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TRANSIENT RESPONSE OF AN INHOMOGENEOUS ELASTIC HALF SPACE TO A TORSIONAL LOAD*

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Transient response of an inhomogeneous elastic half space to an impulsive torsional load is discussed. Variations of rigidity and density are assumed as the functions of depth, $\mu/\mu_0 = \rho/\rho_0 = (1 + \epsilon z)^\nu$. Exact solutions are obtained for two cases, $\nu = \pm 2$. It is found that there exists a disturbance in a region, where no disturbance is observed in the case of a homogeneous solid.

Numerical computations are carried out in detail and they show that the inhomogeneous effect on the response is more remarkable when material parameters increase with depth than when those decrease.

1. INTRODUCTION

There have been few studies concerning the dynamic response of an inhomogeneous elastic solid in contrast to those of a homogeneous elastic solid. In these circumstances we can cite the work of Karal & Keller[1] as a typical ray expansion technique for the inhomogeneous solid. The technique, however, has less advantage for solving the boundary-value problem of the inhomogeneous solid. On the other hand, several problems are solved with use of the integral transform technique. Payton[2] and Eason[3, 4] have considered the transient response of an inhomogeneous elastic bar and have obtained exact solutions for several types of inhomogeneity. Moodie et al.[5-8] have, also, considered the response of an inhomogeneous solid having a cylindrical/spherical cavity. These studies, however, concern only the propagation of a one-dimensional wave, and these solutions can be obtained numerically by the method of characteristics[9] (Recently, Moodie et al.[10] have compared the advantage of the asymptotic inversion of Laplace transform with that of the characteristics).

There are also few studies concerning the multi-dimensional wave propagation in an inhomogeneous elastic solid. Works of Hook[11] and Rao & Goda[12] can be cited as studies related to this type of the problem. Hook has determined a Green function of a scalar potential for a problem of a point source in an infinite solid. Rao & Goda have obtained the surface response of a half space to an impulsive point load by using the numerical technique of Fast Fourier Transform (FFT). These, however, are not so satisfactory that it can not fully explain the effect of the inhomogeneity on the response and on the deformation. Thus, sufficient analytical investigations concerning the multi-dimensional wave propagation in an inhomogeneous elastic solid

are expected.

So, the author[13] has considered the transient response of an inhomogeneous elastic half space to an impulsive anti-plane shear load and has obtained exact solutions for two cases of the inhomogeneity. In this paper the transient response of an inhomogeneous elastic half space to an impulsive torsional load is considered. By assuming that rigidity μ and density ρ vary as functions of depth z ,

$$\mu/\mu_0 = \rho/\rho_0 = (1 + \epsilon z)^\nu$$

the response is analyzed for a case of constant speed of SH-wave. Integral transform technique, used by Eason[14 & 15] for the same problem of the homogeneous solid, is applied and exact solutions are obtained in the form of single integration. Numerical computations are carried out in detail and the effect of the inhomogeneity on the response and on the deformation pattern is discussed.

2. STATEMENT OF PROBLEM

Consider a cylindrical coordinate system (r, θ, z) in an inhomogeneous elastic half space as depicted in Fig. 1. The surface of the space is $z=0$ and $z>0$ is forming its interior. It is assumed that rigidity μ and density ρ vary only with depth z ,

$$\mu \equiv \mu(z), \quad \rho \equiv \rho(z) \dots \dots \dots (1)$$

and that an impulsive torsional load is applied on the surface $z=0$,

$$\sigma_{z\theta} \equiv p(r, t); \quad z=0 \dots \dots \dots (2)$$

It is also assumed that the quiescent condition at initial time $t \leq 0$,

$$u_\theta = \frac{\partial u_\theta}{\partial t} = 0; \quad t \leq 0 \dots \dots \dots (3)$$

Governing equations are given by

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{1}{r^2} u_\theta + \frac{\mu'}{\mu} \frac{\partial u_\theta}{\partial z} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u_\theta}{\partial t^2} \dots \dots \dots (4)$$

$$\sigma_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \quad \sigma_{z\theta} = \mu \frac{\partial u_\theta}{\partial z} \dots \dots \dots (5)$$

where

$$\mu' = d\mu/dz \dots \dots \dots (6)$$

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and c is speed of SH-wave given by

$$c = (\mu/\rho)^{1/2} \quad (7)$$

3. INTEGRAL TRANSFORM

The application of Laplace transform defined by

$$(u_\theta^*, \sigma_{r\theta}^*, \sigma_{z\theta}^*) \equiv \int_0^\infty (u_\theta, \sigma_{r\theta}, \sigma_{z\theta}) e^{-st} dt \quad (8)$$

and of Hankel transform defined by

$$(\bar{u}_\theta^*, \bar{\sigma}_{r\theta}^*) \equiv \int_0^\infty r (u_\theta^*, \sigma_{r\theta}^*) J_1(\xi r) dr \quad (9)$$

$$\bar{\sigma}_{r\theta}^* \equiv \int_0^\infty r \sigma_{r\theta}^* J_2(\xi r) dr \quad (10)$$

converts Eqs. (2), (4) and (5) to

$$\bar{\sigma}_{z\theta}^* = \bar{p}^*(\xi, s); \quad z=0 \quad (11)$$

$$\bar{\sigma}_{r\theta}^* = -\mu \xi \bar{u}_\theta^*, \quad \bar{\sigma}_{z\theta}^* = \mu \frac{d\bar{u}_\theta^*}{dz} \quad (12)$$

$$\frac{d^2 \bar{u}_\theta^*}{dz^2} - \beta^2 \bar{u}_\theta^* + \frac{\mu'}{\mu} \frac{d\bar{u}_\theta^*}{dz} = 0 \quad (13)$$

where

$$\beta = \{\xi^2 + (s/c)^2\}^{1/2} \quad (14)$$

To solve the differential Eq. (13), variations of rigidity and density must be specified. In this paper their variations are assumed as

$$\mu/\mu_0 = \rho/\rho_0 = (1+\varepsilon z)^\nu \quad (15)$$

where μ_0 and ρ_0 are the rigidity and density on the surface, respectively, and ε and ν are parameters of the inhomogeneity. It should be noted that Eq. (15) warrants the constant speed of SH-wave throughout the half space.

Substitution of Eq. (15) into Eq. (13) and a variable transform

$$y = 1 + \varepsilon z \quad (16)$$

yield

$$\frac{d^2 \bar{u}_\theta^*}{dy^2} + \frac{\nu}{y} \frac{d\bar{u}_\theta^*}{dy} - (\beta/\varepsilon)^2 \bar{u}_\theta^* = 0 \quad (17)$$

A general solution of Eq. (17) can be expressed by the modified Bessel functions [16, p. 97] and thus given by

$$\bar{u}_\theta^* = y^{\lambda_1} \{C_1 I_{\lambda_1}(\beta y/\varepsilon) + C_2 K_{\lambda_1}(\beta y/\varepsilon)\} \quad (18)$$

where C_1 and C_2 are unknown functions and

$$\lambda_1 = (1-\nu)/2 \quad (19)$$

Substituting Eq. (18) into Eq. (12), transformed stresses are

$$\bar{\sigma}_{r\theta}^* = -\mu_0 \xi y^{\lambda_1} \{C_1 I_{\lambda_1}(\beta y/\varepsilon) + C_2 K_{\lambda_1}(\beta y/\varepsilon)\} \quad (20)$$

$$\bar{\sigma}_{z\theta}^* = \mu_0 \beta y^{\lambda_2} \{C_1 I_{-\lambda_2}(\beta y/\varepsilon) - C_2 K_{-\lambda_2}(\beta y/\varepsilon)\} \quad (21)$$

$$\lambda_2 = (1+\nu)/2 \quad (22)$$

Thus, the transformed general solutions are given by Eqs. (18, 20 and 21).

Eason [14] and Mitra [17] have already obtained an exact solution of a homogeneous elastic half space with an application of an impulsive torsional load,

$$p(r, t) = p_0 r H(a-r) \delta(t) \quad (23)$$

where p_0 is magnitude of the load, and $\delta(\cdot)$ $H(\cdot)$ are Dirac's delta function and Heaviside's unit step function, respectively. To clarify the effect of the inhomogeneity on the response, we, also, employ the same boundary condition of Eq. (23). Then, Eq. (11)

reduces

$$\bar{\sigma}_{z\theta}^* = \frac{p_0 a^2}{\xi} J_2(\xi a); \quad z=0 \quad (24)$$

4. INVERSION

The unknown functions, C_1 and C_2 , are determined by a condition at infinity,

$$\bar{u}_\theta^* \rightarrow 0, \quad z \rightarrow \infty \quad (25)$$

and by Eq. (24). Transformed displacement $\bar{u}_\theta^*$ is, thus, given by

$$\bar{u}_\theta^* = -\frac{p_0 a^2}{\mu_0} y^{\lambda_1} \frac{K_{\lambda_1}(\beta y/\varepsilon)}{\xi \beta K_{-\lambda_2}(\beta/\varepsilon)} J_2(\xi a) \quad (26)$$

It may be easily deduced from the study of Eason [18, 19], who considered the transient response of an anisotropic body having a cavity, that the exact inversion of Eq. (26) is possible when λ_1 or λ_2 is half of an odd integer,

$$\nu = 2m; \quad m = \pm 1, \pm 2, \dots \quad (27)$$

Then, in this paper two cases, $m = \pm 1$, are considered. (a) is $\nu = 2$ and the rigidity and density increase with depth, (b) is $\nu = -2$ and they decrease with depth.

(a) $\nu = 2$;

Rigidity μ and density ρ vary as

$$\mu/\mu_0 = \rho/\rho_0 = (1+\varepsilon z)^2 \quad (28)$$

Then,

$$\nu = 2, \quad \lambda_1 = -1/2, \quad \lambda_2 = 3/2 \quad (29)$$

Applying the formulas of Bessel function [16, p. 80],

$$\left. \begin{aligned} K_{\pm 1/2}(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \\ K_{\pm 3/2}(z) &= \left(\frac{\pi}{2z}\right)^{1/2} \left(1 + \frac{1}{z}\right) e^{-z} \end{aligned} \right\} \quad (30)$$

to Eq. (26), we get

$$\bar{u}_\theta^* = -\frac{p_0 a^2}{\mu_0 y} \frac{1}{\xi(\beta+\varepsilon)} e^{-\beta^2 J_2(\xi a)} \quad (31)$$

The formal inversion of Eq. (31) is given by

$$\begin{aligned} u_\theta &= -\frac{p_0 a^2}{\mu_0 y} \int_0^\infty J_1(\xi r) J_2(\xi a) d\xi \\ &\quad \times \mathcal{L}^{-1} \left\{ \frac{1}{\beta+\varepsilon} e^{-\beta^2} \right\} \quad (32) \end{aligned}$$

An inversion formula of Eq. (63) in Appendix is applied to Eq. (32) and then an interchange of the order of integration yields

$$\begin{aligned} u_\theta &= -\frac{p_0 a^2 c}{\mu_0 y} H(t-z/c) \left[\int_0^\infty J_1(\xi r) J_2(\xi a) J_0(\xi \sqrt{c^2 t^2 - z^2}) d\xi \right. \\ &\quad \left. - \varepsilon \int_0^{\sqrt{c^2 t^2 - z^2}} \frac{u}{\sqrt{c^2 t^2 - u^2}} \exp\{-\varepsilon(\sqrt{c^2 t^2 - u^2} - z)\} du \right. \\ &\quad \left. \int_0^\infty J_1(\xi r) J_2(\xi a) J_0(\xi u) d\xi \right] \quad (33) \end{aligned}$$

Further, an integral formula obtained by Eason [14],

$$\begin{aligned} \int_0^\infty J_1(\xi r) J_2(\xi a) J_0(\xi \gamma) d\xi &= H(r+a-\gamma) H(a-r) (r/a^2) \\ &\quad + H(\gamma-r-a) H(r+a-\gamma) a^{-1} \chi(r, \gamma) \quad (34) \end{aligned}$$

where

$$\begin{aligned} \chi(r, \gamma) &= \frac{r}{\pi a} \operatorname{sgn}(r-a) \tan^{-1} \left(\left| \frac{r+a}{r-a} \right| \tan \frac{\phi^*}{2} \right) - \frac{1}{\pi} \sin \phi^* - \frac{r}{2\pi a} \phi^* \\ \phi^* &= 2 \sin^{-1} [\{\gamma^2 - (r-a)^2\} / (4ar)]^{1/2} \quad (35) \end{aligned}$$

$$\operatorname{sgn}(r-a) = \begin{cases} +1, & r > a \\ -1, & r < a \end{cases} \quad (37)$$

$$0 < \sin^{-1}(\cdot) < \pi/2 \quad (38)$$

is applied to Eq.(33).

Consequently, the displacement for case (a) of $\nu=2$ is given as

$$u_\theta = -\frac{p_0 a c}{\mu_0 y} \left[H(\tau-z) H(a-r) \frac{r}{a} e^{-\epsilon(\tau-z)} - H(\tau-R_1) H(a-r) \frac{r}{a} \exp[-\epsilon\{\sqrt{\tau^2-(r+a)^2}-z\}] \right. \\ \left. + H(R_1-\tau) H(\tau-R_2) \left\{ \chi(r, \sqrt{\tau^2-z^2}) - \epsilon \int_{r-a}^{\sqrt{\tau^2-z^2}} \frac{u}{\sqrt{\tau^2-u^2}} \chi(r, u) \exp[-\epsilon(\sqrt{\tau^2-u^2}-z)] du \right\} \right. \\ \left. - \epsilon H(\tau-R_1) \int_{r-a}^{\tau+a} \frac{u}{\sqrt{\tau^2-u^2}} \chi(r, u) \exp[-\epsilon(\sqrt{\tau^2-u^2}-z)] du \right] \dots\dots\dots (39)$$

where

$$R_1 = [(r+a)^2 + z^2]^{1/2}, \quad R_2 = [(r-a)^2 + z^2]^{1/2}, \quad \tau = ct \dots\dots\dots (40)$$

(b) $\nu=-2$;

Rigidity μ and density ρ vary as

$$\mu/\mu_0 = \rho/\rho_0 = (1+\epsilon z)^{-2} \dots\dots\dots (41)$$

Then,

$$\nu=-2, \quad \lambda_1=3/2, \quad \lambda_2=1/2 \dots\dots\dots (42)$$

The substitution of the formula of Eq.(30) into Eq.(26) yields

$$\ddot{u}_\theta = -\frac{p_0 a^2}{\mu_0} \frac{y}{\xi \beta} \left(1 + \frac{\epsilon}{\beta y} \right) e^{-\beta z} J_2(\xi a) \dots\dots\dots (43)$$

The formal inversion of the above is

$$u_\theta = -\frac{p_0 a^2}{\mu_0} y \int_0^\infty J_1(\xi r) J_2(\xi a) d\xi \left[L^{-1} \left\{ \frac{1}{\beta} e^{-\beta z} \right\} + \frac{\epsilon}{y} L^{-1} \left\{ \frac{1}{\beta^2} e^{-\beta z} \right\} \right] \dots\dots\dots (44)$$

An inversion formula[20, p. 248]

$$L^{-1} \{ \beta^{-1} e^{-\beta z} \} = c H(t-z/c) J_0(\xi \sqrt{c^2 t^2 - z^2}) \dots\dots\dots (45)$$

and Eq.(72) in Appendix are applied to Eq.(44) and then an interchange of the order of integration yields

$$u_\theta = -\frac{p_0 a^2 c}{\mu_0} y H(t-z/c) \left\{ \int_0^\infty J_1(\xi r) J_2(\xi a) J_0(\xi \sqrt{c^2 t^2 - z^2}) d\xi \right. \\ \left. + \frac{\epsilon}{y} \int_z^\tau du \int_0^\infty J_1(\xi r) J_2(\xi a) J_0(\xi \sqrt{c^2 t^2 - u^2}) d\xi \right\} \dots\dots\dots (46)$$

Further, the integral formula of Eq.(34) is applied and then the displacement for case (b) is given by

$$u_\theta = -\frac{p_0 a c}{\mu_0} y \left[H(R_1-\tau) H(\tau-z) H(a-r) \frac{r}{a} \frac{1+\epsilon \tau}{1+\epsilon z} \right. \\ \left. + H(\tau-R_1) H(a-r) \frac{r}{a} \frac{\epsilon}{y} (\tau - \sqrt{\tau^2 - (r+a)^2}) \right. \\ \left. + H(R_1-\tau) H(\tau-R_2) \left\{ \chi(r, \sqrt{\tau^2-z^2}) + \frac{\epsilon}{y} \int_z^{\sqrt{\tau^2-z^2}} \chi(r, \sqrt{\tau^2-u^2}) du \right\} \right. \\ \left. + \frac{\epsilon}{y} H(\tau-R_1) \int_{r-a}^{\sqrt{\tau^2-(r+a)^2}} \chi(r, \sqrt{\tau^2-u^2}) du \right] \dots\dots\dots (47)$$

where R_1 , R_2 and τ are given in Eq.(40).

5. DISCUSSION OF SOLUTIONS

The following non-dimensionalizations

$$r/a \rightarrow r, \quad z/a \rightarrow z, \quad ct/a \rightarrow \tau, \quad a\epsilon \rightarrow \epsilon$$

are introduced in Eqs.(39) and (47). Then, we get:

(a) $\nu=2$;

$$u_\theta = -\frac{p_0 a c}{\mu_0 y} \left[H(R_1-\tau) H(\tau-z) H(1-r) r e^{-\epsilon(\tau-z)} + H(\tau-R_1) H(1-r) r \right. \\ \times [e^{-\epsilon(\tau-z)} - \exp[-\epsilon\{\sqrt{\tau^2-(r+1)^2}-z\}]] + H(R_1-\tau) H(\tau-R_2) \\ \times \left\{ \frac{1}{\pi} \Phi(r, \sqrt{\tau^2-z^2}) - \frac{\epsilon}{\pi} \int_z^{\sqrt{\tau^2-z^2}} \Phi(r, \sqrt{\tau^2-u^2}) e^{-\epsilon(u-z)} du \right\} \\ \left. - \frac{\epsilon}{\pi} H(\tau-R_1) \int_{r-a}^{\sqrt{\tau^2-(r+1)^2}} \Phi(r, \sqrt{\tau^2-u^2}) e^{-\epsilon(u-z)} du \right] \dots\dots\dots (48)$$

(b) $\nu=-2$;

$$u_\theta = -\frac{p_0 a c}{\mu_0} \left[H(R_1-\tau) H(\tau-z) H(1-r) r (1+\epsilon \tau) + H(\tau-R_1) H(1-r) \epsilon r \right. \\ \times [\tau - \sqrt{\tau^2 - (r+1)^2}] + H(R_1-\tau) H(\tau-R_2) \left\{ \frac{y}{\pi} \Phi(r, \sqrt{\tau^2-z^2}) \right. \\ \left. + \frac{\epsilon}{\pi} \int_z^{\sqrt{\tau^2-(r+1)^2}} \Phi(r, \sqrt{\tau^2-u^2}) du \right\} \\ \left. + \frac{\epsilon}{\pi} H(\tau-R_1) \int_{r-a}^{\sqrt{\tau^2-(r+1)^2}} \Phi(r, \sqrt{\tau^2-u^2}) du \right] \dots\dots\dots (49)$$

where

$$\Phi(r, \tau) = r \operatorname{sgn}(\tau-1) \tan^{-1} \left\{ \left| \frac{r+1}{r-1} \right| \tan \frac{\phi^*}{2} \right\} - \sin \phi^* - \frac{r}{2} \phi^* \dots\dots\dots (50)$$

$$\phi^* = 2 \sin^{-1} [(\tau^2 - (r+1)^2)/(4\tau)]^{1/2} \dots\dots\dots (51)$$

$$R_1 = \sqrt{(r+1)^2 + z^2}, \quad R_2 = \sqrt{(r-1)^2 + z^2} \dots\dots\dots (52)$$

Now, taking the limit $\epsilon \rightarrow 0$ in Eqs.(48) and (49)

$$u_\theta = -\frac{p_0 a c}{\mu_0} [H(R_1-r) H(\tau-z) H(1-r) r \\ + H(R_1-\tau) H(\tau-R_2) \Phi(r, \sqrt{\tau^2-z^2})/\pi] \dots\dots\dots (53)$$

Eq.(53) coincides with the solution of a homogeneous elastic solid obtained by Eason [14]. Comparing Eq.(53) with Eqs.(48) and (49), we can find that the second and fourth terms in Eqs.(48) and (49) represent the disturbances in a region, $\tau > R_1$, where no disturbance is observed in the case of the homogeneous solid given by Eq.(53). This is the most characteristic feature of the inhomogeneous solid and means the following;

In the case of the homogeneous solid there is no disturbance behind a wave emanating from an image source ($r=-1$, $z=0$) and the solid recovers the initial state just after the arrival of the wave. But, in the case of the inhomogeneous solid there exists a disturbance behind the wave and the solid approaches gradually the initial quiescent state after the arrival of the wave.

The jumps on wave fronts, $\tau=R_1$ and R_2 , are easily obtained and are

$$[u_\theta] = 0 \quad \tau \rightarrow R_1, R_2 \dots\dots\dots (54)$$

This shows that wave front jumps in the displacement are zero and are same as in the homogeneous solid.

A one-dimensional torsional wave propagating just below the loaded region ($0 \leq r \leq a$) has a finite jumps at its front,

(a) $\nu=2$;

$$[u_\theta] = -\frac{p_0 a c}{\mu_0} \frac{r}{1+\epsilon \tau}; \quad \tau \rightarrow z \dots\dots\dots (55)$$

(b) $\nu=-2$;

$$[u_\theta] = -\frac{p_0 a c}{\mu_0} r (1+\epsilon \tau); \quad \tau \rightarrow z \dots\dots\dots (56)$$

In case (a) the jump decrease with time, and in case (b) the jump, however, decreases with time.

6. NUMERICAL COMPUTATIONS

Numerical computations of the displacement given by Eqs.(48) and (49) are carried out by setting the inhomogeneous parameter $\epsilon=1$ and 2.

Figs. 2 and 3 show variations of the displacement with time at four observing points, $(r,z)=(0.5, 0)$, $(1.5, 0)$, $(0.5, 0.5)$ and $(1.5, 0.5)$. In these figures the dashed curve is that of the homogeneous solid $\epsilon=0$ and arrowed marks, T_1 and T_2 , show the arrivals of the wave fronts, $\tau=R_1$ and R_2 , respectively. It should be noted that the magnitude of the displacement is greater in case (b) than in (a).

The variations of the remaining disturbance after the arrival of wave front $\tau=R_1$ give the following;

(a) when $\nu=2$, the displacement approaches the initial state from the direction twisted by the load.

(b) when $\nu=-2$, the displacement approaches the initial state from the opposite direction twisted by the load.

Figs. 4, 5 and 6 show equi-displacement curves of non-dimensional displacement

$$-\frac{\mu_0}{\rho_0 a c} u_0$$

for $\tau=0.5, 1$ and 1.5 . In these, (a), (b) and (c) show the cases, (a) $\nu=2$, (b) $\nu=-2$ and (c) homogeneous $\epsilon=0$, respectively. It is interesting that a saddle point of the deformation is formed in an arrowed region in Figs. 5(a) and 6(a). The displacement in in region, $\tau > R_2$ and $0 < r \leq 1$, is independent of depth z in the cases, (b) and (c), but that is dependent in case (a). This fact is also valid in region, $\tau > z$, $\tau < R_2$ and $0 < r \leq 1$.

The deformation gradient becomes greater in two regions. One is near a wave front emanating from image source ($r=-1$, $z=0$) and the other is near point ($r=1$, $z=\tau$). It should noted that case (a) has always a more remarkable feature of the deformation than the other two cases. Thus, the effect of the inhomogeneity becomes clearer when material parameters, μ and ρ , increase with depth z .

7. CONCLUSIONS

The transient response of an inhomogeneous elastic half space to an impulsive torsional load has been considered with a prescribed inhomogeneity given by Eq.(15). Exact solutions are obtained for two cases, $\nu=\pm 2$, and detailed numerical computations are carried out. The results obtained in this paper are summarized as follows;

(1) The inhomogeneous solid defined by Eq. (15) recovers gradually the initial quiescent state after the arrival of a wave emanating from an image source and there exists a disturbance behind the wave. The homogeneous solid, however, makes rapid recovering and has no disturbance behind the wave. This is the most outstanding feature of the response of the inhomogeneous solid.

When the material parameters, μ and ρ , increase with depth, the displacement recovers the initial state from the loaded direction. In the other case of variation

of the material parameters, it recovers from the opposite direction.

(2) When the material parameters increase with depth, a saddle point of the deformation is formed in a region. The region is close to wave front, $\tau=R_2$, and moves into the interior almost along the wave front with passing of time.

(3) Jump in the displacement is zero at two wave fronts, $\tau=R_1$ and R_2 , and thus the wave front singularity is not affected by the inhomogeneity of the solid.

(4) The inhomogeneous effect on the response/deformation is more explicit when the material parameters increase with depth than when those decrease.

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APPENDIX

Consider an inversion of Laplace transform,

$$I_A = \mathcal{L}^{-1} \left\{ \frac{1}{\beta + \varepsilon} e^{-\beta z} \right\} \dots (57)$$

where $\beta = (\xi^2 + s^2/c^2)^{1/2}$.

An inversion formula [20, p. 227]

$$\mathcal{L}^{-1} \{ f^*(\alpha) \} = f(t) - a \int_0^t f(\sqrt{t^2 - u^2}) J_1(au) du \dots (58)$$

$$\alpha = \sqrt{s^2 + a^2} \dots (59)$$

is applied to Eq. (57) with substitution of

$$f^*(\alpha) = (\alpha + c\varepsilon)^{-1} e^{-(\varepsilon/c)z} \dots (60)$$

and with $\alpha = c\xi$.

A simple inversion formula

$$\mathcal{L}^{-1} \{ f^*(s) \} = \mathcal{L}^{-1} \{ (s + c\varepsilon)^{-1} e^{-(\varepsilon/c)z} \} = H(t - z/c) e^{-\varepsilon(ct - z)} \dots (61)$$

is substituted into Eq. (58) and then we get

$$I_A = c \mathcal{L}^{-1} \left[\left(\sqrt{s^2 + (c\xi)^2} + \varepsilon \right)^{-1} \exp \left\{ (-z/c) \sqrt{s^2 + (c\xi)^2} \right\} \right] \\ I_A = cH(ct - z) \left[\exp \{ -\varepsilon(ct - z) \} - \varepsilon c \int_0^t H(\sqrt{t^2 - u^2} - z/c) \right. \\ \left. \times \exp \{ -\varepsilon c(\sqrt{t^2 - u^2} - z/c) \} J_1(c\xi u) du \right] \dots (62)$$

Integrating by parts, Eq. (57) yields

$$I_A = cH(t - z/c) \left\{ J_0(\xi \sqrt{c^2 t^2 - z^2}) \right. \\ \left. - \varepsilon \int_0^{\sqrt{c^2 t^2 - z^2}} \frac{u}{\sqrt{c^2 t^2 - u^2}} \exp \{ -\varepsilon(\sqrt{c^2 t^2 - u^2} - z) \} J_0(\xi u) du \right\} \dots (63)$$

Consider an inversion,

$$I_B = \mathcal{L}^{-1} \left\{ \frac{1}{\beta^2} e^{-\beta z} \right\} \dots (64)$$

An inversion integral along Bromwich contour in Fig. 7 converts Eq. (64) to

$$I_B = \frac{1}{2\pi i} \int_{Br} (\xi^2 + s^2/c^2)^{-1} e^{-\beta z + st} ds \dots (65)$$

$$= \frac{c}{\xi} \sin(ct\xi)$$

$$- \frac{2}{\pi} \frac{c}{\xi} \int_1^\infty \frac{1}{u^2 - 1} \sin(z\xi \sqrt{u^2 - 1}) \sin(ct\xi u) du; \quad t > z/c$$

$$= \frac{c}{\xi} \sin(ct\xi)$$

$$- \frac{2c}{\pi \xi} \int_0^\infty \frac{1}{v \sqrt{v^2 + 1}} \sin(ct\xi \sqrt{v^2 + 1}) \sin(z\xi v); \quad t > z/c \dots (66)$$

The trigonometric function in the integrand is replaced by its integral form

$$\sin(z\xi v) = \xi v \int_0^z \cos(\xi v u) du \dots (67)$$

and an interchange of the order of integration yields

$$I_B = \frac{c}{\xi} \sin(ct\xi) - \frac{2c}{\pi} \int_0^z du \int_0^\infty \frac{1}{\sqrt{v^2 + 1}} \sin(ct\xi \sqrt{v^2 + 1}) \cos(u\xi v) dv; \\ t > z/c \dots (68)$$

Next, integral formula [20, p. 26],

$$\int_0^\infty (a^2 + x^2)^{-1/2} \sin(b \sqrt{a^2 + x^2}) \cos(xy) dx \\ = (\pi/2) H(b - y) J_0(a \sqrt{b^2 - y^2}) \dots (69)$$

is applied to Eq. (68)

$$I_B = \frac{c}{\xi} \sin(ct\xi) - c \int_0^z J_0(\xi \sqrt{c^2 t^2 - u^2}) du; \quad t > z/c \dots (70)$$

Further, the first term in Eq. (70) is replaced with its integral representation [21, p. 671],

$$\sin x = \int_0^x J_0(\sqrt{x^2 - u^2}) du \dots (71)$$

Thus, we get

$$I_B = cH(t - z/c) \int_z^{ct} J_0(\xi \sqrt{c^2 t^2 - u^2}) du \dots (72)$$

Fig. 1 Torsional load applied on the surface of an inhomogeneous elastic half space.

Fig. 2 Variation of displacement with time at the surface.

Fig. 3 Variation of displacement with time at the interior.

Fig. 4 Equi-displacement curve at $\tau=0.5$. Fig. 6 Equi-displacement curve at $\tau=1.5$. Fig. 5 Equi-displacement curve at $\tau=1$.

Fig. 7 Laplace inversion contour used in Eq.(65).