A Seismic Response Analysis of a Cylindrical Liquid Storage Tank Including the Effect of Sloshing*

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A seismic response analysis method for a cylindrical liquid storage tank subjected to a horizontal earthquake is presented in this paper. The kinetic and strain energies of an empty tank shell are estimated assuming it as an axisymmetric shell. The virtual work of liquid pressure exerting on the tank wall is also estimated analytically by assuming that the behaviour of the liquid follows the velocity potential theory which includes the effect of sloshing. The Lagrangian formulation for the coupled vibration is then derived.

Coupling this equation with the Bernoulli equation for the boundary condition of a liquid free surface, an eigenvalue analysis method for the coupled vibration can be obtained. And, a seismic response analysis method is also made using the exciting energy due to a seismic wave. As a result of numerical studies, this solution is proved to be an effective and reasonable method for the seismic response analysis of a cylindrical liquid storage tank.

1. Introduction

A liquid storage tank, including an oil storage tank, is one of the important objectives for an earthquake-proof structure design in several decades. Regarding this seismic response analysis, Housner's⁽¹⁾ theory is well known as a simple and useful design method. Senda and Nakagawa⁽²⁾, and Jacobsen⁽³⁾ have reported a similar analysis method based on the velocity potential theory. Recently, Sogabe et al.⁽⁴⁾ have also proposed an earthquake-proof structure design method considering the transient response of a sine wave on the basis of the velocity potential theory. Moreover, many studies which are related to the structure-fluid coupled vibration problem, such as a rocket missile with liquid fuel, were reported by NASA in the U.S.A. These studies were summarized by Abramson⁽⁵⁾. Mouzakis⁽⁶⁾, Shaaban et al.⁽⁷⁾, Veletsos⁽⁸⁾ and others have published some papers regarding a tank subjected to a horizontal earthquake. The author has also already presented two papers^{(9), (10)} from the same viewpoint.

However, in the above mentioned studies, the impulsive pressure due to the coupled vibration between structure and liquid, and the convective pressure due to the free surface oscillation (i.e. sloshing), were calculated separately or only one of both pressures was calculated. As for studies in which both pressures were considered simultaneously, Bauer and Siekmann⁽¹¹⁾ have proposed a coupled vibration analysis method in which the side and bottom plates of a cylindrical tank are treated as a thin shell and a term of liquid pressure is added to its equilibrium equation. And, the solu-

tion is then obtained by expanding it into series. Furthermore, they showed some numerical examples on only the vertical free vibration problem. Similarly Kondo⁽¹²⁾ has published an analytical solution in which the bottom plate of the tank is assumed to be rigid, the side shell plate to be elastic, and the effect of sloshing is included. Sakai and Sakoda⁽¹³⁾ have proposed a seismic response analysis method based on a finite element method applied to both a tank shell and a liquid.

In the present paper, a seismic response analysis method is proposed in which the tank shell is treated by a finite element method for an axisymmetric shell as in the former paper⁽¹⁰⁾, and furthermore, the effect of the free surface oscillation, that is sloshing, is included. The tank shell is divided into ring shell elements, and then the kinetic and strain energies for each degree of natural modes are calculated. On the other hand, assuming that the liquid in a tank follows the velocity potential theory including the effect of sloshing, the virtual work of liquid on the tank wall is analytically obtained. The eigenvalue analysis for the structure-liquid coupled vibration can be then done using both the equation which is obtained by substituting the above-mentioned energies and work in Lagrange's equation of motion, and one obtained by Bernoulli's equation for the boundary condition of a liquid free surface. Moreover, the exciting energy due to a horizontal seismic wave is obtained, and then a seismic response analysis method is proposed in this paper.

2. Method

2.1 Basic equation

2.1.1 Tank shell

It is considered that a horizontal seismic wave is imposed on the rigid bottom base of a cylindrical liquid storage tank with the variable thickness of side plate in the axial direction as shown in Fig.1.

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Fig.1 Cylindrical liquid storage tank

In this figure, (r, θ, x) denotes a cylindrical co-ordinate system, R the tank radius, l the height, h the wall thickness, H the liquid depth and $\dot{z}$ a horizontal seismic acceleration. The displacement components $u(\theta, x, t)$, $v(\theta, x, t)$ and $w(\theta, x, t)$ of a tank shell are expressed by the following equation using normal functions u_{ks} , v_{ks} and w_{ks} of an empty tank as in the former paper⁽¹⁰⁾.

$$\left. \begin{aligned} u(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} u_{ks}(x) q_{ks}(t) \cos(k\theta), \\ w(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} w_{ks}(x) q_{ks}(t) \cos(k\theta) \\ v(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} v_{ks}(x) q_{ks}(t) \sin(k\theta) \end{aligned} \right\} \dots (1)$$

where, q_{ks} is a normal co-ordinate of the k th circumferential mode and s th axial mode, $\bar{n}$, $\bar{n}$ are the numbers of axial modes and circumferential ones respectively, and t denotes time. The strain energy stored in a cylindrical shell U_E is expressed by the following equation:

$$U_E = \frac{1}{2} \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} K_{ks} q_{ks}^2 \dots (2)$$

where, K_{ks} is obtained from Eq. (5) of the former paper⁽¹⁰⁾.

The kinetic energy K_E of a cylindrical shell is expressed by the following equation also as in the former paper:

$$K_E = \frac{1}{2} \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} M_{ks} \dot{q}_{ks}^2 \dots (3)$$

where, M_{ks} is also obtained by Eq. (7) of the former paper⁽¹⁰⁾, $\dot{q}_{ks}$ is the derivative of q_{ks} with respect to time. The exciting energy F_E due to a horizontal seismic wave $\dot{z}$ exerting the tank bottom base is also expressed as follows:

$$F_E = \left\{ \pi R \rho \int_0^H \sum_{s=1}^{\bar{n}} (w_{1s} - v_{1s}) q_{1s} dx \right\} \dot{z} \quad (\text{for } k = 1) \\ = 0 \quad (\text{for } k \neq 1) \dots (4)$$

where, ρ is the density of a shell.

2.1.2 Liquid in the tank

In the former paper, the velocity potential of the liquid in a tank is obtained by ignoring the effect of sloshing. However, in this paper, its velocity potential Φ shall be obtained including the effect of sloshing. The boundary conditions are given as follows:

$$(i) \quad (\partial \Phi / r \partial \theta)_{\theta=0, \pi} = 0 \dots (5)$$

$$(ii) \quad (\partial \Phi / \partial x)_{x=0} = 0 \dots (6)$$

$$(iii) \quad (\partial \Phi / \partial r)_{r=R} = -\dot{w} - \dot{z} \cos \theta \\ = - \sum_{k=1}^{\bar{n}} \sum_{s=1}^{\bar{n}} w_{ks} \dot{q}_{ks} \cos(k\theta) - \dot{z} \cos \theta \dots (7)$$

The boundary conditions given above are the same as in the former paper. The following boundary condition for a liquid free surface is obtained from the linearized Bernoulli equation as follows:

$$(iv) \quad (\partial^2 \Phi / \partial t^2 + g \partial \Phi / \partial x)_{x=H} = 0 \dots (8)$$

where, g denotes the acceleration of gravity.

The equation of continuity requires that Φ must satisfy the following Laplace equation.

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{\partial \Phi}{r \partial r} + \frac{\partial^2 \Phi}{r^2 \partial \theta^2} + \frac{\partial^2 \Phi}{\partial x^2} = 0 \dots (9)$$

The solution of Eq. (9) is assumed as follows:

$$\Phi = \sum_{k=1}^{\bar{n}} \left(\sum_{p=1}^m \phi_{kp} Q_{kp} + \sum_{s=1}^{\bar{n}} z \phi_{ks} \dot{q}_{ks} \right) - r \dot{z} \cos \theta \dots (10)$$

where, Q_{kp} denotes a normal co-ordinate of sloshing of the k th circumferential mode and p th radial mode. The velocity potential of Eq. (10) is expressed by superposing the velocity potential shown in the former paper on one for a free surface oscillation obtained by assuming the tank wall to be rigid as shown in reference (9). The following expressions satisfy the boundary conditions (5) ~ (7).

$$\left. \begin{aligned} \phi_{kp} &= \phi_p^{(k)} \cos(k\theta) = J_k(\kappa_p^{(k)} r) \cosh(\kappa_p^{(k)} x) \cos(k\theta) \\ \phi_{ks} &= z \phi_s^{(k)} \cos(k\theta) \\ &= - \sum_{i=0}^{\infty} \frac{2 I_k(\lambda_i r / H)}{\lambda_i I_k'(\lambda_i R / H)} \\ &\quad \times \cos\left(\frac{\lambda_i x}{H}\right) \int_0^H w_{1s} \cos\left(\frac{\lambda_i x}{H}\right) dx \cos(k\theta) \end{aligned} \right\} \dots (11)$$

Where,

$$\left[\frac{d}{dr} (J_k(\kappa_p^{(k)} r)) \right]_{r=R} = 0, \quad \lambda_i = \left(i + \frac{1}{2}\right) \pi \\ I_k'(\lambda_i R / H) = I_{k-1}(\lambda_i R / H) - (k H / \lambda_i R) I_k(\lambda_i R / H) \dots (12)$$

and J_k denotes a Bessel function of the first kind of order k , I_k a modified Bessel function of the first kind of order k .

The virtual work δW done by the liquid on the tank wall is given by:

$$\delta W = \int_S ((p)_{r=R} \delta w) dS \quad \dots\dots\dots (13)$$

where, S denotes the contact area between the liquid and the side wall of a tank. The liquid pressure p is expressed as follows:

$$p = \rho_L \frac{\partial \Phi}{\partial t} \quad \dots\dots\dots (14)$$

where, ρ_L denotes the density of the liquid. Substituting Eqs. (1), (10) and (14) into Eq. (13), the following equation is obtained:

$$\begin{aligned} \delta W &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^n Q_{ks} \delta q_k \\ &= \rho_L \int_0^H \int_0^{2\pi} \left[\sum_{k'=1}^{\bar{n}} \left\{ \sum_{p=1}^m (1) \Phi_{k'p} \right\} \ddot{Q}_{k'p} \right. \\ &\quad \left. + \sum_{s'=1}^n (2) \Phi_{ks'} \ddot{q}_{ks'} - R \ddot{z} \cos \theta \right] \\ &\quad \times \left(\sum_{k=1}^{\bar{n}} \sum_{s=1}^n w_{ks} \cos(h\theta) \delta q_k \right) R d\theta dx \quad \dots\dots\dots (15) \end{aligned}$$

where Q_{ks} denotes a generalized force.

2.2 Eigenvalue analysis

It is assumed that the tank is in a free vibration. In this case, z -term in Eq. (7) must be zero as the bottom of the tank keeps still. Therefore, $\ddot{z}$ -term in Eq. (10) and $\ddot{z}$ -term in Eq. (15) can be discarded.

Substituting Eq. (10) with $\ddot{z}$ -term discarded into the boundary condition (8) for a free surface, the following equation is obtained:

$$\begin{aligned} \sum_{k=1}^{\bar{n}} \left\{ \sum_{p=1}^m (1) \Phi_{kp} \ddot{Q}_{kp} + \sum_{s=1}^n (2) \Phi_{ks} \ddot{q}_{ks} \right\} \\ + g \sum_{k=1}^{\bar{n}} \left\{ \sum_{p=1}^m \left(\frac{\partial_1 \Phi_{kp}}{\partial x} \right)_{s=H} \right. \\ \left. \times \dot{Q}_{kp} + \sum_{s=1}^n \left(\frac{\partial_2 \Phi_{ks}}{\partial x} \right)_{s=H} \dot{q}_{ks} \right\} = 0 \quad \dots\dots\dots (16) \end{aligned}$$

As shown in reference (9), after substituting Eq. (11) into Eq. (16), both sides are multiplied by $J_k(\kappa_p^{(k)} r) r$ and are integrated with respect to r from 0 to R . In addition, the following integral formula of Bessel function is applied:

$$\begin{aligned} \int_0^R J_k(\kappa_p^{(k)} r) J_k(\kappa_{p'}^{(k)} r) r dr \\ = \frac{R^2}{2} \left\{ 1 - \left(\frac{k}{\kappa_p^{(k)} R} \right)^2 \right\} \{ J_k(\kappa_p^{(k)} R)^2 \} \quad (\text{for } p=p') \\ = 0 \quad (\text{for } p \neq p') \end{aligned} \quad \dots\dots\dots (17)$$

Then, Integrating Eq. (16) with respect to time for once gives the following equation:

$$[M_1^{(11)}] \{\ddot{Q}_k\} + [K_1^{(11)}] \{Q_k\} + [K_2^{(12)}] \{q_k\} = 0 \quad \dots\dots\dots (18)$$

where, the constant of the integration can be made zero by assuming $Q_{kp} = q_{ks} = 0$ when $\dot{Q}_{kp} = \dot{q}_{ks} = 0$. And where,

$$\left. \begin{aligned} [M_1^{(11)}] &= \begin{bmatrix} & 0 \\ & M_{kp}^L \\ 0 & \end{bmatrix}, \quad [K_1^{(11)}] = \begin{bmatrix} & 0 \\ & K_{kp}^L \\ 0 & \end{bmatrix} \\ M_{kp}^L &= \int_0^R (1) \phi_p^{(k)} \big|_{s=H} J_k(\kappa_p^{(k)} r) r dr, \\ K_{kp}^L &= g \int_0^R \left(\frac{\partial_1 \phi_p^{(k)}}{\partial x} \right)_{s=H} J_k(\kappa_p^{(k)} r) r dr \\ [K_2^{(12)}] &= \left[g \int_0^R \left(\frac{\partial_2 \phi_s^{(k)}}{\partial x} \right)_{s=H} J_k(\kappa_p^{(k)} r) r dr \right] \end{aligned} \right\} \quad \dots\dots\dots (19)$$

Where $\{Q_k\}$ and $\{q_k\}$ denote the vectors which consist of Q_{kp} and q_{ks} as elements respectively, and $[K_2^{(12)}]$ is a matrix of order $p \times s$.

Substituting the strain energy U_E , the kinetic energy K_E of a cylindrical tank and the generalized force Q_{ks} exerted by the liquid on the tank wall into Lagrange's equation of motion, the following equation is obtained:

$$\frac{d}{dt} \left\{ \frac{\partial K_E}{\partial \dot{q}_{ks}} \right\} + \frac{\partial U_E}{\partial q_{ks}} - Q_{ks} = 0 \quad (s=1 \sim n) \quad \dots\dots\dots (20)$$

Also, substituting Eqs. (2), (3) and (15) into Eq. (20) yields,

$$([M_1^{(22)}] + [M_2^{(22)}]) \{\ddot{q}_k\} + [M_2^{(21)}] \{\ddot{Q}_k\} + [K_1^{(22)}] \{q_k\} = 0 \quad \dots\dots\dots (21)$$

Where,

$$\left. \begin{aligned} [M_1^{(22)}] &= \begin{bmatrix} & 0 \\ & M_{ks} \\ 0 & \end{bmatrix}, \quad [K_1^{(22)}] = \begin{bmatrix} & 0 \\ & K_{ks} \\ 0 & \end{bmatrix} \\ [M_2^{(21)}] &= \left[\pi R \rho_L \int_0^H (1) \phi_p^{(k)} \big|_{r=R} (-w_{ks}) dx \right] \\ [M_2^{(22)}] &= \left[\pi R \rho_L \int_0^H (2) \phi_s^{(k)} \big|_{r=R} (-w_{ks}) dx \right] \end{aligned} \right\} \quad \dots\dots\dots (22)$$

and $[M_2^{(21)}]$ is a matrix of order $p \times s$, $[M_2^{(22)}]$ a matrix of order $s \times s$.

By combining Eq. (18) with Eq. (21), the following equation is obtained:

$$([M_1] + [M_2]) \left\{ \begin{matrix} \ddot{Q}_k \\ \ddot{q}_k \end{matrix} \right\} + ([K_1] + [K_2]) \left\{ \begin{matrix} Q_k \\ q_k \end{matrix} \right\} = 0 \quad \dots\dots\dots (23)$$

where,

$$\left. \begin{aligned} [M_1] &= \begin{bmatrix} M_1^{(11)} & 0 \\ 0 & M_1^{(22)} \end{bmatrix}, \quad [M_2] = \begin{bmatrix} 0 & 0 \\ M_2^{(21)} & M_2^{(22)} \end{bmatrix} \\ [K_1] &= \begin{bmatrix} K_1^{(11)} & 0 \\ 0 & K_1^{(22)} \end{bmatrix}, \quad [K_2] = \begin{bmatrix} 0 & K_2^{(12)} \\ 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad \dots\dots\dots (24)$$

Assuming $[M_2] = [K_2] = 0$ in Eq. (23), Eq. (23) yields two separate dynamic equations, one for the free surface oscillation of the liquid in a rigid wall tank, and the other for an empty tank. Substituting

$\{Q_k\} = \{\phi_k\} e^{i\omega_k t}$ into Eq. (23), a characteristic equation is obtained as follows:

$$(-\omega_k^2 ([M_1] + [M_2]) + [K_1] + [K_2]) \{\phi_k\} = 0 \quad \dots\dots\dots (25)$$

Solving the above equation, $m+n$ eigenvalues ω_{kc} and eigenvectors $\{\psi_k\}_c$ for the k th circumferential mode are obtained. Normal functions of the coupled system $(u_k)_c$, $(v_k)_c$, $(w_k)_c$ for the displacement, $(p_k)_c$ for the liquid pressure and $(\eta_k)_c$ for the wave height are obtained as follows:

$$\left. \begin{aligned} (u_k)_c &= \sum_{s=1}^n u_{ks}(\psi_{k,m+s})_c \\ (v_k)_c &= \sum_{s=1}^n v_{ks}(\psi_{k,m+s})_c \\ (w_k)_c &= \sum_{s=1}^n w_{ks}(\psi_{k,m+s})_c \\ (p_k)_c &= -\rho_L \omega_{kc}^2 \left\{ \sum_{p=1}^m \phi_p^{(k)}(\psi_{kp})_c + \sum_{s=1}^n \phi_s^{(k)}(\psi_{k,m+s})_c \right\} \\ (\eta_k)_c &= -\frac{\omega_{kc}^2}{g} \left\{ \sum_{p=1}^m \phi_p^{(k)}(\psi_{kp})_c + \sum_{s=1}^n \phi_s^{(k)}(\psi_{k,m+s})_c \right\}_{z=H} \end{aligned} \right\} \quad (c=1 \sim m+n) \quad (26)$$

2.3 Seismic response analysis

Two kinds of equations, one obtained by substituting Eq. (10) into Eq. (8) with help of Eq. (17), and the other by substituting Eqs. (2) ~ (4) and (15) into Lagrange's equation of motion, are coupled to yield the following coupled dynamic equation:

$$([M_1] + [M_2]) \begin{Bmatrix} \ddot{Q}_k \\ \ddot{q}_k \end{Bmatrix} + ([K_1] + [K_2]) \begin{Bmatrix} Q_k \\ q_k \end{Bmatrix} = -\{F\} \ddot{z} \quad (27)$$

where,

$$\left\{ \begin{aligned} \{F\} &= \begin{Bmatrix} F^{(1)} \\ F^{(2)} \end{Bmatrix} \\ F_p^{(1)} &= -\int_0^R J_1(\kappa_p r) r^2 dr \quad (k=1) \\ &= 0 \quad (k \neq 1) \\ F_s^{(2)} &= \pi R \rho \int_0^1 h(w_{1s} - v_{1s}) dx + \pi R^2 \rho_L \int_0^H w_{1s} dx \quad (k=1) \\ &= 0 \quad (k \neq 1) \end{aligned} \right\} \quad (28)$$

and $\{F\}$ denotes a vector which consists of $F_p^{(1)}$, $F_s^{(2)}$ as elements.

In order to apply a modal analysis method to Eq. (27), the solution of Eq. (27) is assumed to be expressed by:

$$\begin{Bmatrix} Q \\ q \end{Bmatrix} = \sum_{c=1}^{m+n} \{\psi\}_c T_c(t) = [\psi] \{T(t)\} \quad (29)$$

Because the force term in the right hand side of Eq. (27) is to be zero when $k \neq 1$ as is known from Eq. (28), the subscript k shall be omitted hereafter.

In order to orthogonalize Eq. (27), it is impossible to make diagonal matrices by usual means of substituting Eq. (29) into Eq. (27) and pre-multiplying $[\psi]^T$ on both sides because $[M_1] + [M_2]$ and $[K_1] + [K_2]$ are asymmetric matrices. Otherwise, by solving the eigenvalue problem which is obtained by transposing the matrices of Eq. (23), a modal matrix $[\tilde{\psi}]$ corresponding to $[\psi]$ is

obtained the column of which is a modal vector $\{\tilde{\psi}\}_c$. By substituting Eq. (29) into Eq. (27) and pre-multiplying $[\tilde{\psi}]^T$ on both sides, Eq. (27) can be orthogonalized to yield $m+n$ dynamic equations which are independent of each other. Adding damping terms to each equation by ignoring the coupling effect introduced among natural modes, the following approximate equation is obtained:

$$\ddot{T}_c(t) + 2h_c \omega_c \dot{T}_c(t) + \omega_c^2 T_c(t) = -\beta_c \ddot{z} \quad (c=1 \sim m+n) \quad (30)$$

where,

$$\left. \begin{aligned} \omega_c^2 &= K_c / M_c, \quad \beta_c = \{\tilde{\psi}\}_c^T \{F\} / M_c \\ M_c &= \{\tilde{\psi}\}_c^T ([M_1] + [M_2]) \{\psi\}_c \\ K_c &= \{\tilde{\psi}\}_c^T ([K_1] + [K_2]) \{\psi\}_c \end{aligned} \right\} \quad (31)$$

and $T_c(t)$ denotes the normal co-ordinate of the c th coupled vibration mode, h_c the damping ratio of the c th mode, and β_c the participation factor of the c th mode.

When an acceleration response S_{AC} for the c th coupled vibration mode can be obtained from a response spectrum, the response displacement, response acceleration in the radial direction, response pressure and response wave height are given in the following expressions by taking the square root sum of squares method.

$$\left\{ \begin{aligned} w(\theta, x) &= \left[\sum_{c=1}^{m+n} \left\{ \beta_c \frac{S_{Ac}}{\omega_c^2} \sum_{s=1}^n w_s(\psi_{m+s})_c \cos \theta \right\}^2 \right]^{1/2} \\ \ddot{w}(\theta, x) + \ddot{z} \cos \theta &= \left[\sum_{c=1}^{m+n} \left\{ \beta_c S_{Ac} \sum_{s=1}^n w_s(\psi_{m+s})_c \cos \theta \right\}^2 \right]^{1/2} \\ p(r, \theta, x) &= \rho_L \left[\sum_{c=1}^{m+n} \left\{ \beta_c S_{Ac} \left\{ \sum_{p=1}^m \phi_p(\psi_p)_c + \sum_{s=1}^n \phi_s(\psi_{m+s})_c \right\} \cos \theta \right\}^2 \right]^{1/2} \\ \eta(r, \theta) &= \frac{1}{g} \left[\sum_{c=1}^{m+n} \left\{ \beta_c S_{Ac} \left\{ \sum_{p=1}^m \phi_p(\psi_p)_c + \sum_{s=1}^n \phi_s(\psi_{m+s})_c \right\}_{z=H} \cos \theta \right\}^2 \right]^{1/2} \end{aligned} \right\} \quad (32)$$

The other values are similar. Otherwise, these responses can be obtained also by a modal time-history analysis method.

3. Numerical examples

3.1 Comparison among the present solution, the former ones of Ref. (9) and (10)

When the vibration of a tank shell itself becomes predominant, it is referred hereafter to as a bulging mode. And when the free surface oscillation is predominant, it is hereafter referred to as a sloshing mode.

Table 1 shows natural frequencies of sloshing modes obtained by the present solution and the solution of Ref. (9), and ones of bulging modes obtained by the present and the former one. In this numerical example, the tank has : $R=1829$ cm, $h=1219$ cm, $E=206$ GPa ($=2.1 \times 10^{-6}$ kgf/cm²), $\nu=0.3$, $\rho=7.834 \times 10^3$ kg/m³

Table 1 Natural frequencies of the cylindrical liquid storage tank with a free end and a rigid base

No. of mode	Sloshing mode (Hz)				Bulging mode (Hz)						
	1st	2nd	3rd	4th	1st	2nd	3rd	4th	5th	6th	7th
Tank wall : rigid	0.14503	0.26881	0.34039	0.39868	-	-	-	-	-	-	-
Former solution	-	-	-	-	6.3146	11.537	15.551	18.577	21.472	24.348	27.510
Present solution	0.14498	0.26879	0.34037	0.39867	6.3176	11.539	15.553	18.579	21.475	24.353	27.520

($=7.988 \times 10^{-6} \text{ kgf} \cdot \text{s}^2 / \text{cm}^4$) and $\rho_L = 0.9983 \times 10^3 \text{ kg/m}^3$ ($=1.018 \times 10^{-6} \text{ kgf} \cdot \text{s}^2 / \text{cm}^4$) as in the former paper. Regarding the sloshing mode, the present solution gives slightly smaller values than those obtained by assuming the side wall of the tank to be rigid. On the other hand, with regard to the bulging mode, the present solution gives slightly larger values than those in the former paper. These results mean that considering tank flexibility for sloshing of the free surface has the same effect as a slight increase in tank diameter, and considering free surface oscillation for bulging of the tank shell seems to reduce the effect of virtual mass. That is, the degree of a coupled vibration is understood to be very weak. These tendencies of natural frequencies seem to agree well with those in the

free vibration analysis for the vertical direction presented by Bauer et al.⁽¹¹⁾ and Kondo⁽¹²⁾. Moreover, the differences in Table 1 being small, the numerical calculations are worked with the same number of finite elements and the same number of terms involving the Bessel function as in the former paper.

Figure 2 shows the 1st ~ 3rd modes of radial displacement, liquid pressure and wave height obtained by Eq.(26) for the tank having the same conditions as those for Table 1. In this figure, as the dimensions of displacement of tank shell and wave height differ from one of liquid pressure, the unit cm is used for displacement and wave height and the unit kgf/cm^2 for liquid pressure. As the relationship between the wave height η and the liquid

Fig.2 Natural modes of radial component of the displacement, liquid pressure and wave height

pressure p is $p = \rho_L g \eta = 10^{-5} \eta$ according to the above-mentioned dimensions, the liquid pressure and the displacement are expressed with the enlarged scale by order of 10^3 for sloshing modes. From this figure, modes of the wave height and liquid pressure are understood to agree very well with those of Ref. (9) obtained by assuming the tank wall to be rigid. On the other hand, no such enlarged scale is used for bulging modes, so that their wave height are expressed by order of 10^3 greater than those for sloshing modes. In these bulging modes, the liquid pressure and displacement modes agree well with those in the former paper.

For the seismic response analysis using a response spectrum, the characteristics of the response spectrum may not be specified until the concrete stage of design. Therefore, the same acceleration response spectrum of its magnitude of $S_{AC}=100 \text{ cm/s}^2$ (gal) is used for each natural mode $c=1$ to 18, and the same dimensions of a tank as in Table 1 are used. The degree of convergence with respect to the number of natural modes is examined by comparing the result retaining eighteen natural modes $c=1$ to $c=18$ with that retaining $c=1$ to $c=14$, and it has been confirmed that both agree well except for slight minor differences of the acceleration response near the free end of the tank. Figure 3 shows results of the seismic response analysis, where the solid line is for the present solution, the broken line for the former solution, and the double circle for that of Ref. (9). From this figure, the wave height is understood to be little affected by the flexibility of tank wall. There is a considerable difference between the solid line and the broken line concerning the liquid pressure and displacement, although they depend on the characteristics of the seismic wave used for input. It is understood that this difference comes from the response for sloshing because the former paper does not include the effect of sloshing. Besides, when the effect of sloshing is omitted in the present paper, it has been assured that the present

Fig.3 Seismic response analysis by the response spectrum method

solution quite agreed with the former one.

3.2 Effect of the tank sizes (ratio of the tank height to radius)

Table 2 shows the comparison between the present solution and the former one with respect to natural frequencies by taking ratios $H/R=2, 1, 0.5$ as parameters; the remaining conditions of a tank in this numerical example are $R=1000 \text{ cm}$, $h=1 \text{ cm}$, $E=206 \text{ GPa}$ ($=2.1 \times 10^5 \text{ kgf/cm}^2$), $\nu=0.3$, $\rho=7.875 \times 10^3 \text{ kg/m}^3$ ($=8.03 \times 10^{-6} \text{ kgf} \cdot \text{s}^2/\text{cm}^4$), $\rho_L=1.00 \times 10^3 \text{ kg/m}^3$ ($=1.020 \times 10^{-6} \text{ kgf} \cdot \text{s}^2/\text{cm}^4$) and $H=1$ (full tank). The results are the same as in Table 1, and show that the degree of the coupled effect between sloshing and bulging is very small for this range of tank sizes. Figure 4 shows how the first mode of displacement, liquid pressure and wave height for sloshing and bulging change when the ratio H/R is taken 2, 1, and 0.5. As for the sloshing mode, when the ratio H/R is increased, the displacement mode becomes more prominent than the liquid pressure mode, and the liquid pressure mode of the bottom plate becomes immaterial. On the other hand, as for the bulging mode, the present solution

Table 2 Natural frequencies of full tank versus the ratio H/R , 1st circumferential mode (rigid-free)

H/R	No. of mode	Sloshing mode (Hz)			Bulging mode (Hz)				
		1st	2nd	3rd	1st	2nd	3rd	4th	5th
2	Tank wall : rigid	0.21364	0.36381	0.46032	-	-	-	-	-
	Former solution	-	-	-	3.6884	11.436	14.339	17.928	19.630
	Present solution	0.21342	0.36376	0.46029	3.6921	11.439	14.339	17.928	19.630
1	Tank wall : rigid	0.20846	0.36380	0.46032	-	-	-	-	-
	Former solution	-	-	-	7.2886	13.884	19.419	23.493	27.118
	Present solution	0.20838	0.36378	0.46030	7.2912	13.885	19.420	23.493	27.118
0.5	Tank wall : rigid	0.18217	0.36206	0.46023	-	-	-	-	-
	Former solution	-	-	-	11.834	21.306	28.268	33.809	39.297
	Present solution	0.18212	0.36204	0.46022	11.837	21.307	28.269	33.810	39.298

Fig.4 Effect of the ratio H/R on the sloshing and bulging modes

agrees well with the former one except for small differences of the wave height. When the ratio H/R is decreased, the radial deformation near the bottom plate of a tank becomes prominent.

For the seismic response analysis by means of a response spectrum method, the numerical calculation is made with the same acceleration response spectrum value for each degree of freedom assumed as $S_{Ac}=100 \text{ cm/s}^2$ (gal). In Fig.5, the solid line and the broken line show responses from the present solution and those from the former solution, respectively. As the difference between the solid line and the broken line is due to sloshing of the free surface, it is understood that the effect on the response due to sloshing of the free surface increases as the ratio H/R decreases. The conditions of the tank in this figure are the same as those used for Table 2.

3.3 Effect of the thickness of tank side wall

Table 3 shows the comparison between the present solution and the former one with respect to natural frequencies by taking ratios $h/R=2 \times 10^{-3}$, 1×10^{-3} , 0.5×10^{-3} as parameters. The remaining conditions of a tank are the same as those used for Table 2. From this table, it is understood that the difference between the present solution and the former one is very small.

As known from the foregoing, the coupled effect of sloshing and bulging is sufficiently small for practical purposes, and hence it seems adequate to analyze sloshing and bulging separately. However, the present solution seems to be more reasonable because it can give the seismic

responses for sloshing and bulging simultaneously.

4. Conclusions

The following conclusions may be obtained from the foregoing considerations. (1) The natural frequencies for sloshing mode from the present solution are slightly lower than those obtained by assuming the tank wall to be rigid. These results mean that consideration of tank flexibility has the same effect on sloshing mode as a slight increase in tank diameter. On the other hand, as the natural frequencies for bulging mode in the present solution are slightly greater than those from the former paper, it can be said that consideration of free surface oscillation for bulging of the tank wall seems to reduce the effect of virtual mass.

(2) The degree of coupled effect of sloshing and bulging is shown to be very weak according to this investigation. It can be said that the analysis for sloshing of the free surface may be made by assuming the tank wall to be rigid as in Ref. (9), and the analysis for bulging of the tank shell may be made by ignoring the effect of sloshing. However, the present solution seems to be a more reasonable seismic response analysis method because it can give the seismic responses for sloshing and bulging simultaneously.

(3) For an earthquake-proof design, the response due to sloshing of the free surface cannot be omitted as the ratio H/R decreases, that is, the height of a tank becomes smaller relative to its diameter, although it is affected by the characteris-

Table 3 Natural frequencies of full tank versus the ratio h/R , 1st circumferential mode (rigid-free)

h/R	No. of mode	Sloshing mode (Hz)			Bulging mode (Hz)				
		1st	2nd	3rd	1st	2nd	3rd	4th	5th
2×10^{-3}	Tank wall : rigid	0.20846	0.36380	0.46032	-	-	-	-	-
	Former solution	-	-	-	10.216	19.270	26.644	31.808	36.330
	Present solution	0.20842	0.36379	0.46031	10.218	19.270	26.645	31.808	36.330
1×10^{-3}	Tank-wall : rigid	0.20846	0.36380	0.46032	-	-	-	-	-
	Former solution	-	-	-	7.2886	13.884	19.419	23.493	27.118
	Present solution	0.20838	0.36378	0.46030	7.2912	13.885	19.420	23.493	27.118
0.5×10^{-3}	Tank wall : rigid	0.20846	0.36380	0.46032	-	-	-	-	-
	Former solution	-	-	-	5.1930	9.9756	14.084	17.239	20.229
	Present solution	0.20829	0.36375	0.46029	5.1966	9.9772	14.085	17.240	20.230

(a) $H/R = 2$

(b) $H/R = 1$

(c) $H/R = 0.5$

Fig.5 Effect of the ratio H/R on the seismic responses

tics of the seismic wave used for input.

(4) The present solution is an approximate one obtained by using a finite number of natural modes based on an energy method. However, as the present solution has sufficient accuracy for practical purpose and moreover can analyze a tank shell by a finite element method, it is understood that it is a practical and useful method of an earthquake-proof design for an actual cylindrical liquid storage tank whose wall thickness varies with height.

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