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Finite Amplitude Waves of an Elastic Plate Covering the Surface of a Fluid Layer*

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A free surface of an incompressible ideal fluid of finite depth is covered with a thin elastic plate subject to an in-plane force due to the transverse displacement. It is found from the relation between the wave number and the frequency for linear waves that the rigidity and the tensile in-plane force of the plate make the wave profile capillary-like, and the compressive in-plane force makes the wave profile gravity-like. An approximate solution using the method of multiple time scales is sought for a wave with small but finite amplitude. In case of rubber plates the phase speed of non-linear waves increases with an increasing amplitude. It is known from the third order expansion that there are two different waves and three different resonance states.

1. Introduction

A free surface of water in a reservoir is easily contaminated by the dust. In order to prevent such a trouble, covering the water surface with a thin plate is useful. In this case, however, waves can be simultaneously propagated in the plate and near the surface of the fluid under the action of some external disturbance. The free surface of a liquid is extensible and compressible. But, even if an elastic plate covering the surface of a liquid is sufficiently thin, longitudinal deformations can not be caused by comparatively small forces being needed to bend the plate. Then waves on a plate covering the water would be different from waves on the free surface which have been satisfactorily investigated by many workers⁽¹⁾. Thus whether waves on the plate covering the surface of a liquid are gravity-like or capillary-like is one of the questions to be answered. Under what condition such waves occur and what is the relation between the wave velocity and the wave frequency.

In order to study characteristics of waves in our problem, it is assumed that the liquid is incompressible and inviscid, and the plate is elastic and very thin. In this analysis nonlinearity due to longitudinal deformations resulting from a transverse displacement are taken into account, which is well known in the theory of nonlinear flutter problems⁽²⁾. Of course, nonlinear terms associated with fluid motion are also retained.

2. Basic Equations and Boundary Conditions

A liquid layer with a constant depth h is at rest under the action of gravity. The surface of the liquid layer is covered

with a thin elastic plate with a thickness h_p . x -axis is taken along the free surface of the undisturbed liquid layer and y -axis is vertically taken from its surface as shown in Fig.1. The thin elastic plate covering a liquid layer is caused to oscillate under the action of some external disturbance. The motion of the plate and the liquid is assumed to be two-dimensional. We wish to find the height of the plate η as a function of time t and co-ordinate x . The fluid is considered incompressible and inviscid. If the motion of the fluid is started from rest, there will be a velocity potential ϕ . Then the flow of the fluid is governed by the following equation:

$$\left. \begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= 0 \\ -h \leq y \leq \eta \\ -\infty < x < \infty \end{aligned} \right\} \quad \dots\dots\dots(1)$$

We now consider the boundary conditions on this equation. The normal velocity at the wall is equal to zero,

$$\left. \begin{aligned} y = -h: \\ \frac{\partial \phi}{\partial y} &= 0 \end{aligned} \right\} \quad \dots\dots\dots(2)$$

One of the boundary conditions is found by the equation of motion for the elastic plate taking the hydrodynamical forces into account. The elastic plate is subject to a force due to the pressure from the fluid. The pressure acting on the plate is given as

$$\rho g \eta + \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right\}, \quad \dots\dots\dots(3)$$

where, g is the gravitational acceleration

Fig.1 Definition sketch

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and ρ is the density of the fluid. Then the equation of the plate with the density ρ_p , the stiffness D , the initial in-plane force N and Young's modulus E is obtained as follows:

$$y=\eta: \quad D \frac{\partial^4 \eta}{\partial x^4} - \left(N + \frac{E h_p}{2\lambda} \int_0^\lambda \left(\frac{\partial \eta}{\partial x} \right)^2 dx \right) \frac{\partial^2 \eta}{\partial x^2} + \rho_p h_p \frac{\partial^2 \eta}{\partial t^2} + \rho g \eta + \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) = 0, \quad \dots (4)$$

where, λ is the wave length of a disturbance. The kinematic condition for the normal velocity of the fluid on the plate to be equal to the velocity of the plate itself is given as

$$y=\eta: \quad \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}, \quad \dots (5)$$

The conditions (4) and (5) are assumed to hold at the neutral surface of the deformed plate, that is, the thickness of the plate is neglected.

3. Nondimensional differential system

The governing equations obtained in the previous section contain terms with a very large coefficient such as Young's modulus E and with a very small coefficient such as plate thickness h_p . Thus, in order to make easy an estimate of magnitude of each term, the governing equations are nondimensionalized in the following. Distances are nondimensionalized by the wave number of a disturbance K as

$$x = \frac{1}{K} x^*, \quad y = \frac{1}{K} y^*, \quad h = \frac{1}{K} h^*, \quad \eta = \frac{1}{K} \eta^*$$

, velocities by the velocity of the gravity wave $\sqrt{g/K}$, time by $1/\sqrt{gK}$ as $t = t^*/\sqrt{gK}$ and the velocity potential by $\sqrt{g/K}/K$ as $\phi = \sqrt{g/K} \phi^*/K$. Here, the symbol * denotes a nondimensional quantity. If these nondimensional quantities are introduced in our differential system, Eqs. (1)-(5) are rewritten as follows:

$$\phi_{xx} + \phi_{yy} = 0, \quad -h \leq y \leq \eta, \quad -\infty < x < +\infty, \quad \dots (6)$$

$$y=\eta:$$

$$\bar{D} k^4 \eta_{xxxx} - \left(\bar{N} + \frac{E}{4\pi} \int_0^{2\pi} \eta_x^2 dx \right) k^2 \eta_{xx} + \bar{\rho} k \eta_{tt} + \eta + \phi_t + \frac{1}{2} (\phi_x^2 + \phi_y^2) = 0, \quad \dots (7)$$

$$\eta_t + \phi_x \eta_x = \phi_y, \quad \dots (8)$$

$$y=-h:$$

$$\phi_y = 0, \quad \dots (9)$$

where,

$$\bar{D} = E / (12(1-\sigma^2)), \quad E = E / \rho g h_p, \quad \bar{\rho} = \frac{\rho_p}{\rho}, \quad \bar{N} = \frac{N}{\rho g h_p^2}.$$

Here, σ is Poisson's ratio, k is the wave number nondimensionalized by the thickness of the plate, $k = h_p K$. Subscripts x and t in Eqs. (6)-(9) denote differentiations with respect to x and t , respectively. For simplicity, the symbol * is neglected in Eqs. (6)-(9). Eqs. (6)-(9) are nondimensional equations governing the wave motion on the plate covering a liquid layer. The governing equations depend on five dimen-

sionless parameters, i.e. stiffness $\bar{D}$, Young's modulus E , in-plane force $\bar{N}$, density $\bar{\rho}$ and the depth of the liquid layer h which are made dimensionless. This fact is entirely different from the case of the water waves depending only on two dimensionless parameters, that is, the surface tension coefficient and the depth of the layer. Thus our problem is very complicated.

4. Linear Waves

A wave with a small amplitude δ is assumed to arise under the action of some external disturbance. Expanding Eqs. (6)-(9) with respect to δ and retaining only terms of the first power of δ , we have

$$\phi_{xx} + \phi_{yy} = 0, \quad \dots (10)$$

$$y=0:$$

$$\bar{D} k^4 \eta_{xxxx} - \bar{N} k^2 \eta_{xx} + \bar{\rho} k \eta_{tt} + \eta + \phi_t = 0, \quad \dots (11)$$

$$\eta_t - \phi_y = 0, \quad \dots (12)$$

$$y=-h:$$

$$\phi_y = 0, \quad \dots (13)$$

If the wave is initially sinusoidal, the initial condition becomes

$$t=0:$$

$$\eta = \delta \cos x, \quad \dots (14)$$

Solving Eq. (10) under conditions, Eqs. (12) and (13), we have

$$\eta = \delta \cos \theta, \quad \dots (15)$$

$$\phi = \delta A \cosh(y+h) \sin \theta, \quad \dots (16)$$

where, θ is the phase which is related with time and coordinate as

$$\theta = x + \sigma_0 t, \quad \dots (17)$$

σ_0 is the phase velocity. A is a constant to be determined later. Substituting Eqs. (15)-(17) into Eqs. (11) and (12), the solution of the linear wave is obtained as

$$\eta = \delta \cos(x + \sigma_0 t), \quad \dots (18)$$

$$\phi = -\sigma_0 \frac{\cosh(y+h)}{\sinh h} \sin(x + \sigma_0 t), \quad \dots (19)$$

$$\sigma_0^2 = (1 + \bar{N} k^2 + \bar{D} k^4) \frac{\tanh h}{1 + \bar{\rho} h \tanh h}, \quad \dots (20)$$

Fig.2 The relation between phase velocity and wave length in case of a rubber plate

This solution is identical to that of the water wave. It should, however, be noticed that the dispersion relation, Eq. (20), is quite different from that of the water wave. If the density, the stiffness and the in-plane force of the plate tend to zero, Eq. (20) becomes the dispersion relation of the water wave. We examine the characteristics of the linear waves on the basis of Eq. (20) obtained above in the following.

Eq. (20) which relates the phase velocity with the wave length is of importance, because the group velocity giving the velocity of propagation of the wave is found from Eq. (20). Then in the case of parameters corresponding to a plate made of a usual material, the characteristics of the linear wave are examined using Eq. (20). Eq. (20) can be rewritten in the dimensional form as

$$\sigma_0^2 = \frac{\left\{ g \frac{\lambda}{2\pi} + \frac{N}{\rho} \frac{2\pi}{\lambda} + \frac{E h_p^2}{12(1-\sigma^2)\rho} \left(\frac{2\pi}{\lambda} \right)^4 \right\} \tanh \frac{2\pi h}{\lambda}}{\left(1 + \frac{\rho_p h_p}{\rho} \frac{2\pi}{\lambda} \tanh \frac{2\pi h}{\lambda} \right)}, \quad \dots (21)$$

where the symbol $\sim$ is used to identify the dimensional quantity. For the case of the parameters corresponding to a rubber plate, the relations between the phase velocity and the wave length are shown in Fig. 2 in the dimensional form. In this example, the value of $2\pi h/\lambda$ is larger than 2. Then the effects of the depth of the layer on this relation are negligible. The effects of the weight of the plate are also ignored since h_p/λ is very small. In case of the rubber plate, σ_0 is mainly determined by the value in the brace in the numerator on the right-hand side of Eq. (21). When the plate is subject to a tensile in-plane force, i.e. N is positive, only the first term in that brace is dominant on the curves which are upward convex in Fig. 2. Meanwhile, the second and third terms in that brace are dominant on the curves which are downward convex. The first term and the second term in that brace of Eq. (21) correspond to the gravity waves and the capillary waves, respectively. The character of the gravity waves is such that the group velocity is smaller than the phase velocity. In the case of the capillary waves, it is opposite⁽³⁾. The group velocity σ_g is given by

$$\sigma_g = \sigma_0 - \lambda \frac{d\sigma_0}{d\lambda}, \quad \dots (22)$$

The group velocity is represented by the intercept made by the tangent on the σ_0 -axis. Thus on the right side of the point at which σ_0 takes a minimum value in Fig. 2, the group velocity is smaller than the phase velocity, but on the left side vice versa. Therefore, the tensile in-plane force and the stiffness of the plate make the character of the wave on the plate capillary-like. In particular, since the value of the stiffness is always positive, the wave form is made strongly capillary-like by the stiffness. In case of the value of N being negative, the in-plane force tends to make the σ_0 - λ curve upward convex. In such a case, the wave becomes more gravity-like than the water wave on the free surface. It is found from Fig. 2 that if the value of a

compressive in-plane force becomes large, the group velocity can be negative. This property is quite different from that of the water waves on the free surface. The nondimensional in-plane force in Fig. 2 corresponds to the value from 700 dyn/cm to

1000 dyn/cm, that is, of the order of 1 gr-wt/cm. That value is relatively small. Then, it is found that the effect of the in-plane force on the dispersion relation is very strong and the wave motion on a plate covering the water surface is much affected by the slack of the elastic sheet. Fig. 3 corresponds to the case of a steel plate. The phase velocity in this case is larger than that in the rubber plate by a single digit. In this case the wave length at which the phase velocity attains a minimum value is larger than that in the rubber plate. However, the character of the wave motion in case of the steel plate is found from Figs. 2 and 3 to be almost similar to that in case of the rubber plate. An example for the rubber plate covering the liquid layer with a depth of 5 cm is shown in Fig. 4. This example is the case of the liquid layer,

Fig. 3 The relation between phase velocity and wave length in case of a steel plate

Fig. 4 The relation between phase velocity and wave length in case of a rubber plate covering a shallow liquid layer

being shallow. The effects of the depth of the layer on the dispersion relation appear in the region in which the wave motion becomes gravity-like, i.e. in such region that $2\pi h/\lambda$ is less than unity. Since there exists an inflexion point on the $\sigma_0-\lambda$ curve in the gravity-like region as shown in Figs. 2, 3 and 4, two tangents of the $\sigma_0-\lambda$ curve at a point on the σ_0 -axis can be drawn. Therefore, it is found that there are two waves with the same group velocity. This property is similar to that of the gravity wave⁽³⁾

5. Nonlinear Waves

5.1 Solving method

In the previous section, the wave amplitude has been assumed to be infinitesimal. The character of a wave with a small but finite amplitude is examined to the order of δ^3 . The method of multiple time scales⁽⁴⁾ is employed by introducing new time variables

$$\eta(t, x) = \delta\eta_1 + \delta^2\eta_2 + \delta^3\eta_3 + O(\delta^4) \quad (23)$$

η , ϕ and $\partial/\partial t$ are assumed to make the following uniformly valid expansion:

$$\phi(t, x, y) = \delta\phi_1 + \delta^2\phi_2 + \delta^3\phi_3 + O(\delta^4) \quad (24)$$

$$\partial/\partial t = \partial/\partial t_0 + \delta^2 \partial/\partial t_2 + O(\delta^4) \quad (25)$$

$$t_0 = t, \quad t_2 = \delta^2 t \quad (26)$$

Here, η_1 , η_2 and η_3 are functions of t_0 , t_2 and x . ϕ_1 , ϕ_2 and ϕ_3 are functions of t_0 , t_2 , x and y . If these expansions are substituted into the governing equations considered, a differential system governing the linear wave in the previous section is derived to the first order of δ .

5.2 Solution for the second approximation

A differential system for the second approximation is obtained by substituting Eqs. (24)–(26) into Eqs. (6)–(9) and equating coefficients of second power of δ to zero as follows:

$$\phi_{22} + \phi_{2y} = 0, \quad (27)$$

$$y = 0:$$

$$\begin{aligned} & \bar{D}k^2\eta_{2222} - \bar{N}k^2\eta_{222} + \bar{\rho}k\eta_{210t_0} + \eta_2 + \phi_{2t_0} \\ & = -\phi_{1y t_0}\eta_1 - \frac{1}{2}(\phi_{1z}^2 + \phi_{1y}^2), \end{aligned} \quad (28)$$

$$\eta_{2t_0} - \phi_{2y} = -\phi_{1z}\eta_{1z} + \phi_{1y}\eta_{1y}, \quad (29)$$

$$y = -h:$$

$$\phi_{2y} = 0. \quad (30)$$

Taking the initial condition, Eq. (14), into account, the solutions for this differential system are found as

$$\eta_2 = B_2 \cos 2\theta, \quad (31)$$

$$\phi_2 = -\sigma_0 A_2 \frac{\cosh 2(y+h)}{\sinh 2h} \sin 2\theta + C_2(t_0), \quad (32)$$

where, A_2 , B_2 and C_2 are constants. Substituting these solutions into Eqs. (27)–(30), these constants are determined as

$$A_2 = B_2 - \frac{1}{2} \coth h, \quad (33)$$

$$B_2 = \frac{\sigma_0^2}{\gamma^2 - 4\sigma_0^2} \left(\tanh 2h - 2 \coth h - \frac{\tanh 2h}{2 \sinh^2 h} \right), \quad (34)$$

$$C_2 = -\frac{\sigma_0^2}{4} (\coth^2 h - 1) t_0, \quad (35)$$

where,

$$\gamma^2 = 2(16\bar{D}k^4 + 4\bar{N}k^2 - 4\bar{\rho}k\sigma_0^2 + 1) \tanh 2h, \quad (36)$$

5.3 Solutions for third approximation

A differential system for third approximation is derived by the procedure similar to that in section 5.2 as follows:

$$\phi_{322} + \phi_{3yy} = 0, \quad (37)$$

$$y = 0:$$

$$\begin{aligned} & \bar{D}k^4\eta_{33333} - \bar{N}k^2\eta_{333} + \bar{\rho}k\eta_{310t_0} + \eta_3 + \phi_{3t_0} \\ & = \frac{E}{4\pi} \left(\int_0^{2\pi} \eta_{1z}^2 dx \right) k^2\eta_{1zz} - 2\bar{\rho}k\eta_{1t_0t_2} - \phi_{2y t_0}\eta_1 - \phi_{1y t_0}\eta_2 \\ & - \frac{1}{2}\phi_{1y t_0}\eta_1^2 - \phi_{1t_2} - \phi_{1z}\phi_{2z} - \phi_{1y}\phi_{2y} - (\phi_{1z}\phi_{1zy} + \phi_{1y}\phi_{1yy})\eta_1, \end{aligned} \quad (38)$$

$$\eta_{3t_0} - \phi_{3y} = -\eta_{1t_2} - \phi_{1z}\eta_{2z} - \phi_{2z}\eta_{1z} - \phi_{1z}\eta_{1y}\eta_{1z} + \phi_{2yy}\eta_1$$

$$+ \phi_{1yy}\eta_2 + \frac{1}{2}\phi_{1yy}\eta_1^2, \quad (39)$$

$$y = -h:$$

$$\phi_{3y} = 0. \quad (40)$$

Substituting the first and the second approximate solutions into the nonlinear terms on the right-hand side of Eqs. (38) and (39), we obtain:

The right-hand side of Eq. (38)

$$= \left\{ \sigma_0 (\coth h + 2\rho k) \frac{d\sigma_2}{dt_2} + \rho\sigma_0^2 \right\} \cos \theta + q\sigma_0^2 \cos 3\theta, \quad (38)'$$

The right-hand side of Eq. (39)

$$= \left(\frac{d\sigma_2}{dt_2} - \gamma\sigma_0 \right) \sin \theta - s\sigma_0 \sin 3\theta, \quad (39)'$$

where,

$$p = -\frac{\bar{E}k^2}{4\sigma_0^2} + \frac{1}{2}B_2 - \frac{5}{8}\coth h + A_2(1 - \coth h \coth 2h), \quad (41)$$

$$q = \frac{1}{2}B_2 + \frac{1}{8}\coth h + (3 - \coth h \coth 2h)A_2, \quad (42)$$

$$r = \frac{1}{2}(\coth h + 2\coth 2h)B_2 - \frac{1}{2}\coth h \coth 2h + \frac{3}{8}, \quad (43)$$

$$s = \frac{3}{2}B_2 \coth h + 3A_2 \coth 2h + \frac{3}{8}. \quad (44)$$

Thus, the differential system for third approximation contains secular terms of a form proportional to $\sin \theta$ and $\cos \theta$ on the right-hand side of Eqs. (38) and (39). Then, the phase θ is modulated by $\sigma_2(t_2)$. The phases on the right-hand side of Eqs. (38) and (39) are rewritten by

$$\theta = x + \sigma_0 t + \sigma_2(t_2). \quad (45)$$

Since the wave with the phase θ should be eliminated from η_3 , solutions for third approximation are assumed as

$$\eta_3 = B_3 \cos 3\theta, \quad (46)$$

$$\phi_3 = A_{31} \frac{\cosh(y+h)}{\sinh h} \sin \theta + A_{33} \frac{\cosh 3(y+h)}{\sinh 3h} \sin 3\theta, \quad (47)$$

Substituting these solutions into Eqs. (38) and (39) and equating coefficients of the term $\sin \theta$ and $\cos \theta$ to zero, $d\sigma_2/dt_2$ and A_{31} are obtained as

$$\frac{d\sigma_2}{dt_2} = \frac{\sigma_0}{2} \frac{\gamma - p \tanh h}{1 + \rho k \tanh h}, \quad (48)$$

$$A_{31} = -\frac{d\sigma_2}{dt_2} + \gamma\sigma_0. \quad (49)$$

Also, equating coefficients of the term $\sin 3\theta$ and $\cos 3\theta$ to zero, A_{33} and B_3 are obtained as

$$A_{33} = -\sigma_0 \left(B_3 - \frac{1}{3} s \right), \quad (50)$$

$$B_3 = \frac{3\sigma_0^2 (q \tanh 3h - s)}{\gamma_s^2 - 9\sigma_0^2}, \quad (51)$$

where

$$\gamma_s^2 = 3(81\bar{D}k^4 + 9\bar{N}k^2 - 9\sigma_0^2 \rho k + 1) \tanh 3h, \quad (52)$$

Eqs. (45)-(52) are solutions of the order of δ^3 . Since the phase θ is expressed using real time t as

$$\theta = x + \left(\sigma_0 + \frac{d\sigma_2}{dt_2} \delta^2 \right) t,$$

the phase velocity is written as

$$\sigma = \sigma_0 + \frac{d\sigma_2}{dt_2} \delta^2 = \sigma_0 \left(1 + \frac{1}{2} \frac{\gamma - p \tanh h}{1 + \rho h \tanh h} \delta^2 \right), \quad (53)$$

Therefore, the frequency of a wave on the plate covering the water surface is amplitude dependent. In case of a rubber plate as shown in Fig. 2, the value of $d\sigma_2/dt_2$ of the wave with the nondimensional wave length $\lambda=300$, which is capillary-like, is equal to 448.4. A value with $\lambda=1000$, which is gravity-like, is equal to 3276.5. Both these values are positive. Then the wave frequency increases with an increasing wave amplitude.

6. Resonance Condition

Although the second-order solution and the third-order solution obtained in sections 5.2 and 5.3 are valid for a wide range of wave numbers, these solutions are found to break down when the wave number k satisfies the following relations:

$$\gamma_s^2 - 4\sigma_0^2 = 0, \quad \gamma_s^2 - 9\sigma_0^2 = 0, \quad (54)$$

In such a case, it is impossible to express a fundamental wave in terms of a kind of wave as shown in Eq. (15)⁽⁶⁾. This fact is well known in the theory of the mutual action between a gravity wave and a capillary wave. In this case, a fundamental wave should be expressed by the sum of two kinds of waves as follows:

$$a \cos \theta + b \cos 2\theta, \quad a \cos \theta + b \cos 3\theta.$$

Here, the ratio of a to b is a constant. Then two kinds of waves whose phases are different interact with each other and there develops a kind of resonance state. On the basis of the first equation of Eq. (54), we examine under what condition such state can be attained. Rewriting the first equation of Eq. (54) using the previous solutions, we have

$$(16\bar{D}k^4 + 4\bar{N}k^2 + 1) \tanh 2\bar{h}k - 2(1 + 2\bar{\rho}k \tanh 2\bar{h}k)(\bar{D}k^4 + \bar{N}k^2 + 1) \frac{\tanh \bar{h}k}{1 + \bar{\rho}k \tanh \bar{h}k} = 0, \quad (55)$$

where, $\bar{h}$ is the depth of the liquid layer nondimensionalized by the thickness of the plate.

Let us consider the properties of resonance condition, Eq. (55). In case of the liquid layer being deep, i.e. $\bar{h} \rightarrow \infty$, Eq. (55) is reduced to

$$12\bar{D}\rho k^5 + 14\bar{D}k^4 + 2\bar{N}k^2 - 3\bar{\rho}k - 1 = 0, \quad (56)$$

Eq. (56) has always a root irrespective of the sign of $\bar{N}$, the stiffness and the

weight of the plate. Thus, in case of a deep layer a resonance state always develops. A concrete example of resonance state for a rubber plate is shown in Fig. 5. In case of large wave number, i.e. small k , Eq. (55) is reduced to

$$\bar{h} = \frac{-3\rho + \sqrt{9\rho^2 + 12\bar{N}}}{2}, \quad (57)$$

Then a resonance state occurs in the depth given by Eq. (57). If the in-plane force is tensile, Eq. (55) has a root in case of long waves. It is given by a point at which the curve in Fig. 5 intersects $\bar{h}$ -axis. In case of a compressive in-plane force, a resonance state is likely to occur in the region of long waves if the magnitude of the compressive in-plane force is smaller than $3\rho^2/4$. It is given by a point at which the curve in Fig. 6 intersects $\bar{h}$ -axis in case of a rubber plate. Finally, in case of a liquid layer being shallow, i.e. $\bar{h} \rightarrow 0$, we obtain from Eq. (55)

$$k = \sqrt{-\frac{\bar{N}}{5\bar{D}}}. \quad (58)$$

Fig. 5 The resonance condition in case of a rubber plate covering a deep liquid layer

Fig. 6 The resonance condition in the long wave on a rubber plate

Fig.7 The resonance condition in case of a rubber plate subject to a compressive in-plane force

Then, at this wave number a resonance state always occurs if the in-plane force is compressive. An example is shown in Fig.7 for the case of a rubber plate. Thus, there are three kinds of resonance states. They are shown in Figs.5,6 and 7 in a concrete example of the rubber plate.

7. Conclusions

Motion of a thin elastic plate covering a liquid layer with a constant depth is in-

vestigated taking nonlinearities due to the plate and the fluid into account. The fluid is assumed to be ideal, incompressible and irrotational. It is found from the dispersion relation of the linear theory that the stiffness of the plate makes wave motion capillary-like. But wave motion becomes gravity-like or capillary-like depending on whether the in-plane force is tensile or compressive respectively. Introducing the multiple time scales, solutions are obtained taking nonlinearities of the third order of the wave amplitude into account. It is found from these solutions that there develops a resonance state. The presence of that state is shown for the case of a rubber plate.

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