em}$. It should be noted that there exists the maximum of the amplification factor at a certain value of the spiral angle ϵ . In the case of Fig. 8, the angle ϵ is about 14° to 15° . Figure 9 shows the phase velocity c_r belonging to $(\sigma_{ci})_{\max}$ of Fig. 8. The phase velocity c_r for $\epsilon = 14^\circ \sim 15^\circ$ is 0.22 $\sim$ 0.25 in a wide range of the Reynolds numbers R_e .

The numerical example for $2\theta = 30^\circ$ and $S = 3$ mentioned above indicates that the critical Reynolds number R_{ec} varies little

Fig. 6 Critical Reynolds number R_{ec} and Reynolds number R_{em} for the maximum amplification for different values of ϵ
 $2\theta = 30^\circ$, $S = 3$

Fig. 7 Wave numbers σ_c and σ_m of spiral vortices for Reynolds numbers R_{ec} and R_{em} , respectively
 $2\theta = 30^\circ$, $S = 3$

Fig. 8 Amplification factor $(\sigma_{ci})_{\max}$ as a function of spiral angle ϵ
Curve *b*: The maximum amplification factor at Reynolds number R_{em}
 $2\theta = 30^\circ$, $S = 3$

Fig. 9 Phase velocity c_r for amplification factor $(\sigma_{ci})_{\max}$
 $2\theta = 30^\circ$, $S = 3$

in a wide range of the spiral angles ϵ , while the amplification factor $(\sigma_{ci})_{\max}$ varies remarkably as ϵ is changed and it has the maximum at $\epsilon = 14^\circ \sim 15^\circ$. It might be reasonable to assume that the axes of spiral vortices on a rotating cone are directed toward the angle, where the amplification factor becomes a maximum. We have hence $\epsilon = 14^\circ \sim 15^\circ$ for $2\theta = 30^\circ$ and $S = 3$, for which $Re_c = 10.5$ and $Re_m = 42.5$.

Salzberg-Kezios⁽⁶⁾ measured local mass transfer from a rotating naphthalene cone with $2\theta = 30^\circ$ in air stream. From their experimental results, we read a Reynolds number for transition of the boundary layer in a case of rotational speed ratio $S = 3$. The transition Reynolds number defined in the present paper was $30 \sim 40$. On the other hand, a critical Reynolds number for a non-rotating cone ($S = 0$) with $2\theta = 30^\circ$ is estimated to be $Re_c \approx 5.5 \times 10^2$ by using the corresponding shape parameter $H = 2.56$ of the velocity profile in the laminar boundary layer.⁽²⁵⁾ It can be, therefore, said that the Reynolds numbers Re_c and Re_m obtained from the present theory have reasonable values, as compared with the experiment by Salzberg-Kezios.⁽⁶⁾

6. Conclusions

The perturbation equations are given for the instability problem of laminar boundary layer along a rotating cone in axial flow. While the instability for a case of a non-rotating cone in an axial flow is governed by the Orr-Sommerfeld equation, the perturbation equations for the rotating cone include not only the velocity component $U(z)$, but also $V(z)$, so that the perturbation equations become one set of simultaneous ordinary differential equations for the amplitude functions $u(z)$, $v(z)$ and $w(z)$ of velocity perturbations. It is noted that a significant parameter governing the present instability problem is the local rotational speed ratio S .

A numerical example was carried out in a case where the cone included angle 2θ is 30° and the rotational speed ratio S is 3. The stability diagrams show that the spiral angle ϵ is $14^\circ \sim 15^\circ$, the critical Reynolds number Re_c is 10.5 and the Reynolds number Re_m for the maximum amplification is 42.5, which is well consistent with the experiment.

It is, therefore, concluded that the present theory gives reasonable results for studying the effect of rotation on the instability of laminar boundary layers along cone surfaces.

The author is indebted to Mr. Ch. Takamade (Sumitomo Metal Industries, Ltd.) and Mr. M. Kurosawa (Mitsubishi Heavy Industries, Ltd.), who were graduate students at that time, for their aid in numerical calculations.

References

- (1) Leuteritz, R. and Mangler, W., *Untersuch. Mitt. Deut. Luftfahrtforsch.* Nr. 3226, (1945).
- (2) Hess, J. L. and Faulkner, S., *AIAA J.*, Vol. 3, No. 4 (1965-4), p. 767.
- (3) Tien, C. L. and Tsuji, I. J., *Trans. ASME, Ser. C*, Vol. 84, No. 4 (1965-5), p. 184.
- (4) Koh, J. C. Y. and Price, J. F., *Trans. ASME, Ser. C*, Vol. 89, No. 2 (1967-5), p. 139.
- (5) Illingworth, C. R., *Phil. Mag.*, Vol. 44, No. 351 (1953-4), p. 389.
- (6) Salzberg, F. and Kezios, S. P., *Trans. ASME, Ser. C*, Vol. 87, No. 4 (1965-11), p. 469.
- (7) Okamoto, T., et al., *Bull. JSME*, Vol. 19, No. 134 (1976-8), p. 930.
- (8) Tien, C. L. and Campbell, D. T., *J. Fluid Mech.*, Vol. 17, No. 1 (1963-9), p. 105.
- (9) Kreith, F., *Proc. Heat Transfer and Fluid Mech. Inst.*, (1966), p. 29.
- (10) Fukusako, S., et al., *Appl. Sci. Res.*, Vol. 23, No. 3/4 (1970-12), p. 269.
- (11) Kappesser, R., et al., *Appl. Sci. Res.*, Vol. 28, No. 6 (1973-12), p. 442.
- (12) Koyama, H., *Bull. JSME*, Vol. 17, No. 112 (1974-10), p. 1369.
- (13) Koosinlin, M. L., et al., *Trans. ASME, Ser. C*, Vol. 96, No. 2 (1974-5), p. 204.
- (14) Gregory, N., et al., *Phil. Trans.*, Vol. 248, A, No. 943 (1955-7), p. 155.
- (15) Chin, D.-T. and Litt, M., *J. Fluid Mech.*, Vol. 54, No. 4 (1972-8), p. 613.
- (16) Yamashita, I. and Takematsu, M., *Rep. Res. Inst. Appl. Mech., Kyushu Univ.*, Vol. 22, No. 69 (1974), p. 1.
- (17) Kobayashi, R., et al., *Acta Mech.*, Vol. 35, No. 1/2 (1980), p. 71.
- (18) Chandrasekhar, S., *Hydrodynamic and Hydromagnetic Stability*, (1961), p. 272, Oxford Univ. Press.
- (19) Chung, K. C. and Astill, K. N., *J. Fluid Mech.*, Vol. 81, No. 4 (1977-8), p. 641.
- (20) Ludwig, H., *Applied Mechanics, Proc. 11th Int. Congr. Appl. Mech.*, ed. H. Görtler, (1964), p. 1045.
- (21) Nagib, H. M., *Ph. D. Diss., Illinois Inst. Tech.*, (1972).
- (22) Greenspan, H. P., *The Theory of Rotating Fluids*, (1968), p. 5, Cambridge Univ. Press.
- (23) Brown, W. B., *Boundary Layer and Flow Control*, ed. G. V. Lachmann, Vol. 2 (1961), p. 913, Pergamon Press.
- (24) Betchov, R. and Criminale Jr., W. O., *Stability of Parallel Flows*, (1967), p. 74, Academic Press.
- (25) *Laminar Boundary Layers*, ed. L. Rosenhead, (1963), p. 543, Oxford Univ. Press.