Vibrations of Cylindrical Shells with Varying Thickness*

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The asymmetric vibrations of cylindrical shells with variable thickness are studied by the improved shell theory and the classical one. The equations of vibration and the boundary conditions are obtained from stationary conditions of the Lagrangian of vibrations of cylindrical shells. The solutions are obtained in such a case that the thickness of cylindrical shells, built in at one end, varies exponentially along its axis. After numerical calculations, all boundary conditions at the other end are discussed, the effects of parameters on vibrations are investigated and the results by the improved theory are compared with those by the classical one.

1. Introduction

Numerous papers have been written on the vibrations of cylindrical shells, and they are briefly reviewed in Greenspon's paper⁽¹⁾, in which the comments on the classical theory and the improved one are also made. In the studies of the so-called asymmetric vibrations of cylindrical shells by the classical theory, three components of the displacement at any point on a middle surface are used to define the deformation of a shell. According to the improved theory, which is sometimes called the thick shell theory, 5 or 6 components of deformation are adopted to consider the rotatory inertia and the shear deformation.

Among the studies by the classical theory, there is an excellent work by Warburton⁽²⁾, papers by Forsberg⁽³⁾ and the authors⁽⁴⁾. Mirsky⁽⁵⁾ has presented a creative work on the improved theory, in which he has researched the propagation of waves in orthotropic thick cylindrical shells considering 6 components of deformation. Since then, most investigators follow Mirsky's way, and some variations are worked out by Ziv and Perl⁽⁶⁾ (Impulse), Huang⁽⁷⁾ (forced motions of viscoelastic cylindrical shells) and Warburton and Soni⁽⁸⁾ (resonant response). This paper also follows basically Mirsky's method.

As for some other particular topics, Gazis⁽⁹⁾ solved the plane-strain vibration and Nelson et al.⁽¹⁰⁾ presented an approximate solution of a laminated cylinder.

Whereas some parts of machines consist of cylindrical shells with variable, thick wall, most of former researches restrict themselves to vibrations of shells with uniform wall and ones dealing with cylindrical

shells of variable thickness are a few, as one of which is the study on asymmetric vibrations of a cylindrical shell with the thickness varied stepwise, according to the classical theory by the authors⁽¹¹⁾. In former papers, a few of boundary conditions such as supported ends are treated, though the effects of all the boundary conditions should be considered to develop an improved theory.

In this paper, the authors study the vibrations of a cylindrical shell with variable thickness by both the classical theory and the improved one. They elucidate the effects of combinations of boundary condition and of the parameters such as thickness ratio etc. upon frequencies and make a comparison between results by the classical theory and the ones by improved theory.

The following is a summary of the subsequent chapters. In Chapter 2, the Lagrangian of vibrations of a cylindrical shell with variable thickness is given in a general form with five components of deformation. From its stationary condition, the equations of vibration and the boundary conditions are obtained. The solution for the cylindrical shell, of which the thickness

Fig. 1

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is varied exponentially along its axis and one end is built in, are shown in Chapter 3. There are five arbitrary constants in this solution and they coincide with the number of the boundary conditions at the other end. In Chapter 4, numerical results and discussions are presented. The classical theory used in this paper is briefly explained in Appendix 2.

2. Lagrangian, equations of vibration and boundary conditions

We consider the free vibration of a cylindrical shell with variable thickness. The middle surface of the shell is shown in Fig.1. The base circle is taken at the lower end of the middle surface. Its center is denoted by 0, and the radius is R , which is

the representative length. We use the following coordinates system: x' -axis runs along a generator of the middle surface, θ is the angular coordinate and z -axis runs along radius toward the center, which constitutes the right hand system in this sequence as shown in Fig.1. We also use the nondimensional coordinate x where $x' = Rx$ and the nondimensional length μ where μR is the height of the cylindrical shell. Let the thickness at the base circle be h_0 and at any point of shell be $h(x, \theta)$, where $H = h/h_0$. Let the circular frequency and the time be p' and t , respectively. We assume the displacement components at any point along x' , θ and z -axes are expressed as $(\bar{u} + z\bar{u}_0/R) \sin p't$, $(\bar{v} + z\bar{v}_0/R) \sin p't$ and $w \sin p't$, where $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{u}_0$ and $\bar{v}_0$ are functions of x and θ . The normal and shearing strains, ϵ and γ' , and the normal and shearing stresses, σ and τ , obtained by Hooke's law are as follows:

$$\left. \begin{aligned} \epsilon_z &= \frac{1}{R} \left(\frac{\partial \bar{u}}{\partial x} + \frac{z}{R} \frac{\partial \bar{u}_0}{\partial x} \right), & \epsilon_\theta &= \frac{1}{R} \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} + \frac{z}{R} \frac{\partial \bar{v}_0}{\partial \theta} \right) / \left(1 - \frac{z}{R} \right) \\ \epsilon_z &= \frac{\partial \bar{w}}{\partial z} = 0, & \gamma_{z\theta}' &= \frac{1}{R} \left\{ \frac{\partial \bar{v}}{\partial x} + \frac{z}{R} \frac{\partial \bar{v}_0}{\partial x} + \left(\frac{\partial \bar{u}}{\partial \theta} + \frac{z}{R} \frac{\partial \bar{u}_0}{\partial \theta} \right) / \left(1 - \frac{z}{R} \right) \right\} \\ \gamma_{\theta z}' &= \frac{1}{R} \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right) / \left(1 - \frac{z}{R} \right), & \gamma_{zz}' &= \frac{1}{R} \left(\frac{\partial \bar{w}}{\partial x} + \bar{u}_0 \right) \end{aligned} \right\} \dots\dots\dots (1)$$

$$\left. \begin{aligned} \sigma_z &= \frac{E}{1-\nu^2} (\epsilon_z + \nu \epsilon_\theta), & \sigma_\theta &= \frac{E}{1-\nu^2} (\nu \epsilon_z + \epsilon_\theta), & \tau_{z\theta} &= \frac{E \gamma_{z\theta}'}{2(1+\nu)} \\ \tau_{\theta z} &= \frac{k' E \gamma_{\theta z}'}{2(1+\nu)}, & \tau_{zz} &= \frac{k' E \gamma_{zz}'}{2(1+\nu)} \end{aligned} \right\} \dots\dots\dots (2)$$

In the above relations, E , ν and k' are Young's modulus, Poisson's ratio and the shear coefficient, respectively. We assume that $\tau_{\theta z}$ and τ_{zz} have the same shear coefficient. Mirsky's relations⁽⁵⁾ of stresses and strains are slightly different from the above equations. As for displacement along z -axis, he has considered an extra term proportional to z . So, we may call the theory developed here an improved theory.

The Lagrangian in a period is expressed as follows,

$$L = (1/2) \iiint [(\gamma p'^2/g) \{(\bar{u} + z\bar{u}_0/R)^2 + (\bar{v} + z\bar{v}_0/R)^2 + \bar{w}^2\} - \sigma_z \epsilon_z - \sigma_\theta \epsilon_\theta - \sigma_{zz} \epsilon_{zz} - \tau_{z\theta} \gamma_{z\theta}' - \tau_{\theta z} \gamma_{\theta z}' - \tau_{zz} \gamma_{zz}'] R^2 (1 - z/R) dx d\theta dz \dots\dots\dots (3)$$

where γ/g is the density.

Substituting Eqs.(1) and (2) into the above equation and integrating with respect to z , we obtain,

$$\begin{aligned} -L / \left(\frac{D_0}{R^2} \right) &= \frac{1}{2} \int \int H \left[-\alpha_0^4 \left\{ \bar{u}^2 + \bar{v}^2 + \bar{w}^2 + \frac{H^2}{\beta} (\bar{u}_0^2 + \bar{v}_0^2 - 2\bar{u}\bar{u}_0 - 2\bar{v}\bar{v}_0) \right\} \right. \\ &\quad + \beta \left\{ \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right)^2 - 4\zeta \frac{\partial \bar{u}}{\partial x} \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) \right\} \\ &\quad + H^2 \left\{ \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right)^2 - 2 \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{u}_0}{\partial x} + 2 \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) \frac{\partial \bar{v}_0}{\partial \theta} + \left(\frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial \theta} \right)^2 - 4\zeta \frac{\partial \bar{u}_0}{\partial x} \frac{\partial \bar{v}_0}{\partial \theta} \right\} \\ &\quad + \beta \zeta \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial \theta} \right)^2 + H^2 \zeta \left\{ \left(\frac{\partial \bar{u}}{\partial \theta} \right)^2 + 2 \frac{\partial \bar{u}}{\partial \theta} \frac{\partial \bar{u}_0}{\partial \theta} - 2 \frac{\partial \bar{v}}{\partial x} \frac{\partial \bar{v}_0}{\partial x} + \left(\frac{\partial \bar{v}_0}{\partial x} + \frac{\partial \bar{u}_0}{\partial \theta} \right)^2 \right\} \\ &\quad \left. + k' \zeta \left\{ (\beta + H^2) \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right)^2 + \beta \left(\frac{\partial \bar{w}}{\partial x} + \bar{u}_0 \right)^2 \right\} \right] dx d\theta \dots\dots\dots (4) \end{aligned}$$

where, $\alpha_0^4 = \gamma h_0 p'^2 R^4 / g D_0$, $\zeta = (1-\nu)/2$, $\beta = 12R^2/h_0^2$

Taking first variation of Eq.(4) and integrating by parts, we obtain

$$\begin{aligned} \delta L / (D_0/R^2) &= \int_{x_1}^{x_2} \int_{\theta_1}^{\theta_2} (E_1 \delta \bar{u} + E_2 \delta \bar{v} + E_3 \delta \bar{w} + E_4 \delta \bar{u}_0 + E_5 \delta \bar{v}_0) dx d\theta \\ &\quad - \int_{\theta_1}^{\theta_2} (T_1 \delta \bar{u} + T_2 \delta \bar{v} + T_3 \delta \bar{w} + M_1 \delta \bar{u}_0 + M_2 \delta \bar{v}_0) \Big|_{x_1}^{x_2} d\theta - \int_{x_1}^{x_2} (T_4 \delta \bar{u} + T_5 \delta \bar{v} + T_6 \delta \bar{w} + M_3 \delta \bar{u}_0 \\ &\quad + M_4 \delta \bar{v}_0) \Big|_{\theta_1}^{\theta_2} dx \dots\dots\dots (5) \end{aligned}$$

where

$$\begin{aligned}
 H' &= H(\beta + H^2) \\
 E_1 &= \alpha_0^4 H \left(\bar{u} - \frac{H^2}{\beta} \bar{u}_0 \right) + \frac{\partial}{\partial x} \left[\beta H \left\{ \frac{\partial \bar{u}}{\partial x} + (1-2\zeta) \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) \right\} - H^3 \frac{\partial \bar{u}_0}{\partial x} \right] \\
 &\quad + \frac{\partial}{\partial \theta} \left[\zeta \left(\beta H \frac{\partial \bar{v}}{\partial x} + H' \frac{\partial \bar{u}}{\partial \theta} + H^3 \frac{\partial \bar{u}_0}{\partial \theta} \right) \right] \\
 E_2 &= \alpha_0^4 H \left(\bar{v} - \frac{H^2}{\beta} \bar{v}_0 \right) + \frac{\partial}{\partial x} \left[\zeta \left\{ \beta H \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial \theta} \right) - H^3 \frac{\partial \bar{v}_0}{\partial x} \right\} \right] \\
 &\quad + \frac{\partial}{\partial \theta} \left[\beta H (1-2\zeta) \frac{\partial \bar{u}}{\partial x} + H' \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) + H^3 \frac{\partial \bar{v}_0}{\partial \theta} \right] - k' \zeta H' \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right) \\
 E_3 &= \alpha_0^4 H \bar{w} + \frac{\partial}{\partial x} \left[k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial x} + \bar{u}_0 \right) \right] + \frac{\partial}{\partial \theta} \left[k' \zeta H' \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right) \right] \\
 &\quad + \beta H (1-2\zeta) \frac{\partial \bar{u}}{\partial x} + H' \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) + H^3 \frac{\partial \bar{v}_0}{\partial \theta} \\
 E_4 &= \alpha_0^4 \frac{H^3}{\beta} (-\bar{u} + \bar{u}_0) + \frac{\partial}{\partial x} \left[H^3 \left\{ -\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}_0}{\partial x} + (1-2\zeta) \frac{\partial \bar{v}_0}{\partial \theta} \right\} \right] \\
 &\quad + \frac{\partial}{\partial \theta} \left[\zeta H^3 \left(\frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{v}_0}{\partial x} + \frac{\partial \bar{u}_0}{\partial \theta} \right) \right] - k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial x} + \bar{u}_0 \right) \\
 E_5 &= \alpha_0^4 \frac{H^3}{\beta} (-\bar{v} + \bar{v}_0) + \frac{\partial}{\partial x} \left[\zeta H^3 \left(-\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{v}_0}{\partial x} + \frac{\partial \bar{u}_0}{\partial \theta} \right) \right] \\
 &\quad + \frac{\partial}{\partial \theta} \left[H^3 \left\{ \frac{\partial \bar{v}}{\partial \theta} - \bar{w} + (1-2\zeta) \frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial \theta} \right\} \right] - k' \zeta H' \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right)
 \end{aligned} \tag{6}$$

$$\begin{aligned}
 T_1 &= \beta H \left\{ \frac{\partial \bar{u}}{\partial x} + (1-2\zeta) \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) \right\} - H^3 \frac{\partial \bar{u}_0}{\partial x} \\
 T_2 &= \zeta H \left\{ \beta \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial \theta} \right) - H^3 \frac{\partial \bar{v}_0}{\partial x} \right\}, \quad T_3 = k' \beta \zeta H \left(\frac{\partial \bar{w}}{\partial x} + \bar{u}_0 \right) \\
 M_1 &= H^3 \left\{ -\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}_0}{\partial x} + (1-2\zeta) \frac{\partial \bar{v}_0}{\partial \theta} \right\}, \quad M_2 = \zeta H^3 \left(-\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{v}_0}{\partial x} + \frac{\partial \bar{u}_0}{\partial \theta} \right)
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 T_4 &= \zeta \left(\beta H \frac{\partial \bar{v}}{\partial x} + H' \frac{\partial \bar{u}}{\partial \theta} + H^3 \frac{\partial \bar{u}_0}{\partial \theta} \right) \\
 T_5 &= \beta H (1-2\zeta) \frac{\partial \bar{u}}{\partial x} + H' \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) + H^3 \frac{\partial \bar{v}_0}{\partial \theta} \\
 T_6 &= k' \zeta H' \left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v} + \bar{v}_0 \right), \quad M_3 = \zeta H^3 \left(\frac{\partial \bar{v}_0}{\partial x} + \frac{\partial \bar{u}}{\partial \theta} + \frac{\partial \bar{u}_0}{\partial \theta} \right) \\
 M_4 &= H^3 \left\{ \frac{\partial \bar{v}}{\partial \theta} - \bar{w} + (1-2\zeta) \frac{\partial \bar{u}_0}{\partial x} + \frac{\partial \bar{v}_0}{\partial \theta} \right\}
 \end{aligned} \tag{8}$$

From Eq. (5), the equations of vibration are $E_1=E_2=E_3=E_4=E_5=0$ and the boundary conditions at $x=x_1$ and x_2 are $T_1\bar{u}=T_2\bar{v}=T_3\bar{w}=M_1\bar{u}_0=M_2\bar{v}_0=0$. The other boundary conditions at $\theta=\theta_1$ and θ_2 are $T_4\bar{u}=T_5\bar{v}=T_6\bar{w}=M_3\bar{u}_0=M_4\bar{v}_0=0$, but for a closed shell in which $\theta=0\sim 2\pi$, these conditions are always satisfied. As $\bar{u}$, $\bar{v}$ and $\bar{w}$ are linear displacements and $\bar{u}_0/R$ and $\bar{v}_0/R$ are angular ones, so $T_1\sim T_6$, $M_1\sim M_4$ are forces and moments with dimension of length. It may be allowed to call hereafter T and M force and moment, respectively. For reference, the equations of vibration expressed in terms of forces and moments are shown in Appendix 1.

The case where the radius R of the middle surface becomes sufficiently large, is one of special cases of the system mentioned above. When in-plane deformation is very small compared with rotations and when $1 \ll \beta$, we can approximate Eqs. (6), (7) and (8) by utilizing the relations $\partial/R\partial x = \partial/\partial x'$ and $\partial/R\partial \theta = \partial/\partial y'$ as follows,

$$\left. \begin{aligned}
& \frac{\partial}{\partial x'} \left[\beta H \left\{ \frac{\partial \bar{u}}{\partial x'} + (1-2\zeta) \frac{\partial \bar{v}}{\partial y'} \right\} \right] + \frac{\partial}{\partial y'} \left[\beta \zeta H \left(\frac{\partial \bar{v}}{\partial x'} + \frac{\partial \bar{u}}{\partial y'} \right) \right] + \frac{\alpha_0^4 H \bar{u}}{R^2} = 0 \\
& \frac{\partial}{\partial x'} \left[\beta \zeta H \left(\frac{\partial \bar{v}}{\partial x'} + \frac{\partial \bar{u}}{\partial y'} \right) \right] + \frac{\partial}{\partial y'} \left[\beta H \left\{ (1-2\zeta) \frac{\partial \bar{u}}{\partial x'} + \frac{\partial \bar{v}}{\partial y'} \right\} \right] + \frac{\alpha_0^4 H \bar{v}}{R^2} = 0 \\
& \frac{\partial}{\partial x'} \left[k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial x'} + \frac{\bar{u}_0}{R} \right) \right] + \frac{\partial}{\partial y'} \left[k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial y'} + \frac{\bar{v}_0}{R} \right) \right] + \frac{\alpha_0^4 H \bar{w}}{R^2} = 0 \\
& \frac{\partial}{\partial x'} \left[H^3 \left\{ \frac{\partial \bar{u}_0}{\partial x'} + (1-2\zeta) \frac{\partial \bar{v}_0}{\partial y'} \right\} \right] + \frac{\partial}{\partial y'} \left[\zeta H^3 \left(\frac{\partial \bar{v}_0}{\partial x'} + \frac{\partial \bar{u}_0}{\partial y'} \right) \right] \\
& \quad - \frac{k' \zeta \beta H}{R} \left(\frac{\partial \bar{w}}{\partial x'} + \frac{\bar{u}_0}{R} \right) + \frac{\alpha_0^4 H^3}{\beta R^2} \bar{u}_0 = 0 \\
& \frac{\partial}{\partial x'} \left[\zeta H^3 \left(\frac{\partial \bar{v}_0}{\partial x'} + \frac{\partial \bar{u}_0}{\partial y'} \right) \right] + \frac{\partial}{\partial y'} \left[H^3 \left\{ (1-2\zeta) \frac{\partial \bar{u}_0}{\partial x'} + \frac{\partial \bar{v}_0}{\partial y'} \right\} \right] \\
& \quad - \frac{k' \zeta \beta H}{R} \left(\frac{\partial \bar{w}}{\partial y'} + \frac{\bar{v}_0}{R} \right) + \frac{\alpha_0^4 H^3}{\beta R^2} \bar{v}_0 = 0
\end{aligned} \right\} \dots\dots\dots (9)$$

$$\left. \begin{aligned}
T_1 &= \beta H \left\{ \frac{\partial \bar{u}}{\partial x'} + (1-2\zeta) \frac{\partial \bar{v}}{\partial y'} \right\} R, & T_2 &= \zeta \beta H \left(\frac{\partial \bar{v}}{\partial x'} + \frac{\partial \bar{u}}{\partial y'} \right) R \\
T_3 &= k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial x'} + \frac{\bar{u}_0}{R} \right) R, & M_1 &= H^3 \left\{ \frac{\partial \bar{u}_0}{\partial x'} + (1-2\zeta) \frac{\partial \bar{v}_0}{\partial y'} \right\} R \\
M_2 &= \zeta H^3 \left(\frac{\partial \bar{v}_0}{\partial x'} + \frac{\partial \bar{u}_0}{\partial y'} \right) R, & T_4 &= \zeta \beta H \left(\frac{\partial \bar{v}}{\partial x'} + \frac{\partial \bar{u}}{\partial y'} \right) R \\
T_5 &= \beta H \left\{ (1-2\zeta) \frac{\partial \bar{u}}{\partial x'} + \frac{\partial \bar{v}}{\partial y'} \right\} R, & T_6 &= k' \zeta \beta H \left(\frac{\partial \bar{w}}{\partial y'} + \frac{\bar{v}_0}{R} \right) R \\
M_3 &= \zeta H^3 \left(\frac{\partial \bar{v}_0}{\partial x'} + \frac{\partial \bar{u}_0}{\partial y'} \right) R, & M_4 &= H^3 \left\{ (1-2\zeta) \frac{\partial \bar{u}_0}{\partial x'} + \frac{\partial \bar{v}_0}{\partial y'} \right\} R
\end{aligned} \right\} \dots\dots\dots (10)$$

First two of Eq. (9), containing $\bar{u}$ and $\bar{v}$ only, are the equations of the in-plane vibration of a plate with variable thickness. The other three, including $\bar{w}$, $\bar{u}_0$ and $\bar{v}_0$ are ones of the out-plane vibrations of that plate, and they coincide with Mindlin's equations⁽¹²⁾. This is also the case for the boundary conditions (10). It is easy to express these equations in polar coordinates and we can obtain the well-known equations of vibration and the boundary conditions⁽¹³⁾.

In Appendix 2, another special case is discussed, in which $\gamma_{\theta z} = \gamma'_{z\chi} = 0$. We have obtained the equations of vibration and the boundary conditions of the classical theory in a similar manner to this chapter and they are also shown in Appendix 2.

3. Solution

We consider a cylindrical shell ($\theta=0 \sim 2\pi$, $x=0 \sim \mu$), of which the thickness varies exponentially along x -axis as $H=\exp(-kx)$. The shell is built-in at one end where $x=0$ and it vibrates with $2n$ nodal generators ($n=2, 3, 4, \dots$). We assume that $\bar{u}$, $\bar{w}$, $\bar{u}_0 = \{u(x), w(x), u_0(x)\} \sin n\theta$ and $\bar{v}$, $\bar{v}_0 = \{v(x), v_0(x)\} \cos n\theta$. Substituting these expressions into equations $E_1=E_2=\dots=E_5=0$, we obtain the following equations, using $K=k'\zeta$, $\alpha^4=\alpha_0^4/\beta$ and $\phi=e^{-2kx}/\beta$

$$\left. \begin{aligned}
& \frac{d^2 u}{dx^2} - k \frac{du}{dx} + \{\alpha^4 - n^2 \zeta (1+\phi)\} u - n(1-\zeta) \frac{dv}{dx} + nk(1-2\zeta)v \\
& \quad - (1-2\zeta) \left(\frac{dw}{dx} - kw \right) - \phi \left\{ \frac{d^2 u_0}{dx^2} - 3k \frac{du_0}{dx} + (\alpha^4 + n^2 \zeta) u_0 \right\} = 0 \\
& n(1-\zeta) \frac{du}{dx} - nk \zeta u + \zeta \frac{d^2 v}{dx^2} - k \zeta \frac{dv}{dx} + \{\alpha^4 - (n^2 + K)(1+\phi)\} v \\
& \quad - n(1+K)(1+\phi)w - \zeta \phi \frac{d^2 v_0}{dx^2} + 3k \zeta \phi \frac{dv_0}{dx} - \{(\alpha^4 + n^2) \phi + K(1+\phi)\} v_0 = 0 \\
& (1-2\zeta) \frac{du}{dx} - n(1+K)(1+\phi)v + K \frac{d^2 w}{dx^2} - Kk \frac{dw}{dx} + \{\alpha^4 - (1+n^2 K)(1+\phi)\} w \\
& \quad + K \left(\frac{du_0}{dx} - ku_0 \right) - n\{\phi + K(1+\phi)\} v_0 = 0 \\
& - \frac{d^2 u}{dx^2} + 3k \frac{du}{dx} - (\alpha^4 + n^2 \zeta) u - \frac{K}{\phi} \frac{dw}{dx} + \frac{d^2 u_0}{dx^2} - 3k \frac{du_0}{dx} \\
& \quad + \left\{ (\alpha^4 - n^2 \zeta) - \frac{K}{\phi} \right\} u_0 - n(1-\zeta) \frac{dv_0}{dx} + 3nk(1-2\zeta)v_0 = 0 \\
& - \zeta \frac{d^2 v}{dx^2} + 3k \zeta \frac{dv}{dx} - \left\{ \alpha^4 + n^2 + K \left(1 + \frac{1}{\phi} \right) \right\} v - n \left(1 + K + \frac{K}{\phi} \right) w \\
& \quad + n(1-\zeta) \frac{du_0}{dx} - 3nk \zeta u_0 + \zeta \frac{d^2 v_0}{dx^2} - 3k \zeta \frac{dv_0}{dx} + \left\{ \alpha^4 - n^2 - K - \frac{K}{\phi} \right\} v_0 = 0
\end{aligned} \right\} \dots\dots\dots (11)$$

Expanding ϕ and K/ϕ in above equations, we obtain

$$\phi = \sum_{n=0}^{\infty} F_n x^n, \quad K/\phi = \sum_{n=0}^{\infty} F'_n x^n, \quad F_n = (-2k)^n / \beta n!, \quad F'_n = \beta K (2k)^n / n!$$

From condition that $u=v=w=u_0=v_0=0$ at $x=0$, we can express the displacements in the following series:

$$u = \sum_{p=1}^{\infty} A_p x^p, \quad v = \sum_{p=1}^{\infty} B_p x^p, \quad w = \sum_{p=1}^{\infty} C_p x^p, \quad u_0 = \sum_{p=1}^{\infty} D_p x^p, \quad v_0 = \sum_{p=1}^{\infty} E_p x^p \dots\dots\dots(12)$$

In the above expressions, A_p , B_p etc. are unknown constants and $A_q=B_q=C_q=D_q=E_q=0$ for $q \leq 0$. Substituting Eq.(12) into Eq.(11), we obtain the following equations, from which A_p, B_p etc. are determined,

$$\left. \begin{aligned} \sum_{p=0}^{\infty} [(p+2)(p+1)A_{p+2} - (p+2)(p+1)F_0 D_{p+2} + f_1] x^p &= 0 \\ \sum_{p=0}^{\infty} [(p+2)(p+1)\zeta B_{p+2} - (p+2)(p+1)\zeta F_0 E_{p+2} + f_2] x^p &= 0 \\ \sum_{p=0}^{\infty} [(p+2)(p+1)KC_{p+2} + f_3] x^p &= 0 \\ \sum_{p=0}^{\infty} [(p+2)(p+1)(-A_{p+2} + D_{p+2}) + f_4] x^p &= 0 \\ \sum_{p=0}^{\infty} [\zeta(p+2)(p+1)(-B_{p+2} + E_{p+2}) + f_5] x^p &= 0 \end{aligned} \right\} \dots\dots\dots(13)$$

In the above equations, $f_1 \sim f_5$ are linear functions of A, B etc. with suffixes under $(p+1)$, as follows. We use the following $G_1 \sim G_{15}$ for convenience's sake:

$$\left. \begin{aligned} G_1 &= \alpha^4 - n^2 \zeta (1 + F_0), & G_2 &= n(1 - \zeta), & G_3 &= 1 - 2\zeta, & G_4 &= \alpha^4 + n^2 \zeta \\ G_5 &= k\zeta, & G_6 &= \alpha^4 - (n^2 + K)(1 + F_0), & G_7 &= n(1 + K)(1 + F_0) \\ G_8 &= (\alpha^4 + n^2)F_0 + K(1 + F_0), & G_9 &= \alpha^4 + n^2 + K, & G_{10} &= \alpha^4 - (1 + n^2 K)(1 + F_0) \\ G_{11} &= n(F_0 + K(1 + F_0)), & G_{12} &= \alpha^4 - n^2 \zeta - F_0', & G_{13} &= \alpha^4 + n^2 + K + F_0' \\ G_{14} &= n(1 + K + F_0'), & G_{15} &= \alpha^4 - n^2 - K - F_0' \\ f_1 &= -k(p+1)A_{p+1} + G_1 A_p - (p+1)G_2 B_{p+1} + nkG_3 B_p \\ &\quad - G_8 \{(p+1)C_{p+1} - kC_p\} + F_0 \{3(p+1)kD_{p+1} - G_4 D_p\} - F_p (2D_2 - 3kD_1) \\ &\quad - \sum_{q=1}^{p-1} \{n^2 \zeta A_q + (q+2)(q+1)D_{q+2} - 3(q+1)kD_{q+1} + G_4 D_q\} F_{p-q} \\ f_2 &= (p+1)G_2 A_{p+1} - nG_5 A_p - (p+1)G_5 B_{p+1} + G_6 B_p - G_7 C_p \\ &\quad + 3(p+1)G_5 F_0 E_{p+1} - F_p (2\zeta E_2 - 3G_5 E_1) - G_8 E_p - \sum_{q=1}^{p-1} \{(n^2 + K)B_q + n(1 + K)C_q\} F_{p-q} \\ &\quad + \zeta(q+2)(q+1)E_{q+2} - 3G_5(q+1)E_{q+1} + G_9 E_q\} F_{p-q} \\ f_3 &= (p+1)G_3 A_{p+1} - G_7 B_p - (p+1)KkC_{p+1} + G_{10} C_p + K\{(p+1)D_{p+1} - kD_p\} \\ &\quad - G_{11} E_p - \sum_{q=1}^{p-1} \{n(1 + K)B_q + (1 + n^2 K)C_q + n(1 + K)E_q\} F_{p-q} \\ f_4 &= 3k(p+1)(A_{p+1} - D_{p+1}) - G_4 A_p + G_{12} D_p - (p+1)F_0' C_{p+1} - F_p' C_1 \\ &\quad - (p+1)G_2 E_{p+1} + 3nkG_3 E_p - \sum_{q=1}^{p-1} \{(q+1)C_{q+1} + D_q\} F_{p-q}' \\ f_5 &= 3(p+1)G_5 (B_{p+1} - E_{p+1}) - G_{13} B_p + G_{15} E_p - G_{14} C_p + (p+1)G_2 D_{p+1} \\ &\quad - 3nG_5 D_p - \sum_{q=1}^{p-1} (B_q + nC_q + E_q) F_{p-q}' \end{aligned} \right\} \dots\dots\dots(14)$$

In the expressions of f_1, f_2 and f_4 shown above, $F_p (2D_2 - 3kD_1)$, $F_p (2\zeta E_2 - 3G_5 E_1)$ and $F_p' C_1$ are zero when $p=0$. We can calculate $A_2, B_2, \dots, A_3, \dots$ from Eq.(13) as follows, where A_1, B_1, C_1, D_1 and E_1 are arbitrary.

$$\left. \begin{aligned} A_{p+2} &= (f_1 + F_0 f_4) / (p+2)(p+1)(F_0 - 1), & B_{p+2} &= (f_2 + F_0 f_5) / \zeta(p+2)(p+1)(F_0 - 1) \\ C_{p+2} &= -f_3 / K(p+2)(p+1), & D_{p+2} &= (f_1 + f_4) / (p+2)(p+1)(F_0 - 1) \\ E_{p+2} &= (f_2 + f_5) / \zeta(p+2)(p+1)(F_0 - 1) \end{aligned} \right\} \dots\dots\dots(15)$$

Similarly, forces and moments $T_1 \sim M_2$ are expressed as follows,

$$\left. \begin{aligned} T_1/\beta e^{-kx} &= \sum_{p=0}^{\infty} \left\{ (p+1)(A_{p+1} - F_0 D_{p+1}) - G' B_p - G_3 C_p - F_p D_1 - \sum_{q=1}^{p-1} (q+1) D_{q+1} F_{p-q} \right\} x^p \\ T_2/\zeta \beta e^{-kx} &= \sum_{p=0}^{\infty} \left\{ n A_p + (p+1)(B_{p+1} - F_0 E_{p+1}) - F_p E_1 - \sum_{q=1}^{p-1} (q+1) E_{q+1} F_{p-q} \right\} x^p \\ T_3/k' \zeta \beta e^{-kx} &= \sum_{p=0}^{\infty} \left\{ (p+1) C_{p+1} + D_p \right\} x^p \\ M_1/e^{-3kx} &= \sum_{p=0}^{\infty} \left\{ (p+1)(-A_{p+1} + D_{p+1}) - G' E_p \right\} x^p \\ M_2/\zeta e^{-3kx} &= \sum_{p=0}^{\infty} \left\{ (p+1)(-B_{p+1} + E_{p+1}) + n D_p \right\} x^p \end{aligned} \right\} \dots\dots\dots (16)$$

where $G' = n(1-2\zeta)$.

Table 1. The values of μ for k , where μ is the nondimensional length and k is the rate of variable thickness along height. ($n=2$, $\alpha=1.2$, $\beta=1000$. The first mode of vibration in the improved theory)

k	$T_3 T_2 M_1$				$T_3 v M_1$			
0.02	0.295 1	0.297 3	0.307 0	0.309 5	0.311 7	0.314 0	0.331 2	0.334 0
0.1	0.295 7	0.297 9	0.307 4	0.309 9	0.312 1	0.314 5	0.331 3	0.334 2
0.2	0.296 3	0.298 5	0.308 0	0.310 5	0.312 7	0.315 1	0.331 5	0.334 4
0.4	0.297 6	0.299 9	0.309 0	0.311 6	0.313 7	0.316 2	0.331 8	0.334 8
0.6	0.298 8	0.301 1	0.310 0	0.312 6	0.314 7	0.317 2	0.332 1	0.335 1
0.8	0.300 0	0.302 4	0.310 9	0.313 5	0.315 6	0.318 2	0.332 3	0.335 3
1.0	0.301 1	0.303 5	0.311 7	0.314 4	0.316 4	0.319 1	0.332 4	0.335 5
	$T_3 T_2 u_0$				$T_3 v u_0$			
0.02	0.353 2	0.356 4	0.363 8	0.367 2	0.379 7	0.382 8	0.398 6	0.402 0
0.1	0.352 7	0.355 9	0.363 2	0.366 5	0.378 7	0.381 9	0.397 0	0.400 4
0.2	0.352 2	0.355 3	0.362 3	0.365 7	0.377 5	0.380 7	0.395 2	0.398 6
0.4	0.351 1	0.354 2	0.360 7	0.364 1	0.375 2	0.378 5	0.391 6	0.395 0
0.6	0.350 0	0.353 1	0.359 2	0.362 5	0.373 1	0.376 3	0.388 3	0.391 7
0.8	0.348 9	0.352 0	0.357 7	0.361 0	0.371 0	0.374 2	0.385 1	0.388 6
1.0	0.347 9	0.351 0	0.356 2	0.359 6	0.369 0	0.372 3	0.382 2	0.385 7
	$w T_2 M_1$				$w v M_1$			
0.02	0.570 0	0.572 7	0.570 5	0.573 3	0.613 1	0.613 7	0.614 9	0.615 4
0.1	0.566 7	0.569 4	0.567 1	0.569 9	0.608 6	0.609 2	0.610 2	0.610 8
0.2	0.562 6	0.565 3	0.563 0	0.565 7	0.603 1	0.603 8	0.604 5	0.605 2
0.4	0.554 6	0.557 3	0.554 9	0.557 5	0.592 4	0.593 3	0.593 5	0.594 4
0.6	0.546 8	0.549 6	0.547 1	0.549 8	0.582 1	0.583 3	0.583 0	0.584 1
0.8	0.539 4	0.542 1	0.539 6	0.542 3	0.572 3	0.573 6	0.573 0	0.573 8
1.0	0.532 1	0.534 9	0.532 3	0.535 0	0.562 9	0.564 4	0.563 5	0.565 0
	$w T_2 u_0$				$w v u_0$			
0.02	0.667 4	0.669 2	0.667 4	0.669 2	0.710 8	0.710 8	0.711 0	0.711 0
0.1	0.661 1	0.662 9	0.661 1	0.662 9	0.703 2	0.703 2	0.703 4	0.703 4
0.2	0.653 5	0.655 3	0.653 5	0.655 3	0.694 0	0.694 0	0.694 1	0.694 1
0.4	0.639 1	0.641 0	0.639 1	0.641 0	0.676 6	0.676 7	0.676 7	0.676 8
0.6	0.625 7	0.627 6	0.625 7	0.627 6	0.660 4	0.660 7	0.660 5	0.660 8
0.8	0.613 1	0.615 1	0.613 1	0.615 1	0.645 3	0.645 8	0.645 4	0.646 0
1.0	0.601 4	0.603 4	0.601 3	0.603 3	0.631 2	0.632 1	0.631 5	0.632 1

The boundary condition at the lower end is built-in. $T_3 T_2 M_1$ means that three of the five boundary conditions at the upper end are $T_3 = T_2 = M_1 = 0$. The combinations of boundary conditions are 8. The combinations of the other two are 4, that is $T_1 = M_2 = 0$, $u = M_2 = 0$, $T_1 = v_0 = 0$ and $u = v_0 = 0$, and four columns are correspondent to these conditions in sequence from left to right, respectively. Comparing the groups $w T_2 M_1$ and $w T_2 u_0$, we find μ in column $w M_2$ is larger than that in $T_1 v_0$. In group $w T_2 u_0$ differences of μ between groups $T_1 M_2$ and $T_1 v_0$, or between $w M_2$ and $w v_0$ are very small and they appear at fifth or sixth digits.

Fig.2 Effects of k on the convergency of solutions.

In the expressions of T_1 and T_2 , $F_p D_1$ and $F_p E_1$ are zero when $p=0$. Let the solutions u, v etc., obtained by putting $A_1=1$ and $B_1=C_1=D_1=E_1=0$, be u_1, v_1 etc. and let those obtained by putting $B_1=1$ and $C_1=D_1=E_1=A_1=0$, be u_2, v_2 etc., and so on. Five independent solutions used in numerical calculations are such linear combinations of above solutions as follows.

$$\left. \begin{aligned} u &= H_1 u_1 + H_2 u_2 + H_3 u_3 + H_4 u_4 + H_5 u_5 \\ v &= H_1 v_1 + H_2 v_2 + H_3 v_3 + H_4 v_4 + H_5 v_5 \end{aligned} \right\} \dots\dots (17)$$

where, $H_1 \sim H_5$ are arbitrary constants. At the upper end, we can obtain displacements from Eq.(17) and forces and moments by substituting Eq.(17) into Eq.(16). Then, we obtain from the boundary conditions at the upper end a frequency equation by making the determinant of order 5×5 equals to zero. In the classical theory, the frequency equation is obtained from the determinant of order 4×4

4. Numerical calculations

In numerical calculations according to both the improved theory and the classical one, with the values $n, \alpha, \beta, \zeta, k$ and k' given, we search for values of nondimensional height μ , which make the frequency equation zero. The boundary conditions are built-in at the lower end and all combinations of boundary conditions are considered at the upper end. The values of ζ and k' used are $0.35 (\nu=0.3)$ and $\pi^2/12$ (the value of Mirsky), respectively. The capacity of the computers used being restricted to about 100 terms of coefficients A_p, B_p etc. for u, v etc., we are able to calculate the value of μ in the range of $n=2 \sim 6, \alpha^4=0.4 \sim 40, \beta=200 \sim 1000$ and $k=0.02 \sim 1.0$.

The values of β and k have large effect on the convergency of series solutions of u, v etc., but the values of n and α in the range mentioned above have little effect. The relations between number p of coefficient A_p and the common logarithms of its absolute value are shown in Figs.2 and 3. The Figures are for the case obtained by putting $A_1=1$ and $B_1=\dots=0$. The A_p, B_p, C_p etc. for

Fig.3 Effects of β on the convergency of solutions.

Table 2. The value of μ for boundary condition that $T_1 u=0$ ($n=2, k=0.6, \beta=1000$. The first mode of vibration in the classical theory)

α	$T_2=T_3=M_1=0$		$w=v=w'=0$	
	$T_1=0$	$u=0$	$T_1=0$	$u=0$
1.0	0.466 6	0.438 1	0.859 9	0.876 4
1.2	0.329 0	0.322 3	0.703 1	0.693 9
1.4	0.263 0	0.260 5	0.591 3	0.582 1
1.6	0.222 2	0.221 1	0.511 6	0.505 0
1.8	0.193 8	0.193 2	0.452 2	0.447 5

In case $w=v=w'=0$, μ for $u=0$ is larger than that for $T_1=0$, when α is smaller than unity.

Fig.4 Effects of n and β on frequencies (both built-in, $k=0.6$ first mode of vibration) Figures 2 and 6 are of values of n .

other cases indicate a similar tendency. In both the classical theory and the improved one, the convergency becomes better as β becomes smaller and also as k becomes smaller (that is, the thickness of cylindrical shells tends to be uniform). The range of β within which the series converge according to the classical theory seems larger than that according to the improved theory. From our calculations with OKITAC system 50 (model 40), it seems that the upper limit of β according to the improved theory is about 1000 when $k=0.6$, while it is more than $10^4 \sim 10^5$ according to the classical theory.

The boundary conditions at the lower end are built-in and those at the upper end are selected from among 32 kinds of combinations such that $T_1 u = T_2 v = T_3 w = M_1 u_0 = M_2 v_0 = 0$. In Table 1, the values μ is the first mode of vibration are shown for 32 kinds of boundary conditions, where $n=2$, $\alpha=1.2$, $\beta=1000$ and $k=0.02$ (nearly uniform thickness) ~ 1.0 . These values are calculated to 7 digits but only 4 digits of them are shown in the Table. It is clear from the table that the condition that $T_3 w = 0$ has the largest effect on the length μ (hence on the frequency) of the cylindrical shell and that the length μ for $T_3 = 0$ is always shorter than the one for $w = 0$. The conditions effective next to it are that $T_2 v = M_1 u_0 = 0$. The conditions that $T_1 u = M_2 v_0 = 0$ have so little effect on the length μ that a minor difference occurs only at the third digit. According to the classical theory, effects of boundary conditions are similar to those mentioned above. Among the boundary conditions shown in Appendix 2, the condition that $T_3 w = 0$ is most effective followed by the condition that $T_2 v = M_1 \frac{dw}{dx} = 0$. The condition that $u_0 = 0$ for the improved theory corresponds to the one that $\frac{dw}{dx} = 0$ for the classical theory. In Table 2, an example of the effect of condition that $T_1 u = 0$ according to the classical theory is shown. Therefore it seems to be sufficient in practice to consider 8 kinds of boundary conditions for both the improved theory and the classical one. A similar tendency appears in mode shapes.

Fig.5 Similar to Fig.4 (built-in-supported, $k=0.6$)

For example, when one of the boundary conditions is $T_1 = 0$, the displacement u at the upper end becomes very small and vice versa.

Let us consider the effects of each parameter on the length μ of a cylindrical shell or on the nondimensional frequency α ($=\sqrt[4]{\alpha_0^4/\beta}$). The effect of k , the ratio of variation of thickness, is shown in Table 1. For other values of n , α and β , the tendency of μ is similar to that in Table 1. In this table, though the authors can not explain why, μ becomes larger as k increases when μ is small (or when the frequency is low in case of constant length), but reversely when μ is large μ decreases as k increases.

Fig.6 Similar to Fig.4 (built-in-free, $k=0.6$)

Fig.7 Effects of n on frequencies (built-in-free) Figures 2,6,...are values of n .

The columns in this table are arranged in the order of conditions that $T_1=M_2=0$, $u=M_2=0$, $T_1=v_0=0$ and $u=v_0=0$. For an arbitrary k , in general, μ becomes larger in the sequence mentioned above. But in the group of wT_2u_0 , this tendency is disturbed and it becomes evident for large k . The group of wT_2M_1 also has a similar tendency.

For brevity, we refer to the conditions at upper end that $u=v=w=u_0=v_0=0$ as both built-in, to ones that $u=v=w=M_1=M_2=0$ as built-in-supported and to ones that $T_1=T_2=T_3=M_1=M_2=0$ as built-in-free. The effects of β and n on the frequency are shown in Figs.4 through 7. In the case that $n=2$ and 6, the relation between β and $\sqrt[4]{\alpha_0^4/\beta}$ for both

built-in shell is shown in Fig.4, one for built-in-supported shell in Fig.5 and one for built-in-free shell in Fig.6. The curves in these figures shift left as β increases. The curves for $n=3,4$ and 5 come between those for $n=2$ and 6 at nearly equal intervals. The curves for larger values of n are shown in Fig.7 (built-in-free shell). These curves are all for the first mode of vibration.

The differences depending on the used theory, improved or classical, are shown in Fig.8 (both built-in), Fig.9 (built-in-supported) and Fig.10 (built-in-free) where $n=2$ and 6, $k=0.6$ and $\beta=1000$. The lower four curves in figures are for the first mode and the upper four are for the second mode.

Fig.8 Comparison depending on the used theory, improved or classical (both built-in)

Fig.10 Similar to Fig.8 (built-in-free)

Fig.9 Similar to Fig.8 (built-in-supported)

Fig.11 Frequencies of a cylindrical shell with nearly uniform thickness (built-in-supported)

In the second mode, the ranges shown in figures are the calculable limits of our computer. These curves shift left in the numbering order of the figures, and the tendency of going down to the right becomes remarkable for built-in-free shell. Though the frequencies obtained from the improved theory come close to those from the classical theory as μ increases from a small value, the former is considerably lower than the later and the difference expands as n increases and as the shell vibrates in higher modes. In the axially symmetric modes of vibration of a cylindrical shell with variable thickness, there is, on the contrary, little difference in frequencies for lower modes depending on the theory used. The curves for $n=3,4$ and 5, come, as mentioned above, between those for $n=2$ and 6 at nearly equal intervals. Owing to poor convergency of solutions when the upper end of a cylindrical shell is free and the value of β is about 1000 or over 1000, the authors can not obtain the frequency of the second mode for $n=2$ from the classical theory and it is not shown in Fig.10.

The comparison of frequencies of the improved theory for $k=0.02$ and 0.6 is shown in Fig.11. The boundary conditions are built-in supported and $\beta=500$.

The authors have calculated modes, forces and moments in the first mode of vibration for 32 kinds of boundary conditions where $n=2$, $\alpha=1.5$, $k=0.6$ and $\beta=1000$. Figs.12

and 13 indicate these values of a built-in-free cylindrical shell in the first and second modes of vibration. They are obtained from the improved theory.

5. Conclusions

We have studied the free vibration of a cylindrical shell with variable thickness according to both the improved theory and the classical one. Because of complexity of the equations of the improved theory, we have assumed that the displacement component along z (that is, along radius) is independent of z . The authors think that they should investigate about it in future. Whereas the combinations of boundary conditions at one end are 32 for improved theory and 16 for the classical theory, they are classified into 8 main groups.

The effects of each parameter on frequencies are explained in the previous chapter. Summing up the results obtained there, we conclude the vibrations of a cylindrical shell with variable thickness can be analyzed accurately by the classical theory, in case the length is long, the thickness is thin, the number of nodal generators is a few and the shell vibrates in lower modes. We must adopt the improved theory in other cases. The authors wish to express their appreciation to Mrs.T.Yamaguchi in the faculty of engineering of Yamagata University for her kind help in numerical calculations.

Fig.12 The displacements, forces and moments of a built-in-free cylindrical shell (the first mode, $\mu=0.215555$)

Fig.13 Similar to Fig.12 (the second mode, $\mu=0.513913$)

Appendix 1 Equations of vibration expressed in terms of forces and moments

$$\begin{aligned}\alpha_0^4 H(\bar{u} - H^2 \bar{u}_0 / \beta) + \delta T_1 / \partial x + \delta T_4 / \partial \theta &= 0, & \alpha_0^4 H(\bar{v} - H^2 \bar{v}_0 / \beta) + \delta T_2 / \partial x + \delta T_5 / \partial \theta - T_6 &= 0 \\ \alpha_0^4 H \bar{w} + \delta T_3 / \partial x + \delta T_6 / \partial \theta + T_5 &= 0, & \alpha_0^4 (H^3 / \beta) (\bar{u}_0 - \bar{u}) + \delta M_1 / \partial x + \delta M_3 / \partial \theta - T_3 &= 0 \\ \alpha_0^4 (H^3 / \beta) (\bar{v}_0 - \bar{v}) + \delta M_2 / \partial x + \delta M_4 / \partial \theta - T_6 &= 0\end{aligned}$$

Appendix 2 The classical theory

Let the displacements along x' , θ and z in Fig.1 be $\bar{u}(x, \theta)$, $\bar{v}(x, \theta)$ and $\bar{w}(x, \theta)$, respectively. To simplify expressions, we omit the factor $\sin \theta$ in them. After treating similarly to chapter 2, we obtain, using the nomenclature in chapter 2.

$$\begin{aligned}\varepsilon_z &= \frac{1}{R} \frac{\partial \bar{u}}{\partial x} - \frac{z}{R^2} \frac{\partial^2 \bar{w}}{\partial x^2}, & \varepsilon_\theta &= \frac{1}{R} \left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w} \right) - \frac{z}{R^2} \left(\frac{\partial^2 \bar{w}}{\partial \theta^2} + \frac{\partial \bar{v}}{\partial x} \right) \\ \gamma_{x\theta}' &= \frac{1}{R} \left(\frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial \theta} \right) - \frac{2z}{R^2} \left(\frac{\partial^2 \bar{w}}{\partial x \partial \theta} + \frac{\partial \bar{v}}{\partial x} \right) \\ \sigma_x &= \frac{E}{1-\nu^2} (\varepsilon_z + \nu \varepsilon_\theta), & \sigma_\theta &= \frac{E}{1-\nu^2} (\nu \varepsilon_z + \varepsilon_\theta), & \tau_{x\theta} &= \frac{E}{2(1+\nu)} \gamma_{x\theta}' \\ L &= (1/2) \int \int \int [(\gamma p^2 / g) (\bar{u}^2 + \bar{v}^2 + \bar{w}^2) - \sigma_x \varepsilon_x - \sigma_\theta \varepsilon_\theta - \tau_{x\theta} \gamma_{x\theta}'] R^2 dx d\theta dz \\ \delta L / (D_0 / R^2) &= \int \int [E_1 \delta \bar{u} + E_2 \delta \bar{v} + E_3 \delta \bar{w}] dx d\theta - \int [T_1 \delta \bar{u} + T_2 \delta \bar{v} + T_3 \delta \bar{w} + M_1 \delta (\partial \bar{w} / \partial x)] \Big|_{x=0}^{x=\mu} d\theta \\ &\quad - \int [T_4 \delta \bar{u} + T_5 \delta \bar{v} + T_6 \delta \bar{w} + M_2 \delta (\partial \bar{w} / \partial \theta)] \Big|_{\theta=0}^{\theta=2\pi} dx - [4\zeta H^3 (\partial^2 \bar{w} / \partial x \partial \theta + \partial \bar{v} / \partial x) \delta \bar{w}] \Big|_{x=0, \theta=0}^{x=\mu, \theta=2\pi} \\ T_1 &= \beta H \{ \partial \bar{u} / \partial x + (1-2\zeta) (\partial \bar{v} / \partial \theta - \bar{w}) \} \\ T_2 &= \beta \zeta H \{ \partial \bar{v} / \partial x + \partial \bar{u} / \partial \theta + 4\zeta H^3 (\partial^2 \bar{w} / \partial x \partial \theta + \partial \bar{v} / \partial x) \} \\ T_3 &= -(\partial / \partial x) [H^3 \{ \partial^2 \bar{w} / \partial x^2 + (1-2\zeta) (\partial^2 \bar{w} / \partial \theta^2 + \partial \bar{v} / \partial \theta) \}] - (\partial / \partial \theta) \{ 4\zeta H^3 (\partial^2 \bar{w} / \partial x \partial \theta + \partial \bar{v} / \partial x) \} \\ M_1 &= H^3 \{ \partial^2 \bar{w} / \partial x^2 + (1-2\zeta) (\partial^2 \bar{w} / \partial \theta^2 + \partial \bar{v} / \partial \theta) \} \\ T_4 &= \beta \zeta H \{ \partial \bar{v} / \partial x + \partial \bar{u} / \partial \theta \} \\ T_5 &= \beta H \{ (1-2\zeta) \partial \bar{u} / \partial x + \partial \bar{v} / \partial \theta - \bar{w} \} + H^3 \{ (1-2\zeta) \partial^2 \bar{w} / \partial x^2 + \partial^2 \bar{w} / \partial \theta^2 + \partial \bar{v} / \partial \theta \} \\ T_6 &= -(\partial / \partial x) \{ 4\zeta H^3 (\partial^2 \bar{w} / \partial x \partial \theta + \partial \bar{v} / \partial x) \} - (\partial / \partial \theta) [H^3 \{ (1-2\zeta) \partial^2 \bar{w} / \partial x^2 + \partial^2 \bar{w} / \partial \theta^2 + \partial \bar{v} / \partial \theta \}] \\ M_2 &= H^3 \{ (1-2\zeta) \partial^2 \bar{w} / \partial x^2 + \partial^2 \bar{w} / \partial \theta^2 + \partial \bar{v} / \partial \theta \} \\ E_1 &= \alpha_0^4 H \bar{u} + \delta T_1 / \partial x + \delta T_4 / \partial \theta (=0), & E_2 &= \alpha_0^4 H \bar{v} + \delta T_2 / \partial x + \delta T_5 / \partial \theta (=0) \\ E_3 &= \alpha_0^4 H \bar{w} + \delta T_3 / \partial x - \delta^2 M_2 / \partial \theta^2 + T_5 - M_2 = \alpha_0^4 H \bar{w} - \delta^2 M_1 / \partial x^2 + \delta T_6 / \partial \theta + T_5 - M_2 (=0)\end{aligned}$$

The strains in above equations are quoted from Timoshenko's textbook⁽¹⁴⁾.

Putting $\gamma_{\theta z}' = \gamma_{xz}' = 0$ and $1 \pm z/R \neq 1$ in Eq.(1) shown in chapter 2, we obtain $\bar{v}_0 = -\partial \bar{w} / \partial \theta$ and $\bar{u}_0 = -\partial \bar{w} / \partial x$. Substituting these $\bar{u}_0$ and $\bar{v}_0$ into ε_x , ε_θ and $\gamma_{x\theta}'$ in Eq.(1), we find that they coincide with ε_x and ε_θ in this appendix, while the last term $\partial \bar{v} / \partial x$ of $\gamma_{x\theta}'$ differs by (1/2). Therefore the special case of the improved theory is slightly different from the equations in the classical theory. We can solve the equations of vibration, $E_1 = E_2 = E_3 = 0$, in a similar way to chapter 3 and so we omit the detail. There exist 4 arbitrary constants in the solution.

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