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Stress Analysis of Thin Rim Spur Gears

by Finite Element Method*

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The root stresses of thin rim spur gears are investigated by means of the two-dimensional finite element method (FEM) with triangular elements.

The effects of boundary constraints for gear models and of adjacent teeth on the root stress distribution are discussed. The gear tooth model with a single tooth is shown to be available for a sufficiently precise evaluation of root stresses. A suitable rim thickness of gear model for the stress analysis by FEM is proposed.

The true root stresses are computed and measured for gears of different rim thicknesses. The true stresses computed by FEM are in a fairly good agreement with measured ones. On the basis of these investigations the effects of rim thickness on the root stresses and on the critical sections in thin rim spur gears are also discussed.

1. Introduction

With recently increased requirements for speedup and light weight, thin rim gears have come increasingly into use and their load bearing capacity is now a matter of great concern.

In order to advance the study on the bending fatigue strength of thin rim gears, it would be necessary, first of all, to determine the root fillet stresses produced by an application of load. The stress state on the root fillet in a thin rim gear is considered to be appreciably different from that in a solid gear because of its larger elastic deformation of the rim part. It might be therefore reasonable to apply the finite element method (FEM)^{(1)~(3)} to such a stress analysis considering the deformation of the rim part.

In the present paper, the effects of constraint conditions for gear models and of adjacent teeth on the root stress distribution are investigated. The gear tooth model with a single tooth is shown to be available for a sufficiently precise evaluation of root stresses. The true stresses on the root fillet of gear tooth were computed by FEM and measured with gears of different rim thicknesses. The computed stresses are in a fairly good

agreement with measured ones. On the basis of these investigations the effects of rim thickness on the root stresses and on the position of the critical sections in thin rim gears are also discussed.

2. Stress computation by finite element method

2.1 Triangular element and displacement function

Figure 1 shows a typical triangular element m of a thin plate experiencing plane stresses. The element is referenced to an orthogonal coordinate system, and its nodes are marked by i, j, k in the counterclockwise direction. Let the coordinates of each node be (x_i, y_i) , (x_j, y_j) , (x_k, y_k) and the components of displacements and the nodal forces in x and y directions at the node i be u_i , v_i and X_i , Y_i respectively. The displacement distribution within the element must clearly be a function of x and y . Denoting the displacements in x and y directions at any

Fig.1 Nodal displacements and forces for triangular element

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point (x, y) within the triangular element by u and v respectively, the displacement functions can be written as Eq.(1).

$$\left. \begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \right\} \quad (1)$$

where α_1 through α_6 are constants yet to be determined.

2.2 Computer procedures for finite element analysis

Root stresses on the fillet of gear teeth were analysed as a two-dimensional elastic problem by means of the FEM with typical triangular elements. Element and node numbers, coordinates of each node and data of material properties are necessary as input data for the computation by FEM. The input data increase almost proportionately to the number of elements. Therefore it takes much time and labor to prepare the input data by hand and the accuracy of data thus obtained is generally lower. Hence the partition of elements for the regular part was carried out by the electronic computer and for the irregular part by hand in the present analysis. The rim part of a gear model was partitioned by the computer with a family of straight lines passing through the center of the gear and circular arcs with its center in the center of the gear.

The computer program consists of a main program and one for an automatic partition.

2.3 Number of elements and accuracy of computed results

The accuracy of computed results by FEM depends generally on the number, size and type of elements. Figure 2 shows, as an example, the relation between the deflection at the free end and the number of elements for a cantilever with rectangular cross section subjected at the free end to the load (2) . The theoretical values in this figure were obtained by the following Eq.(2) (4) .

Fig.2 Effect of number of elements on calculated values of deflection (2)

$$w = \frac{Pl^3}{3EI} \left\{ 1 + 0.71 \left(\frac{t}{l} \right)^2 - 0.1 \left(\frac{t}{l} \right)^3 \right\} \quad (2)$$

The error in the computed results by FEM is lower than 5% for the number of elements 400 as seen from Fig.2. Although it may be unreasonable to apply the results of Fig.2 directly to the stress analysis of a complicated-shaped tooth, they are considered to give a standard for the determination of the number of elements in the stress analysis. The number of elements in the present analysis was determined referring to Fig.2.

2.4 Partition of gear tooth model and constraint conditions

The finite element analysis was made on a gear model with a fan-shaped hole as shown in Fig.5 under the loading condition that P_n/b (load per unit face width) = 1 kg/mm was applied to the tip of the tooth. The main dimensions of the gear model are module $m = 4.5$, pressure angle $\alpha_0 = 20^\circ$, number of teeth $z = 72$. The analysis was made on gear models with a single tooth and with three teeth, the middle tooth of which was loaded.

Figure 3(a), (b) show mesh patterns for the loaded tooth (a) and its adjacent teeth (b) of the model of rim thickness $t_r = 2m$. The numbers of elements of each tooth (a), (b) are 350 and 301 respectively.

Figure 4 shows the mesh pattern for a half of the model and the total numbers of elements and nodes are 1032 and 574

Fig.3 Mesh pattern for gear teeth

Fig.4 Finite element model of gear with a fan-shaped hole

respectively. The angle between the both ends of the model about its center is 25° corresponding to five teeth (Fig.5). In the stress analysis by FEM, each node on the both ends of the model was constrained to have zero displacement as shown in Fig. 4. Furthermore, in order to examine the effects of the rim thickness l_w and boundary constraint conditions on the computed stresses, the stress analysis was made on single tooth models under two different constraint conditions shown in Fig.6(a), (b), varying the rim thickness in steps. The mesh pattern of the tooth is the same

Section AOA

Fig.5 Gear with a fan-shaped hole

Fig.6 Constraint conditions for gear tooth model

Table 1 Rim thickness and number of elements and nodes

Rim thickness l_w	Total no. of elements	Total no. of nodes
m	564	330
1.5m	604	351
2m	644	372
3m	724	414
4m	804	456
6m	964	540
8m	1124	624
10m	1284	708

as one shown in Fig.3(a) and the relationship between rim thicknesses and the numbers of elements and nodes is shown in Table 1.

3. Experimental procedure and test gear

3.1 Test gear

The root fillet stresses on the test gear tooth shown in Fig.5 were measured with a strain-gage method by carrying out a static bending test and compared with computed values by FEM. The main dimensions of the test gear are module $m = 4.5$, pressure angle $\alpha_0 = 20^\circ$, number of teeth $z = 72$ and face width $b = 30$ mm. The gear is made of S35C steel and hobbled after normalizing.

3.2 Experimental procedure

The root stresses were measured for various rim thicknesses by thinning the rim of the gear with a fan-shaped hole by an end mill. Figure 7 shows the pasted positions of strain gages. Strain gages 1, 2 were pasted on the root fillets on both the tensile and the compressive sides at the middle of the face width in tooth height direction. The applied load (per unit face width) was chosen $P_n/b = 6.67$ kg/mm within the elastic range so that the tooth might not yield. The static bending testing machine used consists of an oil

Fig.7 Positions of strain gages

- ① Pulsating pressure generator
- ② Pressure controller
- ③ Loading apparatus
- ④ Manual oil pump

Fig.8 Static bending testing machine

pressure generator (manual oil pump) ④ and a loading apparatus ③ as shown in Fig.8. The test gear is supported with a shaft and three teeth of an internal gear fixed to the bed of the testing machine. The oil supplied from the manual pump is introduced into a hydraulic cylinder and it pushes the piston and the tooth is thus loaded. The applied load value is given from the oil pressure detected with a pressure transducer attached to the upper part of the cylinder.

4. Computed and experimental results and discussions

4.1 Effect of adjacent teeth on root stresses

Figure 9 (a), (b) show the root stresses computed by FEM for one tooth and three teeth models of rim thickness $l_w = 2m$. In this figure, the values of the principal stresses which occur along the tooth profile are represented by the length of the straight lines normal to the tooth surface through the centroid of each surface element. The tensile and compressive stresses are indicated by the signs $\oplus$ and $\ominus$ respectively. The positions of maximum root stresses lie in the neighborhood of the critical section determined by Hofer's method for the both cases as shown in Fig.9.

The root stress values at the critical section of loaded tooth are $\sigma_t = 0.647 \text{ kg/mm}^2$ on tensile side and $\sigma_c = -1.045 \text{ kg/mm}^2$ on compressive side in one tooth model (a), and $\sigma_t = 0.642 \text{ kg/mm}^2$ and $\sigma_c = -1.047 \text{ kg/mm}^2$ in three teeth model (b). The differences between stress values in one tooth and three teeth models are only 0.7% on tensile side and 0.2% on compressive side. The measured results of root stresses on tensile and compressive sides are $\sigma_{t \text{ meas.}} = 0.667 \text{ kg/mm}^2$ and $\sigma_{c \text{ meas.}} = -1.058 \text{ kg/mm}^2$ respectively. Thus the root stresses computed by FEM are in a fairly good agreement with measured ones.

Fig.9 Stress distributions on root fillet

The root stresses in the case of rim thickness $l_w = m$ are also computed. In this case, the differences between stress values in one tooth and three teeth models are 3.8% on tensile side and 1.6% on compressive side. The effect of adjacent teeth on root stresses is negligibly small, though it becomes slightly larger with a decreasing rim thickness. Hence one tooth model is considered to be satisfactory for root stress analysis by FEM.

As is seen from Fig.9, the tensile stress occurs on the right adjacent tooth of the loaded tooth and the compressive stress on the left adjacent one. This phenomenon is observed in the case of $l_w = m$. This might be attributed to the fact that the amount of deformation of rim part in a thin rim gear is larger compared with the case of a solid gear. When such a thin rim gear is used as a driven gear, the tensile stress occurs at the root fillet of loaded side before the beginning of engagement and the compressive stress after the end of engagement. Thus stress conditions come to be partly reversed. Therefore the effects of these stress conditions must be taken into consideration in the bending strength design of thin rim gears.

Fig.10 Relation between rim thickness and root stress distributions

4.2 Effect of rim thickness on root stresses

4.2.1 Constraint at both ends of rim

Figure 10 shows the root stress distributions computed by FEM for one tooth models of different rim thicknesses.

Figure 11 shows the relation between the root stresses at the critical section determined by Hofer's method and the rim thickness. The abscissa denotes the ratio of rim thickness to module (l_w/m) and the ordinate the root stresses (compressive stress : absolute value). The root stresses measured for gears of different rim thicknesses and those computed for a solid gear by the formulas proposed by Aida and Terauchi⁽⁵⁾⁽⁶⁾, which are given by Eqs.(3)~(5) using the symbols as shown in Fig.12, are also indicated. These computation and measurement of root stresses are carried out under a normal tooth load $P_n/b = 6.67$ kg/mm.

$$\sigma_t = (1 + 0.08 \frac{s}{p})(0.66A_b + 0.40 \sqrt{A_b^2 + 36A_{\tau}^2} + 1.15A_c) \frac{P_n}{b} \quad (3)$$

$$\sigma_c = (1 + 0.08 \frac{s}{p})(-0.66A_b - 0.40 \sqrt{A_b^2 + 36A_{\tau}^2} + 1.40A_{c1} + A_{b2}) \frac{P_n}{b} \quad (4)$$

where

$$\left. \begin{aligned} A_b &= \frac{6l \sin \theta}{s^2}, & A_c &= -(1 + \frac{6\gamma}{s}) \frac{\cos \theta}{s} \\ A_{\tau} &= \frac{\sin \theta}{s}, & A_{c1} &= -\frac{\cos \theta}{s} \\ A_{b2} &= \frac{6\gamma \cos \theta}{s^2} \end{aligned} \right\} \quad (5)$$

It is seen from Fig.11 that the root stresses computed by FEM are in a fairly good agreement with the measured ones on both tensile and compressive sides, and those values are almost constant in the range $l_w \geq 8m$, being about the same as those of a solid gear. The root stresses on compressive side show a tendency to increase with a decreasing rim thickness

Fig.11 Relation between rim thickness and root stresses

and those on tensile side have an extreme value at about $l_w = 1.5m$. The reasons may be explained as follows.

Figure 13 shows each load component which acts on a tooth. The normal tooth load P_n is resolved into components P_1 and P_2 normal and parallel to the center line of the tooth respectively. The root stress due to P_n is considered to have a composition of bending stress σ_{b1} (b) due to P_1 , compressive stress σ_{cr} (c) to P_2 and bending stress σ_{b2} (d) to P_2 . Figure 14 shows each root stress component computed by FEM with l_w/m on the abscissa.

Figure 14(a) shows each root stress component on tensile side. It is seen from this figure that the tensile bending stress σ_{b1} and the compressive stress σ_{cr} increase sharply with a decreasing rim thickness. This might be due to the larger deformation of the rim part with a decreasing rim thickness. The $\sigma_b (= \sigma_{b1} + \sigma_{b2})$ increases with a decreasing rim thickness, since the compressive bending stress σ_{b2} varies slightly. The variation of the true root stress $\sigma_t (= \sigma_b + \sigma_{cr})$ on tensile side is dominated by those of σ_b and σ_{cr} . The true stress σ_t on the tensile side takes an extreme value at $l_w = 1.5m$, because the increasing rates of σ_{cr} and σ_b differ from each other. On the compressive side the compressive stress $\sigma_c (= \sigma_b + \sigma_{cr})$ shows a tendency to increase

Fig.12 Dimensions for calculation of root stresses

Fig.13 Load components on tooth

monotonously, since both the compressive stresses σ_b and σ_{cr} increase with a decreasing rim thickness as shown in Fig.14(b).

4.2.2 Constraints at ends and inside of rim

Figure 15 shows the root stresses σ_t on tensile side and σ_c (absolute values) on compressive side at the critical section determined by Hofer's method, which were computed by FEM for the case where the nodal displacements are constrained at ends and inside of the rim (solid lines) and only at ends (broken lines). It is seen from Fig.15 that both the tensile and the compressive root stresses (absolute values) decrease with a decreasing rim thickness for the case where the nodal displacements are constrained at ends and inside of the rim. This might be because the deformation of rim part becomes smaller with a decreasing rim thickness due to the constraint at the inside of the rim. Root stresses on tensile and compressive sides are constant in the range $l_w \geq 10m$, being almost equal to those for the case where the nodal displacements are constrained only at ends of the rim. Therefore it is sufficient for the computation of root stresses of a solid gear by FEM to take an arc length corresponding to five teeth as a rim length and $10m$ as a rim thickness. In this case, computed results are considered to be the same for each boundary constraint condition in Fig.6(a) and (b).

In order to investigate the effects of the rigidity of gear body on root stresses, the root stresses on tensile side were measured for gear G.O with a fan-shaped hole and webbed gear G.N of different rim thicknesses and the results are shown in Fig.16. The root stresses of G.O show a tendency to increase and those of G.N to decrease with an increasing rim thickness (with an increasing rigidity of gear body). This might be because the effect of compressive stress σ_{cr} due to P_2 (Fig.14) in G.N is smaller than in G.O, since G.N has a web. Figure 17 shows the measured results of root stresses for gears with various web structures. The main dimensions of gears are the same as those shown in the caption of Fig.16 and the rim thickness l_w of G.O and G.N are $2m$ (m : module). It is seen from this figure that the root stresses are different according to the types of web structure.

4.3 Effect of rim thickness on position of maximum root stress

Tangential angle θ (deg) is defined

Fig.14 Root stresses due to each load component ($P_N/b = 1 \text{ kg/mm}$)

as the angle between the center line of the tooth and the tangent to the fillet curve as shown in Fig.18(a). The position of each element on fillet surface is represented by the tangential angle θ .

Figure 18 shows the relations between the root stresses and the tangential angle θ . In the range $l_w \geq 1.5m$, the maximum root stresses on tensile side occur at the position $\theta = 30^\circ$, which is considered generally to be critical in the root fillet

Fig.15 Relation between rim thickness and root stresses ($P_N/b = 1 \text{ kg/mm}$)

of a solid gear. But when the rim thickness decreases still further to $l_w = m$, the position of maximum root stress fairly moves from the position $\theta = 30^\circ$, toward the tooth bottom ($\theta = 55^\circ$) and the maximum stress value becomes larger. This might be because the bending stress σ_b becomes larger than the compressive one σ_{cr} , as the rim thickness decreases to $l_w = m$ as mentioned in the foregoing section and because the position of maximum stress in σ_b moves toward the tooth bottom.

The absolute values of root stresses on compressive side increase with a decreasing rim thickness. The position of maximum root stress is nearly at $\theta = 30^\circ$ for $l_w \geq 6m$, and moves toward the tooth bottom, yielding $\theta = 55^\circ$, when the rim thickness decreases still further to $l_w = 2m$.

5. Conclusions

The main results obtained from this investigation are summarized as follows.

(1) One tooth model is satisfactory for root stress analysis by finite element method because the effects of adjacent teeth on root stresses are negligibly small.

(2) It might be sufficient for root stress analysis of a solid gear by finite element method to take arc length corresponding to five teeth as a rim length and $l_w = 10m$ as a rim thickness.

(3) For thin rim gears (driven), the tensile stress occurs at the root fillet of loaded side of the gear tooth before the beginning of engagement and the compressive stress after the end of engagement, due to the effects of elastic

$m = 4.5$, $\alpha_0 = 20^\circ$, $z = 72$, $b = 30$ mm,
 b_w : web thickness mm, l_w : rim thickness mm,
 $P = 200$ kg

Fig. 16 Effect of rim thickness on root stresses of gear with a fan-shaped hole and of webbed gear

Fig. 17 Measured root stresses of gears with various structures

deformation of rim part. Hence the stress conditions in the root fillet of thin rim gears under normal running conditions come to be partly reversed.

(4) The root stresses are different according to the gear structures. For a gear with a fan-shaped hole, the root stress (absolute value) on compressive side increases with a decreasing rim thickness and that on tensile side has an extreme value at about $l_w = 1.5m$.

(5) The position of maximum root stress in a thin rim gear shows a tendency to move toward the tooth bottom with a decreasing rim thickness.

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(a)

(b)

Fig. 18 Relation between root stresses and tangential angle

investigation.

The numerical computations in this investigation were carried out by using FACOM M-190 system of Data Processing Center of Kyoto University.

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