Influence of Compressibility of Oil on the Step Response of Hydraulic Servomechanism

1st Report

Basic Equations and Responses under Simple Loads

By Eizo URATA** and Toshikumu ROYAMA***

A step response of hydraulic servomechanism under influence of compressibility of oil is studied. Firstly basic equations for a closed loop system under a general loading condition are studied. Secondly time dependent variation of pressure in the system and displacement with no-load condition are studied. An exact solution in an analytical form is given for this case. The compressibility of oil considerably influences pressures in the system, whereas its influence on the displacement response is small.

The response under constant loading force is similar to the response without loading. When a few parameters are transformed suitably, the response with constant load or the Coulomb friction coincides with the response of the no-load system.

1. Introduction

Analysis in early studies on the dynamics of the hydraulic servomechanism were performed with linearized approximations to nonlinear characters of valve and loads. Exact analyses without linear approximation were given by Turnbull(1) and by Urata(2) for systems in which the compressibility of oil was negligible.

Effect of the compressibility of oil on the dynamic characteristics of the hydraulic servomechanism has been studied, by a few researchers, utilizing linearized analysis(3), describing functions(4), and direct numerical solutions by electronic computers(5), (6).

Any generally applicable analysis or result on this subject has, however, not yet been given. The purpose of this paper is firstly to determine the dynamic characteristics of the hydraulic servomechanism under loads and other conditions considering the influence of the compressibility of oil and secondly to express the results in a unified form.

Characters of the step response under various loads are inspected and examples of numerical solutions by computer are shown in this study. Solutions thus obtained can be compared with solutions already given by linear approximation and/or on in-compressibility assumption. Thus we can define the situations in which the approximation and the assumption are justified.

The hydraulic servomechanism treated in this paper is shown schematically in Fig. 1. A four way pilot valve is fixed on a servocylinder. Hence the displacement of the cylinder automatically forms a unity feedback corresponding to an input on valve spool. A hydraulic servomechanism having another configuration may be analyzed as shown in this paper since the system shown in Fig.1 expresses the fundamental construction.

Nomenclature

$A_1,A_2,A$ : Effective area of piston
$C_1,C_2$ : Cavity volumes formed in cylinder chambers

* Received 14th October, 1979.
** Associate Professor, College of Engineering, Kanagawa University, Rokkakubashi 3-27, Kanagawa-ku, Yokohama, 221 Japan.
*** Kojimachi Post Office, Kudan 4-5-9, Chiyoda-ku, Tokyo, 104 Japan.
Ideal conditions are assumed that the pilot valve allows no internal leakage flow and thatappings between spool land and valve body are zero at all control edges. Relative displacement of valve body and spool \( z = s - y \) is positive when left side of central port opens. Inversely, \( x \) is negative when right side of the central port opens. Valve opening is given by \( |x| \) in both instances. Loads and driving force are positive for rightward direction. Fluids in cylinder chambers and valve chambers are agitated normally by some external random vibration. Hence the fluid pressure in the system can not drop below a certain value. The lowest limit of the pressure should be constant during the small time interval in which the servomechanism is operated. Under compulsory displacement of the cylinder a cavity should be formed in the cylinder chamber, while the pressure itself remains at the constant value \( p_s \). The value \( p_a \) with oil is higher than the vapour pressure and is defined as a pressure at which a vigorous release of dissolved air and other gases occurs. Any visible error will not be induced by the assumption that \( p_a \) is approximately 1 bar because the supply pressure of the hydraulic servomechanism is sufficiently high. Relations of variables will be found under heretofore mentioned conditions.

The equations of continuity in the cylinder chambers are given by:

\[
\begin{align*}
V_1 \frac{dp_1}{dt} & = Q_1 - A_1 \frac{dy}{dt} - c_1 (p_a - p_l) - E_1 p_1; \\
V_2 \frac{dp_2}{dt} & = Q_2 + A_2 \frac{dy}{dt} + c_2 (p_a - p_l) - E_2 p_2; \\
\frac{dc_1}{dt} & = - \left[ Q_1 - A_1 \frac{dy}{dt} - c_1 (p_a - p_l) - E_1 p_1 \right]; \\
\frac{dc_2}{dt} & = - \left[ Q_2 + A_2 \frac{dy}{dt} + c_2 (p_a - p_l) - E_2 p_2 \right];
\end{align*}
\]

Equations (1) and (2) have been used usually to express continuity without cavity. Although the density of oil changes with pressure, it can be assumed constant because the change in the range of supply pressures is small. Equations (3) and (4) express continuity relations when cavities appear. It should be noted that the character of seal may induce an external leakage coefficient different from values in Eqs.(1) and (2) because the air may come into the cylinder from outside. Therefore new leakage coefficients are used in Eqs.(3) and (4). However the leakage flows are extremely small in practical systems, hence the external leakage coefficients can be assumed zero.

In accordance with prescribed rules for symbols, the flows through the valve are:

\[
\begin{align*}
& x \geq 0; \quad Q_1 = c_{wz} \operatorname{sgn} \left( p_a - p_l \right) \sqrt{2} \left| p_a - p_l \right| / \rho, \\
& x < 0; \quad Q_1 = c_{wz} \operatorname{sgn} \left( p_a - p_s \right) \sqrt{2} \left| p_a - p_s \right| / \rho.
\end{align*}
\]

Fluid flows inversely when \( p_a > p_s \) or \( p_a > p_s > p_l \) holds. This state appears when the cylinder is moved by external force or

\[
\begin{align*}
F &= M \frac{dy}{dt^2} + K \frac{dy}{dt} + \left( c_f + k_y + \operatorname{sgn} \left( \frac{dy}{dt} \right) \right) F_r + F_s + f(t); \\
F &= A_1 p_1 - A_2 p_2 = A (p_a - p_l) \quad \text{(when } A_i = A_s = d) \quad \text{(9)}
\end{align*}
\]

\[
\text{inertial force.}
\]

The equation of motion is written as follows:

\[
\text{Average forces acting on the piston are equal.}
\]
where \( F \) is the driving force, \( M \) is the mass, \( r \) is the coefficient of the resistance proportional to square of velocity, \( \alpha \) is the coefficient of viscous resistance, and \( k \) is the spring constant. The 5th term of right-hand side of Eq. (9) expresses the Coulomb friction, \( F_0 \) is a constant resistance, and \( f(x) \) is an external force. The volume of oil in cylinder chambers varies with relative displacement of the piston. It means that parameters of the hydraulic servomechanism take different values depending on its starting position. Influence of volume change of cylinder chamber shall be taken into account when the compressibility of oil is not negligible.

\[
P_i = p_i / \rho_i \quad (i = 1, 2, \text{or } v) \quad X = x - x_0, \quad Y = y - y_0,
\]

\[
\lambda = \frac{A_{z0}}{V_i + V_2 / \rho_i}, \quad c_i = \frac{p_i c_i}{(A_{z0} / T)}, \quad e_i = \frac{p_i E_i}{(A_{z0} / T)}
\]

\[
v_i = \frac{V_i}{V_1 + V_2} \quad (i = 1, 2), \quad c_i = \frac{C_i}{A_{z0}} \quad (i = 1, 2)
\]

\[
a_0 = F_0 / A_{P0}, \quad a_1 = F_1 / A_{z0}, \quad a_2 = c_2 T / A_{P0}, \quad a_3 = \frac{r(z_0 / T)^3}{A_{P0}}
\]

In these relations \( a_0, T \) and \( A_p \) are taken as units for displacement, time and pressure respectively. The physical meanings of each nondimensional parameter and variable are rather obvious, and only an explanation for \( \lambda \) will be given here. The parameter \( \lambda \) is a measure of compressibility in the following sense. If a volume when \( X \geq 0 \)

\[
q_1 = \sqrt{2 (1 - Y) \text{sgn} (1 - P_1) \sqrt{1 - P_1}} \quad (15)
\]

\[
q_2 = \sqrt{2 (1 - Y) \text{sgn} (P_2 - P_0) \sqrt{P_2 - P_0}} \quad (16)
\]

when \( X < 0 \)

\[
q_1 = \sqrt{2 (1 - Y) \text{sgn} (P_1 - P_0) \sqrt{P_1 - P_0}} \quad (17)
\]

\[
q_2 = \sqrt{2 (1 - Y) \text{sgn} (1 - P_2) \sqrt{1 - P_2}} \quad (18)
\]

Substitution of Eqs. (13) and (15) into Eqs. (1)-(4) gives the following four equations:

\[
\frac{dP_1}{dt} = \frac{2}{v_1} \left[ q_1 A_1 \frac{dY}{dt} - c_1 (P_1 - P_2) - e_1 P_1 \right] \quad (c_1 \equiv 0, P_1 > P_0) \quad (19)
\]

\[
\frac{dP_2}{dt} = \frac{2}{v_1} \left[ q_2 A_2 \frac{dY}{dt} + c_2 (P_1 - P_2) - e_2 P_2 \right] \quad (c_2 \equiv 0, P_2 > P_0) \quad (20)
\]

\[
\frac{dc_1}{dt} = -q_1 A_1 \frac{dY}{dt} + c_1 (P_1 - P_2) + e_1 P_1 \quad (P_1 \equiv P_0, c_1 \geq 0) \quad (21)
\]

\[
\frac{dc_2}{dt} = -q_2 A_2 \frac{dY}{dt} - c_2 (P_1 - P_2) + e_2 P_2 \quad (P_2 \equiv P_0, c_2 \geq 0) \quad (22)
\]

The equation of motion becomes

\[
a_{0} \frac{d^2 Y}{dt^2} + a_{1} \frac{dY}{dt} + a_{2} \frac{dY}{dt} + a_{3} \text{sgn} \left( \frac{dY}{dt} \right) + a_{4} = \frac{A_1}{A} P_1 - \frac{A_2}{A} P_2 \quad (23)
\]

In the following analysis we assume normally a symmetric cylinder where

\[
A_1 = A_2 = A \quad \text{holds.} \quad \text{Input signal } u \text{ is assumed sufficiently small so that}
\]

\[
z_0 < V_1 / A \quad \text{holds, and the change of the volume of cylinder chambers is negligible. This assumption would not be justified if } P_0 \text{ were lower than zero. When } P_0 \text{ is not zero, all pressures should be transformed to } P_1 - P_0 \text{ and } P_2 - P_0, \text{ then the new variables are substituted instead of } P_1 \text{ or } P_2. \text{ Hence we can assume } P_0 = 0 \text{ without loss of generality.}
\]
Initial conditions are:
\[ Y = 0, \frac{dY}{d\tau} = 0; \text{ at } \tau = 0 \]  
...(26)
Initial values of \( p_1 \) and \( p_2 \) are free under restrictions that Eq. (26), the equation of motion and the equation of continuity are able to be satisfied.

3. Responses under Simple Loads

for \( 1 - Y \geq 0 \)
\[
\frac{dP}{d\tau} = \sqrt{2} (1 - Y) (\sqrt{1 - P} - \sqrt{P}) - (e_1 + e_2) P \lambda 
\]
...(27)
\[
\frac{dY}{d\tau} = \frac{\sqrt{2} (1 - Y) (e_2 \sqrt{1 - P} + e_1 \sqrt{P}) + (e_1 e_2 - e_2 e_1) P}{\sqrt{1 - P} - \sqrt{P}}
\]  
...(28)

for \( 1 - Y < 0 \)
\[
\frac{dP}{d\tau} = \sqrt{2} (1 - Y) (\sqrt{P} - \sqrt{1 - P}) - (e_1 + e_2) P \lambda 
\]
...(27a)
\[
\frac{dY}{d\tau} = \frac{\sqrt{2} (1 - Y) (e_2 \sqrt{1 - P} + e_1 \sqrt{P}) + (e_1 e_2 - e_2 e_1) P}{\sqrt{1 - P} - \sqrt{P}}
\]  
...(27b)

The condition of equilibrium is given by
\[ Y = 1 \text{ and } P = 0. \]
An exception appears when \( e_1 = e_2 = 0 \) holds. In the instance the equilibrium condition for \( Y \) remains at unity but the condition for \( P \) depends on the initial condition. When \( Y \) is not greater than unity, we have
\[
\frac{dP}{d\tau} = \sqrt{2} (1 - Y) (\sqrt{P} - \sqrt{1 - P}) - (e_1 + e_2) P \lambda 
\]
...(29)
An integral of the above equation is
\[
\frac{1}{2} \log |\sqrt{1 - P} - \sqrt{P}| = \frac{1}{2} \log |\sqrt{1 - P} - \sqrt{P}| 
\]  
...(30)

This result is the same as the response with incompressible fluid. When \( P = P_0 \) holds through all time interval of transient response, the influence of the compressibility disappears. The response in this case agrees with the incompressible fluid. The phase plane trajectories are drawn in Fig. 2 using Eq. (29) or (30). All points on the line \( Y = 1 \) are equilibrium points.

The trajectories are symmetric with two lines \( Y = 1 \) and \( P = 1/2 \) in a symmetric system of \( \nu / \nu_0 = 1 \). A point symmetry according to point \( (Y = 1, P = 1/2) \) results when \( \nu / \nu_0 \neq 1 \) holds.

We prove next how the equilibrium point is given by \( Y = 1 \) and \( P = 0 \) when external leakage flow exists. The equilibrium point under assumption \( Y = 0 \) gives

\[
\frac{e_1 + e_2}{\sqrt{1 - P} - \sqrt{P}} = \frac{(e_1 e_2 - e_2 e_1)}{\nu_1 \sqrt{1 - P} + \nu_2 \sqrt{P}}
\]  
...(32)

Fig. 2 Phase plane trajectory of no-load system \((\phi_1 = \phi_2 = 0)\)

Fig. 3 Pressure response of no-load system \((\phi_1 = \phi_2 = 0)\)
This equation will be solved for $P$. A real
solution of $P$ requires that the following
inequality hold.

\[
J = \left\{ \frac{1}{(\varepsilon_1 + \varepsilon_2)\rho_1} \left| 1 + \frac{(\varepsilon_1 + \varepsilon_2)\rho_1}{\varepsilon_1 \varepsilon_2 - \varepsilon_1 \rho_1} \right| \right\}
\]

\[= \frac{-e_1 e_2}{(e_1 e_2 - e_1 \rho_1)} > 0 \] \(\cdots(33)\)

No real solution $P$ exists because the in-
equality can not hold physically. The only
possible solution is $P = 0$ when $Y_1 = 1$ is
assumed. For equilibrium point of $Y_1 = 1$ at $P = 0$, we have the following equation from a
set of perturbation equations around the
point.

\[
\frac{dP}{d(Y)} = -\frac{\lambda}{\sqrt{2}} \frac{dY}{dP} + (\varepsilon_1 + \varepsilon_2)P \] \(\cdots(34)\)

The point is one of elementarily singular
points. The characters of the equilibrium
point are investigated using the routine method.
This equilibrium point is a stable node except when

\[\varepsilon_1 \varepsilon_2 - e_1 \rho_1 > 0 \] \(\cdots(35)\)

and

\[\sqrt{2} (1 - \sqrt{\varepsilon_1})^2 < \lambda \varepsilon_1 < (1 + \sqrt{\varepsilon_1})^2 \sqrt{2} \] \(\cdots(36)\)

hold; stable spiral is obtained in this
case. Since $\varepsilon_1$ and $\varepsilon_2$ are very small
quantities in practical systems, solutions
assuming $\varepsilon_1 = \varepsilon_2 = 0$ will introduce no serious
error in short duration, for example $\tau < 5$.

The change of $P$ approaching zero occurs
thereafter far slowly.

The value of $P$ at $\tau = 0$, namely $P_0$, can take
an arbitrary value for $0 \leq P \leq 1$.

Figures 3 and 4 show numerical results of $P$ and $Y$ for $\tau$ with $E = 0.9$ and with $\lambda = 0.1$,
0.316, 1, 5.16 and 10. Variation of $Y$
shown in Fig. 4 reveals that the influence of $\lambda$
is moderate. Difference between $\lambda = 10$ and $\lambda = \infty$ is invisible. Variation of $P$ with change of $\lambda$ is rather severe: $P \neq 0$ at $\tau = 0.2$ under $\lambda = 10$; $P < 0.5$ when $\tau = 10$ under $\lambda = 0.1$ and the variation is very slow.

Usually only output $Y$ is deemed as the
response of a servomechanism. Variation
of $Y$ from the incompressible system is
small even if $\lambda = 0.1$, though $P$ changes
seriously with $\lambda$. This fact is a reason
why the incompressible theory has in good
agreement with experiments. If $P$ has been
sought or a pressure control had been mainly
the aim, the incompressible theory would
have been unsatisfactory. Figure 5 shows
influence of $P_0$ on $P$ with $\lambda = 0.1$. Output
$Y$ is not shown because the influence is so
small. Difference of final value of $P$ depending
on $P_0$ is here clearly shown. Since $e_1 e_2 = 0$ is assumed in this example, $\tau = \infty$
does not result in $P = 0$. Influence of
leakage is so small that it can not appear
in time interval of several multiples of
the time constant $T$. Hence curves in Fig.
5 indicating a change in $\tau < 10$ do not show
any visible influence of leakage.

3.2. Constant Load and the Coulomb Friction.
A constant load means

\[A(p_1 - p_2) = F_0 \] \(\cdots(37)\)

An example of load in this category is a
hung weight with a negligibly small accer-
leration. Though initial condition of
pressure is different depending on whether
the load is added to the system in $\tau < 0$ or in
$\tau > 0$, the response of displacement is scar-
cely affected by this difference.

The results of this section can be
utilized when the Coulomb friction is ex-
pressed approximately in the following sim-
ple form:

\[A(p_1 - p_2) = \text{sgn} (dy/dt) F_0 \] \(\cdots(38)\)

Practically no overshoot in response of the
present system has to be considered unlike
in the case treated in the previous section.
Therefore the sign function in the first of
Eq. (38) is not necessary. Since

\[P_1 - P_2 = a_1 + e_1 \]

we can assume $a_0 = 0$ without loss of genera-
ity. Since

\[
\frac{dP_1}{d\tau} = \frac{dP_2}{d\tau} \]

holds, we have the following relations from
Eqs. (19) and (20).

\[\frac{dP_1}{d\tau} = \sqrt{2 (1 - Y)} \lambda (\sqrt{1 - P_1} - \sqrt{P_1 - a_1}) \]

\[-\lambda (e_1 + e_2) P_1 + \lambda e_1 \] \(\cdots(39)\)

\[\frac{dY}{d\tau} = \sqrt{2 (1 - Y)} (e_1 \sqrt{1 - P_1} + e_1 \sqrt{P_1 - a_1}) \]

\[-(e_1 e_2 - e_1 \rho_1) P_1 - a_1 (c_1 + e_1 e_2) \] \(\cdots(40)\)
The equilibrium point is obtained by putting zeros at the lefthand sides of Eqs. (39) and (40). This calculation is performed usually by numerical method. It should be noted that $Y$ is not unity at $\tau = 0$ if a leakage flow exists. A permanent error is observed in this case. The equilibrium state is reached at valve opening just to compensate for the leakage and sustain the load. Values at $\tau = 0$ bear suffix $m$.

(a) Without external leakage ($e_i = 0$)

\[ Y_m - 1 = \gamma + \gamma + \tau = -a_i c_i / \sqrt{1-a_i} \]

\[ P_m = (1-a_i)/2, \quad P_m = (1-a_i)/2 \]  \hspace{1cm} (41)

(b) Without internal leakage ($c_i = 0$)

when $a_i > 0$, we have

\[ Y_m - 1 = a_i c_i / \sqrt{1-a_i}, \quad P_m = a_i \]

\[ P_m = 0 \]  \hspace{1cm} (42)

when $a_i < 0$, we have

\[ Y_m - 1 = a_i c_i / \sqrt{1-a_i}, \quad P_m = 0, \]

\[ P_m = -a_i \]  \hspace{1cm} (43)

We would have $Y = 1$ if all the leakages were zero. Final pressures $P_m$ and $P_m$ depend on initial values as observed in the no-load system. Ideal response is obtained when leakage does not exist. In this instance, the following transformations

\[ P^* = (P_i - a_i)/(1-a_i), \quad \tau^* = \tau / \sqrt{1-a_i} \]

\[ \lambda^* = \lambda / (1-a_i) \]  \hspace{1cm} (44)

give

\[ \frac{dP^*}{d\tau^*} = -\sqrt{2} (1-Y) \lambda^* (\sqrt{1-P^2} - \sqrt{P^*}) \]

\[ \frac{dY}{d\tau^*} = -\sqrt{2} (1-Y) (c_i \sqrt{1-P^2} + c_i \sqrt{P^*}) \]

\[ \lambda^* = \lambda / (1-a_i) \]  \hspace{1cm} (45)

Thus the system of equations is reduced to the same one for the no-load system. Observing the transformation for $\tau$, it is seen that the response of the new system is delayed by $1/\sqrt{1-a_i}$, times from response of the no-load system. These characters are common to the incompressible system.

As has been shown, the response under constant or Coulomb friction loads are analogous with the no-load response in many phases. But there is a serious difference that a permanent error occurs with a loaded system.

4. Conclusions

This paper presents firstly a system of basic equations to analyze the influence of the compressibility of oil on the hydraulic servomechanism. Secondly the characteristics of a no-load system providing the fundamentals of the system dynamics are given. Thirdly characters of a loaded system are analytically reduced to those of a no-load system, and systems analogous with the no-load system are studied.

Influence of the compressibility on displacement response under these loads is normally small. It is concluded that calculated response assuming incompressibility can give good approximation in many instances. The responses under more complex loads will be treated in the following parts of this study.

References


