Two Dimensional Flow between Eccentric Rotating Cylinders*

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This study is concerned with the flow of a viscous incompressible fluid between eccentric cylinders which rotate with respective angular velocities. The flow pattern between cylinders, and the torque and flow force exerted on each cylinder are discussed for a wide range of radius ratios and eccentricities. The revolutionary motion of the rotating inner cylinder around the center of the stationary outer cylinder is also discussed.

The analysis is based both on the Stokes approximation and on the numerical solution of the Navier-Stokes equations. The flow patterns obtained from the analysis are compared with experimental results.

1. Introduction

The two dimensional viscous flow between eccentric rotating cylinders has been studied by many researchers⁽¹⁾⁻⁽³⁾ with special interest in the lubrication of a journal bearing. The Reynolds approximation can be applied to the flow for very small clearance ratios, neglecting the curvature and inertia effects. For a large clearance ratio, studies based on the Stokes approximation, neglecting the inertia effect, were reported early by Jeffery⁽⁴⁾ and later by Kamal⁽⁵⁾, Ashino⁽⁶⁾ and others.

In this paper, in order to estimate the inertia effect, the numerical solutions of the Navier-Stokes equations are derived for various clearance ratios and eccentricities. The flow pattern between cylinders and the flow force on each cylinder are presented for various boundary conditions by both the Stokes solutions and the numerical solutions. The revolutionary motion of the rotating inner cylinder due to the flow force is also treated.

The flow patterns derived from the analysis are compared with the experimental results obtained by a flow visualization technique.

The discussion is limited to the two dimensional flow, and does not concern the stability problem.⁽⁷⁾⁽⁸⁾

2. Coordinate system and basic equations

2.1 Rotating bipolar coordinate system

Two dimensional flow between eccentric cylinders shown by solid lines in Fig. 1 (a) is described conveniently in the following bipolar coordinate system.

$$\eta' = \log \{ (z' - a) / (z' + a) \} \quad \dots\dots\dots (1)$$

where $z' = x' + iy'$ and $\eta' = \alpha + i\beta$. This transformation maps the lines of $\alpha = \text{constant}$ into the circles in z' plane whose centers are at $(-a \cos \alpha, 0)$ and maps the lines of $\beta = \text{constant}$ into the circles which pass two common fixed points A, B [OA = OB = a ($a > 0$)] and with their center at $(0, a \cos \beta)$. These families of circles constitute a bipolar coordinate system. Then the region between the eccentric circles in the z' plane corresponds to the rectangular region of $\alpha_1 \leq \alpha \leq \alpha_2$ and $-\pi \leq \beta \leq \pi$ in the η' plane. The radii r_1 and r_2 of the inner and outer cylinders are given by

$$r_1 = -a(1/\sinh \alpha_1), \quad r_2 = -a(1/\sinh \alpha_2) \quad \dots\dots\dots (2)$$

$$2a = (c/\varepsilon) [\{ (r_1 + r_2)/c - \varepsilon^2 \} - 4\varepsilon^2 r_1^2 / c^2]^{1/2} \dots\dots (3)$$

where c is radial clearance ($c = r_2 - r_1$), and e displacement of the center of inner cylinder from the center of outer cylinder, ε eccentricity ratio ($\varepsilon = e/c$). Let θ_i be the angle between the radius r_i ($i=1, 2$) and the x' axis, then the following is written;

Fig. 1 Bipolar coordinate system rotating with constant angular velocity

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$$\theta_i = \sin^{-1} \{ -\sinh \alpha_i \sin \beta / (\cosh \alpha_i - \cos \beta) \} \dots\dots\dots (4)$$

So far a usual bipolar coordinate system is treated. In this paper, in order to examine the revolutionary motion of the rotating inner cylinder due to the flow force, the $z(x, y)$ plane is used instead of the z' plane (see Fig.1). The y axis passes through the center C_2 of the outer cylinder. Now the mapping function is

$$\left. \begin{aligned} \eta &= \log \{ [z - a(1 + \coth \alpha_2)] / [z - a(1 - \coth \alpha_2)] \} \\ z &= x + iy \end{aligned} \right\} \dots\dots\dots (5)$$

The transformations to the Cartesian coordinate system are given by

$$\left. \begin{aligned} x &= \{ (-a \sinh \alpha) / (\cosh \alpha - \cos \beta) \} + a \coth \alpha_2 \\ y &= a \sin \beta / (\cosh \alpha - \cos \beta) \end{aligned} \right\} \dots\dots\dots (6)$$

The flow is described referring to a rotating bipolar coordinate system which rotates around the origin of the x, y plane (the center of outer cylinder) with constant angular velocity Ω . The relation of Eqs. (2), (3) and (4) still hold for the rotating coordinate system.

The stream function which satisfies the above continuity equation, is defined by

$$u = (1/a)(\partial \psi / \partial \beta), \quad v = -(1/a)(\partial \psi / \partial \alpha) \dots\dots\dots (10)$$

The equations of motion are obtained in the following form with respect to ψ .

$$\left. \begin{aligned} (1/h^2) \{ (\partial^2 \psi / \partial \alpha \partial \beta) (\partial \psi / \partial \beta) - (\partial^2 \psi / \partial \beta^2) (\partial \psi / \partial \alpha) \} - (1/h^3) (\partial h / \partial \alpha) \{ (\partial \psi / \partial \alpha)^2 + (\partial \psi / \partial \beta)^2 \} + 2\Omega (\partial \psi / \partial \alpha) + \nu (\partial \zeta / \partial \beta) &= -(1/\rho) (\partial P / \partial \alpha) + \Omega^2 h^2 \{ \cos \beta \times (\cosh \alpha_2 \cosh \alpha - \sinh \alpha) - \coth \alpha_2 \} \quad (11.a) \\ (1/h^2) \{ (\partial^2 \psi / \partial \alpha \partial \beta) (\partial \psi / \partial \alpha) - (\partial^2 \psi / \partial \alpha^2) (\partial \psi / \partial \beta) \} - (1/h^3) (\partial h / \partial \beta) \{ (\partial \psi / \partial \alpha)^2 + (\partial \psi / \partial \beta)^2 \} + 2\Omega (\partial \psi / \partial \beta) - \nu (\partial \zeta / \partial \alpha) &= -(1/\rho) (\partial P / \partial \beta) + \Omega^2 h^2 \sin \beta \times \{ \coth \alpha_2 \sinh \alpha - \cosh \alpha \} \quad (11.b) \end{aligned} \right\} \dots\dots\dots (11)$$

The variable ζ in the above equation indicates the vorticity defined by

$$\zeta = -\nabla^2 \psi, \quad \text{where } \nabla^2 = (1/h^2) \{ (\partial^2 / \partial \alpha^2) + (\partial^2 / \partial \beta^2) \} \dots\dots\dots (12)$$

Taking the curl of each term of Eq. (7), we get the following vorticity equation.

$$(\partial \psi / \partial \alpha) (\partial \zeta / \partial \beta) - (\partial \psi / \partial \beta) (\partial \zeta / \partial \alpha) = -\nu h^2 \nabla^2 \zeta = \nu h^2 \nabla^4 \psi \dots\dots\dots (13)$$

It is also noted that the above equation does not include the parameter Ω . This means that the streamlines and the vorticity bearing on the adopted coordinate system are governed by the same equation (13), whether the system rotates or not.

3. Results from Stokes approximation

The flow patterns and the flow forces are analyzed on the Stokes approximation. The results will be compared with the results of the numerical solutions of Eqs. (12) and (13) in the next chapter.

3.1 Solution of the Stokes equations

If the inertia terms are negligibly small as compared with the viscous term, Eqs. (11) and (13) reduce to

$$\left. \begin{aligned} -(1/\rho) (\partial P / \partial \alpha) &= \nu (\partial \zeta / \partial \beta) = \nu (\partial / \partial \beta) (-\nabla^2 \psi) \quad (11.a)' \\ -(1/\rho) (\partial P / \partial \beta) &= \nu (\partial \zeta / \partial \alpha) = \nu (\partial / \partial \alpha) (-\nabla^2 \psi) \quad (11.b)' \end{aligned} \right\} \dots\dots\dots (11)'$$

$$\text{and } \nabla^4 \psi = 0 \dots\dots\dots (13)'$$

Equation (13)' was solved formerly by Jeffery. He gave the expressions of the torque exerted on rotating cylinders, but did not refer to the flow patterns between the cylinders.

2.2 Basic equations of flow

On the assumption of the incompressible steady flow, the Navier-Stokes equation and the continuity equation are expressed in the coordinate system rotating with constant angular velocity Ω as follows;

$$\begin{aligned} (1/2) \nabla (\mathbf{q} \cdot \mathbf{q}) + (\nabla \times \mathbf{q}) \times \mathbf{q} + 2\Omega \times \mathbf{q} + \Omega \\ \times (\Omega \times \mathbf{r}) = -(1/\rho) \nabla P - \nu \nabla \times (\nabla \times \mathbf{q}) \dots\dots\dots (7) \end{aligned}$$

$$\text{and } \nabla \cdot \mathbf{q} = 0 \dots\dots\dots (8)$$

where $\mathbf{q}$ is the velocity vector and $\mathbf{r}$ the position vector, P the pressure, ρ the density of fluid, ν the kinematic viscosity, ∇ operator of gradient. The third term and the fourth term in the left hand side of Eq. (7) mean the Coriolis force and the centrifugal force, respectively. Since the flow is two dimensional, $\mathbf{q}$ and $\mathbf{r}$ are two dimensional vectors $\mathbf{q}(u, v)$ and $\mathbf{r}(x, y)$, and Ω becomes a scalar Ω .

Equations (7) and (8) can be written in the following form with the α, β components of the vectors. The continuity equation is

$$\partial / \partial \alpha (hu) + \partial / \partial \beta (hv) = 0 \dots\dots\dots (8)'$$

where,

$$\begin{aligned} h &= \sqrt{(\partial x / \partial \alpha)^2 + (\partial y / \partial \beta)^2} \\ &= a / (\cosh \alpha - \cos \beta) \dots\dots\dots (9) \end{aligned}$$

Recently, Kamal⁽⁵⁾ discussed the separation in the flow field, solving Eq. (13)' in almost the same manner as Jeffery⁽⁴⁾. Ashino⁽⁶⁾ also obtained the solution by another method. Here we present the Stokes solution under the following general boundary condition, and describe various flow patterns. The boundary condition is,

$$\left. \begin{aligned} v_{\alpha=\alpha_i} &= -\left\{ \left(\frac{1}{h} \right) \left(\frac{\partial \phi}{\partial \alpha} \right) \right\}_{\alpha=\alpha_i} = V_i \\ u_{\alpha=\alpha_i} &= 0 \quad (i=1, 2) \end{aligned} \right\} \dots\dots (14)$$

This means that the inner and outer cylinders rotate at tangential velocities V_1 and V_2 with respect to the coordinate system. As shown by Kamal, through the transformation $\phi = h\psi$, Eq. (13)' reduces to a fourth order differential equation with constant coefficients. The solution is assumed to have a representation in the form:

$$\phi(\alpha, \beta) = \sum_{n=0}^{\infty} F_n(\alpha) \cos n\beta + \sum_{n=1}^{\infty} G_n(\alpha) \sin n\beta$$

After several calculations the solution is obtained as a finite sum as follows;

$$\begin{aligned} \phi &= \phi/h = A_0 \sinh \alpha + B_0 \cosh \alpha + D_0 \alpha \sinh \alpha + E_0 \alpha \cosh \alpha \\ &+ (A_1 + B_1 \alpha + D_1 \sinh 2\alpha + E_1 \cosh 2\alpha) \cos \beta \quad (15 \cdot a) \end{aligned}$$

where

$$\begin{aligned} A_0 &= [(Q/a)(-\alpha_1 \cosh^2 \alpha_2 + \alpha_2 \cosh^2 \alpha_1 - \cosh \alpha_1 \cosh \alpha_2 \sinh \delta) \\ &+ V_1(\alpha_2 \delta \cosh \alpha_1 - \alpha_1 \sinh \delta \cosh \alpha_2) + V_2(-\alpha_1 \delta \cosh \alpha_2 \\ &+ \alpha_2 \sinh \delta \cosh \alpha_1)] / (\delta^2 - \sinh^2 \delta) \\ B_0 &= [(Q/a)(-\alpha_2 \delta + \alpha_1 \sinh \alpha_2 \cosh \alpha_2 - \alpha_2 \sinh \alpha_1 \cosh \alpha_1 \\ &+ \cosh \alpha_1 \sinh \alpha_2 \sinh \delta) + V_1(\alpha_1 \sinh \delta \sinh \alpha_2 - \alpha_2 \delta \sinh \alpha_1) \\ &+ V_2(-\alpha_2 \sinh \delta \sinh \alpha_1 + \alpha_1 \delta \sinh \alpha_2)] / (\delta^2 - \sinh^2 \delta) \\ D_0 &= [(Q/a)(-\cosh^2 \alpha_1 + \cosh^2 \alpha_2) + V_1(-\delta \cosh \alpha_1 + \sinh \delta \cosh \alpha_2) \\ &+ V_2(\delta \cosh \alpha_2 - \sinh \delta \cosh \alpha_1)] / (\delta^2 - \sinh^2 \delta) \\ E_0 &= [(Q/a)(\delta + \sinh \alpha_1 \cosh \alpha_1 - \sinh \alpha_2 \cosh \alpha_2) + V_1(\delta \sinh \alpha_1 \\ &- \sinh \delta \sinh \alpha_2) + V_2(-\delta \sinh \alpha_2 + \sinh \delta \sinh \alpha_1)] / (\delta^2 - \sinh^2 \delta) \\ A_1 &= (Q/2a)(2\alpha_2 \cosh \delta + \sinh \delta) / (\delta \cosh \delta - \sinh \delta) \\ B_1 &= (-Q/a) \cosh \delta / (\delta \cosh \delta - \sinh \delta) \\ D_1 &= (Q/4a)(\sinh 2\alpha_1 - \sinh 2\alpha_2) / \{\sinh \delta(\delta \cosh \delta - \sinh \delta)\} \\ E_1 &= (Q/4a)(-\cosh 2\alpha_1 + \cosh 2\alpha_2) / \{\sinh \delta(\delta \cosh \delta - \sinh \delta)\} \quad (15 \cdot b) \\ \delta &= \alpha_1 - \alpha_2 \\ Q &= -a \{ V_1(\delta \sinh \alpha_1 - \sinh \delta \sinh \alpha_2) + V_2(-\delta \sinh \alpha_2 + \sinh \delta \sinh \alpha_1) \} \\ &\times (\delta \cosh \delta - \sinh \delta) / \{ (\delta + \sinh \alpha_1 \cosh \alpha_1 - \sinh \alpha_2 \cosh \alpha_2) \\ &\times (\delta \cosh \delta - \sinh \delta) - \cosh \delta(\delta^2 - \sinh^2 \delta) \} \quad (15 \cdot c) \end{aligned} \dots\dots (15)$$

The parameter Q in the above equations indicates the flow rate between the cylinders. Let $V_2 = 0$, then Eq. (15) agrees with Kamal's solution. From Eqs. (15.a) and (12), the expression of the vorticity is obtained in the following form.

$$\begin{aligned} \zeta &= (-2/a) [(B_0 + D_0 + A_1) + \{(-D_0 + 2E_1) \cosh \alpha + 2D_1 \sinh \alpha\} \cos \beta \\ &- (D_1 \sinh 2\alpha + E_1 \cosh 2\alpha) \cos 2\beta] \dots\dots\dots (16) \end{aligned}$$

3.2 Flow pattern and flow force

3.2.1 Streamlines and separation point

Figure 2 shows the streamlines calculated from Eq. (15). Figures (a), (b) and (c) show the flow patterns for the case of inner cylinder rotating ($V_2 = 0$), outer cylinder rotating ($V_1 = 0$) and both cylinders rotating, respectively. Since the equation for streamline [Eq. (13)], is independent of the rotation of coordinate system, the flow pattern shown in Fig. (c) can be explained either as the flow where both the inner and outer cylinders rotate at velocities V_1 and V_2 about their fixed centers, or as the flow, viewed from the coordinate system rotating with the rate Ω , where the rotating inner cylinder revolves with the rate $\Omega (= -V_2/r_2)$, keeping the eccentricity constant. Figure 2 is the result for the radius ratio $\kappa = r_1/r_2 = 1/4$ and the eccentricity ratio $\epsilon = e/(r_2 - r_1) = 2/3$. The flow patterns are symmetric with respect to the line connecting the centers of both cylinders, because Eq. (15.a) is an even function of β . Figures (a) and (b) show the occurrence of a separation from the inner or outer cylinder wall. Bentwich and Ashino discussed the separation of the flow for the case of inner cylinder rotating. The relation between the angle position θ_s of the separation point and the eccentricity ratio ϵ is shown in Fig. 3 for various values of radius ratio κ . The figure includes the results for both cases of outer cylinder rotating and inner cylinder rotating. The solid lines indicate the result for the case of inner

cylinder rotating, which coincides with Ashino's result, and the broken lines indicate the result for the case of outer cylinder rotating. It has been reported that for the case of inner cylinder rotation, a separation does not occur for the value of eccentricity ratio less than $\epsilon=0.3$. Figure 3 shows that the critical value becomes nearly 0.15 for the case of outer cylinder rotating. It is noted that the values of θ_s is calculated from Eq. (4) with the value of β determined by the following

Fig. 2 Streamlines from Stokes approximation at $\kappa=1/4$, $\epsilon=2/3$ (values of stream function at both cylinder walls are given)

condition of separation point.

$$(\partial v / \partial \alpha)_{\alpha=\alpha_i}=0 \quad \text{or} \quad (\zeta)_{\alpha=\alpha_i}=0$$

3.2.2 Flow forces exerted on cylinders

The pressure in the flow field between cylinders is given from Eqs. (11)' and (16) in the following form.

$$P = -(2\mu/a) [\{ (-D_0 + 2E_1) \sinh \alpha + 2D_1 \cosh \alpha \} \sin \beta - (D_1 \cosh 2\alpha + E_1 \sinh 2\alpha) \sin 2\beta] + P_0 \dots\dots\dots (17)$$

where P_0 is a constant value and μ is the viscosity ($\mu = \rho\nu$). The flow forces exerted on the cylinders can be obtained with the values of stress tensor $\sigma_{\alpha\alpha}$, $\tau_{\alpha\beta}$ on the cylinder wall, which are given by

$$\left. \begin{aligned} (\sigma_{\alpha\alpha})_{\alpha=\alpha_i} &= -P_i, \\ (\tau_{\alpha\beta})_{\alpha=\alpha_i} &= \mu \{ \zeta_i + (2/a) V_i \sinh \alpha_i \} = \tau_i \end{aligned} \right\} \quad (18)$$

Let $(f_i)_x$ and $(f_i)_y$ be the components of flow forces on cylinders in the x and y direction, respectively. Then they are obtained in the following forms.

$$\begin{aligned} (f_1)_x &= - \int_{-\pi}^{\pi} (P_1 \cos \theta_1 + \tau_1 \sin \theta_1) h_1 d\beta = 0 \\ (f_1)_y &= - \int_{-\pi}^{\pi} (-P_1 \sin \theta_1 + \tau_1 \cos \theta_1) h_1 d\beta \\ &= 4\pi\mu D_0 \end{aligned}$$

$$(f_2)_x = 0, \quad (f_2)_y = -4\pi\mu D_0 \quad \dots\dots (19)$$

where $h_i d\beta$ means an infinitesimal arc segment of the cylinder wall. It is seen from Eq. (19) that the flow force on each cylinder is zero in the x direction (the direction of the eccentricity), and has same absolute value with opposite sign in the y direction. The moment by the flow force about the center of each cylinder, M_i , is given by

Fig. 3 Separation position θ_s versus eccentricity ratio ϵ

$$\left. \begin{aligned}
 M_1 &= r_1 \int_{-\pi}^{\pi} \tau_1 h_1 d\beta = -4\pi\mu a (1/\sinh^2 \alpha_1) [(B_0 + D_0 + A_1) - V_1 \sinh \alpha_1 \\
 &\quad + \{(-D_0 + 2E_1) \cosh \alpha_1 + 2D_1 \sinh \alpha_1\} (\sinh \alpha_1 + \cosh \alpha_1) \\
 &\quad - (D_1 \sinh 2\alpha_1 + E_1 \cosh 2\alpha_1) (\cosh 2\alpha_1 + \sinh 2\alpha_1)] \quad (20 \cdot a) \\
 M_2 &= -r_2 \int_{-\pi}^{\pi} \tau_2 h_2 d\beta = 4\pi\mu a (1/\sinh^2 \alpha_2) [(B_0 + D_0 + A_1) - V_2 \sinh \alpha_2 \\
 &\quad + \{(-D_0 + 2E_1) \cosh \alpha_2 + 2D_1 \sinh \alpha_2\} (\sinh \alpha_2 + \cosh \alpha_2) \\
 &\quad - (D_1 \sinh 2\alpha_2 + E_1 \cosh 2\alpha_2) (\cosh 2\alpha_2 + \sinh 2\alpha_2)] \quad (20 \cdot b)
 \end{aligned} \right\} \dots\dots\dots (20)$$

Equations (19) and (20) yield

$$M_1 + M_2 = e(f_1)_y \dots\dots\dots (21)$$

This means that the Moments M_1 and M_2 do not balance in general ($M_1 + M_2 \neq 0$).

3.3 Revolutionary motion of rotating inner cylinder in fixed outer cylinder

Equation (19) holds for any rotating coordinate system with constant angular velocity Ω . From this, it can be known that if the tangential velocities of cylinders, V_1 and V_2 , are adjusted so that the condition $(f_1)_y = 0$, i.e., $D_0 = 0$ [c.f., Eq. (15)] is fulfilled, no resisting and accelerating flow force is exerted on the inner cylinder. Therefore, when the inner cylinder rotates with tangential velocity V_1 , which satisfies the above condition, in the fixed outer cylinder, it revolves around the center of the outer cylinder along a circle of radius e at the angular velocity which coincides with that of the rotating coordinate system $\Omega_0 = -V_2/r_2$. From Eqs. (15) and (2), the condition for the steady revolutionary motion mentioned above is reduced to the following form.

$$-V_2/V_1 = r_1/r_2 \dots\dots\dots (22)$$

It is also expressed in the form

$$\Omega_0 = \omega r_1^2 / (r_1^2 + r_2^2) = \omega \kappa^2 / (\kappa^2 + 1) \dots\dots (23)$$

where ω is the angular velocity of the inner cylinder with respect to the stationary coordinate system. The revolution rate Ω_0 is proportional to the rotation rate ω , and depends only on the radius ratio of cylinders. [As is well known in two dimensional potential theory of fluid dynamics, a vortex

filament with strength Γ revolves with the angular velocity $\Omega' = \Gamma / 2\pi(r_2^2 - e^2)$, if it exists in a cylinder with the eccentric displacement e' . Here, Ω' depends on the value of e' . It is interesting that there exist the following similarities between the above expression derived from the potential theory and Eq. (23) from the Stokes approximation: (i) The direction of the revolution is same as that of the rotation, and (ii) the revolution rate is proportional to the rotation rate.] Figure 4 shows the relation between the flow force $(f_1)_y$ and the revolution rate Ω of inner cylinder for various values of radius ratio κ . As seen in the figure, For $\Omega > \Omega_0$, the flow force is resistive to the revolutionary motion, and for $\Omega < \Omega_0$, it becomes accelerative. It is noted that the value of Ω_0/ω increases with the radius ratio κ , and tends to the value $1/2$ in the limit of $\kappa = 1$.

4. Numerical analysis and experiment on the inertia effect

4.1 Difference approximation and method of computation

The finite difference approximation is used to obtain the numerical solutions of the basic equations (12) and (13) under the boundary condition of Eq. (14). Figure 5 shows the region of interest which is divided in a finite difference mesh of cells. The region is thus divided into 20 equal increments in the α direction ($I = 1 \sim 21$), and into 80 increments in the β direction ($J = 1 \sim 80$). In computation, equations are normalized by the following nondimensional quantities.

Fig. 4 Relation between Ω/ω and $(f_1)_y$ at $\varepsilon = 2/3$

Fig. 5 Calculation region divided in a mesh of cells

$$\Psi = \phi/DV_i, \quad Z = D\zeta/V_i, \quad H = h/D, \quad R_e = DV_i/\nu \text{ (Reynolds number)}, \quad s = \Delta\beta/\Delta\alpha$$

$$\bar{Q} = Q/DV_i, \quad \bar{V}_2 = V_2/V_i, \quad \bar{V}_1 = V_1/V_i, \quad \bar{P} = P/\rho V_i^2, \quad \bar{\Omega} = D\Omega/V_i$$

where V_i is the representative velocity (for the case of inner cylinder rotating or not, $i=1$ or 2 , respectively), and

$$\Delta\alpha = (\alpha_2 - \alpha_1)/20, \quad \Delta\beta = \pi/40, \quad D = 2r_2 \dots\dots\dots (24)$$

Equations (12) and (13) are written in the following finite difference form.

$$Z_{I,J} = [1/\{2(s^2+1)\}] \{s^2(Z_{I+1,J} + Z_{I-1,J}) + Z_{I,J+1} + Z_{I,J-1} - R_e[s/\{8(s^2+1)\}]\{(\Psi_{I,J+1} - \Psi_{I,J-1})(Z_{I+1,J} - Z_{I-1,J}) - (\Psi_{I+1,J} - \Psi_{I-1,J})(Z_{I,J+1} - Z_{I,J-1})\} \dots\dots\dots (25)$$

$$\Psi_{I,J} = [1/\{2(s^2+1)\}] \{s^2(\Psi_{I+1,J} + \Psi_{I-1,J}) + (\Psi_{I,J+1} + \Psi_{I,J-1}) + H_{I,J}^2(\Delta\beta)^2 Z_{I,J}\} \dots\dots (26)$$

where I, J indicates the number of mesh points. Since Ψ and Z are periodic functions with respect to β , the above equations are applied by putting $J-1=80$ at the points of $J=1$, and $J+1=1$ at the points of $J=80$. The boundary condition of Eq. (14) is expressed in the following finite difference form.

$$\Psi_{1,J} = \bar{Q} = \text{constant}, \quad \Psi_{21,J} = 0, \quad \left. \begin{aligned} Z_{1,J} &= -(1/H_{1,J}^2)(-\Psi_{3,J} + 8\Psi_{2,J} - 7\Psi_{1,J} + 6H_{1,J}\bar{V}_1\Delta\alpha)/(2\Delta\alpha^2) \\ Z_{21,J} &= -(1/H_{21,J}^2)(-\Psi_{19,J} + 8\Psi_{20,J} - 7\Psi_{21,J} - 6H_{21,J}\bar{V}_2\Delta\alpha)/(2\Delta\alpha^2) \end{aligned} \right\} \dots\dots\dots (27)$$

The above equations for Z are derived on the assumption that the velocity profile is quadratic in the vicinity of the cylinder wall. In addition, Eq. (11) is written as;

$$\left. \begin{aligned} \left(\frac{\partial P}{\partial \alpha}\right)_{I,J} &= \frac{1}{2H_{I,J}^2} \left[\left\{ \left(\frac{\Psi_{I+1,J} - \Psi_{I-1,J}}{\Delta\alpha} \right)^2 + \left(\frac{\Psi_{I,J+1} - \Psi_{I,J-1}}{\Delta\beta} \right)^2 \right\} \frac{f_{I,J}}{2} \right. \\ &\quad - \frac{(\Psi_{I,J+1} - \Psi_{I,J-1})(\Psi_{I+1,J+1} - \Psi_{I+1,J-1} - \Psi_{I-1,J+1} + \Psi_{I-1,J-1})}{4\Delta\alpha\Delta\beta^2} \\ &\quad + \left. \frac{(\Psi_{I+1,J} - \Psi_{I-1,J})(\Psi_{I,J+1} - 2\Psi_{I,J} + \Psi_{I,J-1})}{\Delta\alpha\Delta\beta^2} \right] - \frac{1}{R_e} \frac{Z_{I,J+1} - Z_{I,J-1}}{2\Delta\beta} \\ &\quad - \left[\bar{Q} \frac{\Psi_{I+1,J} - \Psi_{I-1,J}}{\Delta\alpha} + \bar{Q}^2 H_{I,J}^2 \{ \cos \beta (\sinh \alpha - \coth \alpha_2 \cosh \alpha) + \coth \alpha_2 \} \right] \dots\dots\dots (28 \cdot a) \\ \left(\frac{\partial P}{\partial \beta}\right)_{I,J} &= \frac{1}{2H_{I,J}^2} \left[\left\{ \left(\frac{\Psi_{I+1,J} - \Psi_{I-1,J}}{\Delta\alpha} \right)^2 + \left(\frac{\Psi_{I,J+1} - \Psi_{I,J-1}}{\Delta\beta} \right)^2 \right\} \frac{g_{I,J}}{2} \right. \\ &\quad - \frac{(\Psi_{I+1,J} - \Psi_{I-1,J})(\Psi_{I+1,J+1} - \Psi_{I+1,J-1} - \Psi_{I-1,J+1} + \Psi_{I-1,J-1})}{4\Delta\alpha^2\Delta\beta} \\ &\quad + \left. \frac{(\Psi_{I,J+1} - \Psi_{I,J-1})(\Psi_{I+1,J} - 2\Psi_{I,J} + \Psi_{I-1,J})}{\Delta\alpha^2\Delta\beta} \right] + \frac{1}{R_e} \frac{Z_{I+1,J} - Z_{I-1,J}}{2\Delta\alpha} \\ &\quad - \left[\bar{Q} \frac{\Psi_{I,J+1} - \Psi_{I,J-1}}{\Delta\beta} + \bar{Q}^2 H_{I,J}^2 \sin \beta (\cosh \alpha - \coth \alpha_2 \sinh \alpha) \right] \dots\dots\dots (28 \cdot b) \end{aligned} \right\} \dots\dots\dots (28)$$

where,

$$f_{I,J} = \{-\sinh \alpha / (\cosh \alpha - \cos \beta)\}_{I,J}$$

$$g_{I,J} = \{-\sin \beta / (\cosh \alpha - \cos \beta)\}_{I,J}$$

$$[f, g] \neq, f = (1/h)(\partial h / \partial \alpha),$$

$$g = (1/h)(\partial h / \partial \beta).$$

An outline of the method for obtaining the numerical solution is given in the following. (i) Arbitrary initial values are given to $\Psi_{I,J}$ and $Z_{I,J}$. At the boundaries, a certain constant value is given to $\Psi_{1,J}$ and zero to $\Psi_{21,J}$. (ii) Calculate $\Psi_{I,J}$ except at boundaries by Eq. (26). (iii) Then calculate $Z_{I,J}$ and $Z_{21,J}$ at boundaries by Eq. (27). (iv) Calculate $Z_{I,J}$ of the inside region by Eq. (25). (v) Repeat the steps (ii) ~ (iv) until the values of $\Psi_{I,J}$ and $Z_{I,J}$ satisfy the following condition of convergence.

$$\max |\Psi_{I,J}^n - \Psi_{I,J}^{n-1}| < 10^{-5}$$

$$\max |Z_{I,J}^n - Z_{I,J}^{n-1}| < 10^{-5}$$

where n is the number of repetitions. (iv) Using Eq. (28.b), evaluate the integral

$$\Delta \bar{P} = \oint_c (\partial \bar{P} / \partial \beta) d\beta.$$

If $\Delta \bar{P} \neq 0$, vary the initial value of $\Psi_{1,J}$ by small increment, and repeat the steps (i) ~ (v) until $\Delta \bar{P} = 0$ is attained. The step (vi) ensures the continuity of the pressure in the flow field. [Since Eqs. (25) ~ (27) give an infinite number of solution, a unique solution cannot be obtained without consideration of pressure continuity.] The integral $\Delta \bar{P}$ is taken along the circle at $I=11$ (see Fig. 5), and in steps (ii) ~ (iv) the successive over-relaxation method is used. From the results of numerical solutions for Ψ and Z , the pressure distribution can be obtained by Eq. (28), and the flow

forces of $(f_i)_x$, $(f_i)_y$ and M_i are also derived by respective formulae.

4.2 Flow visualization

Experiments are conducted in order for comparison with the results of the numerical analysis. The flow between the eccentric cylinders with radii $r_1 = 27.5\text{mm}$ and $r_2 = 110\text{mm}$ is examined. For the observation of the flow pattern a visualization technique (aluminium tracer method) is used. Dilute glycerine is used as the working fluid and both cylinders are made of transparent acrylic resin. In the experiments, the outer cylinder is fixed, while the inner cylinder is rotated. Figure 6 shows a schema of the experimental apparatus. The rotation speed can be adjusted by controller 2. In order to realize the revolutionary motion due to the flow force as purely as possible, the driving device is made such that its driving torque does not influence the revolutionary motion of inner cylinder. (It is conformed that the rotating cylinder driven by this device does not move when it is set into another vessel of enough large scale.) The distance between the axes A and B is adjusted by a screw C. A screw D constrains the arm E and the fixed axis A. If the screw D is loosened, the inner cylinder can revolve with constant eccentricity. The rotation rate of the inner cylinder is measured by a digital counter 5 through a light source 3, a small mirror attached on the rotating axis and a photoelectric cell 4 system. A projector 6 is used to take photographs of the flow pattern. The camera is placed on a turntable which can rotate synchronously with the revolution rate of inner cylinder, in order to obtain a flow pattern viewed from the rotating coordinate system.

4.3 The inertia effect

Fig. 6 Schema of experimental apparatus

The flow in the case of inner cylinder rotating at a fixed position is first treated. Figures 7(a) and (b) show the streamlines calculated for the Reynolds numbers $Re = 50$ and $Re = 300$ [$\kappa = 1/4$, $\epsilon = 2/3$, the corresponding Stokes solution is already presented in Fig. 2(a)]. As the Reynolds number increases, the flow pattern becomes asymmetric with respect to the line connecting both centers of cylinders, and the separation region becomes large. The center of the eddy in the separation region moves

(a) $\kappa = 1/4$, $\epsilon = 2/3$, $Re = 50$

(b) $\kappa = 1/4$, $\epsilon = 2/3$, $Re = 300$

(c) $\kappa = 0.227$, $\epsilon = 2/3$, $Re = 50$

Fig. 7 Numerical solutions of streamlines for the case of inner cylinder rotating in fixed outer cylinder [(a), (b)] and photograph of flow pattern [(c)]

in the direction of the rotation of cylinder. Figure 7(c) is a photograph of the flow pattern. It agrees well with the corresponding calculated pattern in Fig. 7 (a). Figure 8(a) and (b) show the calculated pressure distribution at each cylinder wall. The result from the Stokes solution is also given in the figure [Eq. (17)]. For comparison with the Stokes solution, the pressure is represented in terms of $\text{Re}(\bar{P}-\bar{P}_{1,1})$ and $\text{Re}(\bar{P}-\bar{P}_{21,1})$ where $\bar{P}_{1,1}$ and $\bar{P}_{21,1}$ indicate the nondimensional pressure at the point $(I=1, J=1)$ and the point $(I=21, J=1)$, respectively. The pressure distribution from the Stokes solution is point symmetric with respect to $\theta=\pi$. It is seen from Fig. 8 that the inertia force makes the pressure distribution asymmetric.

Figure 9 shows the x, y components of the flow force, $(f_1)_x, (f_1)_y$, calculated for various Reynolds numbers [the flow force is nondimensionalized by multiplying $1/(2\rho \times V_1^2 r_2)$], i.e., $F_1 = f_1/(2\rho V_1^2 r_2)$.

The value of $(f_1)_x$ is not equal to zero due to the inertia effect. There exists an extreme value of $(f_1)_x$ at certain Reynolds

Fig. 8 Pressure distribution on inner and outer cylinder walls

number.

Figure 10(a) shows the flow force $(f_1)_x$ [represented by $\text{Re}(f_1)_x$] versus the eccentricity ratio ϵ for a constant value of the radius ratio κ . For small eccentricity the flow force attracts the inner cylinder toward the outer cylinder but for large eccentricity the direction of flow force reverses. Figure 10(b) shows the moments M_1 and M_2 versus the eccentricity ratio ϵ . [M_i is nondimensional moment $M_i/(\rho V_1^2 (2r_2)^2)$]. The difference between the moments on both cylinders increases with the value of ϵ . In addition, it is confirmed in the numerical analysis that the relation of Eq. (22) still holds even if the inertia effect exists, and that the relation of $(f_1)_x = -(f_2)_x$ also holds. Calculated flow patterns for various values of the eccentricity ratio ϵ are shown in Fig. 11. The separation region appears at a certain value of ϵ , and expands as ϵ increases, which agrees with the result from the Stokes solution. The flow pattern obtained from the numerical analysis is close to the one from the Stokes approximation, when the eccentricity ratio ϵ is small. A photograph corresponding to

Fig. 9 Flow force exerted on inner cylinder versus Reynolds number (value of Stokes solution is denoted by $\bullet$)

Fig. 10 (a) Flow force exerted on inner cylinder versus eccentricity ratio ϵ ($\kappa=1/4, Re=200$)
(b) Moments on both cylinders versus ϵ

Fig. 11 Flow patterns for various values of eccentricity ratio ε ($\kappa = 1/4$, $Re = 200$)

Fig. 12 Calculated streamlines for the case of inner cylinder fixed in rotating outer cylinder ($\kappa = 1/4$, $\varepsilon = 2/3$, $Re = 50$)

Fig.11(c) is shown in Fig.11(d).

Figure 12 shows a calculated result of streamlines for the case that the outer cylinder rotates and the inner cylinder is fixed. It is seen from comparison with Fig.2(b) that the center of the eddy in the separation region moves in the direction

Fig. 13 Revolution rate of rotating inner cylinder versus Reynolds number ($\kappa = 1/4$, $\varepsilon = 2/3$)

of the rotation of outer cylinder, and the flow pattern turns asymmetric.

The revolutionary motion of the rotating

inner cylinder in the fixed outer cylinder is discussed in the following. In the numerical analysis the angular velocity Ω_0 of the revolution is determined such that $(f_1)_y$ reduces to zero, through the calculation for various values of $-V_2/V_1$. Figure 13 shows the value of Ω_0/ω versus the Reynolds number $[\omega = \Omega_0 + V_1/r_1, \omega$: rotation rate with respect to the stationary coordinate system]. In the figure both the value of Ω_0/ω obtained from the Stokes solution and the value of Ω'/ω are presented. Here Ω' is the revolution rate of vortex filament in an ideal fluid, which is obtained on the assumption of $\Gamma = 2\pi r_1^2 \omega$ (Γ : strength of vortex filament). As seen in the figure, the revolution rate increases under the influence of inertia. In the experiment on the revolution rate Ω_0 , which is carried out for various values of eccentric displacement e , the result shows that Ω_0 increases in proportion to ω , and also increases with e . However, the result from the numerical analysis that Ω_0 increases under the influence of inertia, cannot be verified quantitatively by the experiment. The photograph of the flow pattern taken by the camera set on the turntable, agrees with the result of the numerical analysis viewed from the rotating coordinate system [the pattern is similar to that of Fig. 2(c)].

5. Conclusion

This paper discusses a two dimensional flow between eccentric rotating cylinders for various boundary conditions, and examines mainly the inertia effect on the flow

pattern and the flow force. It should be noted that although it is known that the flow between eccentric rotating cylinders becomes unstable at a certain value of Reynolds number, the flow remains almost two dimensional for the Reynolds numbers up to 250 in this experiment for a geometric range of $\kappa = 1/4 \sim 1/2$ and $\varepsilon = 1/6 \sim 5/6$.

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