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New Proposal of Crack Energy Density Concept as a Fundamental Fracture Mechanics Parameter*

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Crack energy density concept is newly proposed as one of the most important parameters in fracture mechanics, and the role of it is discussed. This parameter represents the state of the work absorbed (strain energy in case of the continuum) at crack tip per unit area of crack plane generally, and it should be called, as it were, strain energy area density especially for the case of the continuum.

The introduction of the concept enables us to make a unified smooth description of the fracture condition from completely elastic (brittle) fracture to ductile fracture with large scale yielding, and also to solve some ambiguous problems in fracture mechanics. For example, the physical meanings of $Q_c(K_c)$ criteria; J_c criteria and COD criteria on fracture are explained clearly through the concept.

1. Introduction

The so-called fracture mechanics was initiated with Griffith's theory based on energy balance⁽¹⁾, and was modified to the Orowan-Irwin theory^{(2),(3)} so as to make it applicable to pseudo-brittle fracture. And it has been developed quite rapidly with the establishment of the concept of small scale yielding⁽⁴⁾ and introduction of the concepts of the stress intensity factor K and the energy release rate G by Irwin⁽⁵⁾. At present, having taken the COD⁽⁶⁾ and J -integral⁽⁷⁾ proposed as parameters for fracture with large scale yielding into the system, it appears to have almost established the system as a powerful engineering methodology for evaluation of the strength and fracture of material.

On the other hand, when it is examined closely, it becomes apparent that the present system of fracture mechanics involves some problems in its basic points. For example, if we take the fracture criterion, even in case of the most fundamental $Q_c(K_c)$ criteria for the small scale yielding, the relation to sufficient condition has not been made clear, although it is deemed as necessary condition by the energy balance consideration. Moreover as for COD or J criteria, even the interpretation on the energy balance has not been made, and the physical background to use these quantities is not established⁽⁸⁾. After all, it seems that the fracture mechanics up to now has been supported by enormous data with these basic problems still pending.

Then, what is the reason for the situation? Author believes it is that the fracture mechanics so far has lacked the most fundamental concept concerned in the physical

background of fracture. The aim of this paper is to propose a new concept as a major parameter of fracture mechanics and to show the principle and the role of the concept. In the previous papers^{(9),(10)}, author has discussed the physical meaning of J -integral, and departing from usual meaning of J -integral as the energy release rate, he has shown that physically the integral has a meaning of strain energy area density. Furthermore, for the case of continuum it was shown that by considering the strain energy area density as a fracture mechanics parameter, the necessary and sufficient condition of fracture would become clear, and many obscure points in fracture mechanics would be clarified. In this paper, the concept of "crack energy density" which is a more generalized concept of strain energy area density for the continuum is proposed as a major parameter of fracture mechanics, and the role of it in fracture mechanics is discussed. The reason why the parameter defined with continuum is available to actual material in which the crack tip would not satisfy the condition of the continuum is explained and also some of basic errors in understanding of fracture mechanics are pointed out.

2. Crack energy density $\mathcal{E}$

All of the fracture mechanics parameters in the past (except COD) are the quantities defined in the continuum model, and accordingly evaluation by elastic and plastic theory to continuum is possible. This fact is one of the main factors which caused the fracture mechanics to be developed as a practical technique. As the crack energy density is defined as a physical quantity, it can be defined in models other than a continuous body model by considering each mechanism as will be shown in the section 4. In this section, at first the definition for the continuum model is demonstrated, taking above circumstances into account. The crack energy density is

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generally defined as "the quantity expressed per unit area of the work done on the point before deformation of crack tip in a crack plane by outside of the plane through deformation". Thus, as the work done on a given part by the outside is represented as strain energy for the continuum, by defining σ_{ij} , and ϵ_{ij} is the tensor components of the stress and the strain respectively, and the strain energy density W as

$$dW = \sigma_{ij} d\epsilon_{ij}, \quad W = \int dW \dots\dots\dots (1)$$

the crack energy density for the plane containing a crack is defined by

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \int_{\Gamma_0} W dx_2 \dots\dots\dots (2)$$

(Refer to Fig.1(a), (b)). Here the quantity defined by Eq.(2) is considered. Since $W dx_2 = W \cdot 1 \cdot dx_2 \cdot 1$, $W dx_2$ represents strain energy per volume enclosed by the solid line in Fig.2(a) and (b) (note that unit thickness is considered), and in case of Fig.2(a) $\int_{\Gamma_0} W dx_2$ represents the strain energy per volume enclosed by the broken line contributing to transfer of Γ_0 part of the notch to the right (direction). In case of Fig. 2(b), the integral of left side part of the

Fig.1 Integration path

circular notch in $\int_{\Gamma_0} W dx_2$ will be negative and it means that the strain energy per volume at left side enclosed with broken line, contributing to making the left side part of Γ_0 move to right is negative. That is, further it means that it (left side part integral) is the quantity contributing to making the Γ_0 part move to left. As a whole $\int_{\Gamma_0} W dx_2$ over whole circle of the circular notch is the strain energy contributing to making the Γ_0 part per whole volume enclosed with broken line (sum of the volumes of the left and right sides) move to right. Now, when $\rho \rightarrow 0$ in Fig.2(a) or (b), the notch turns to a crack and the volume enclosed with broken line in each case is reduced to an area shown with broken line in Fig.2(c). Thus $\mathcal{E}$ defined by Eq.(2) turns out what represents strain energy per unit area shown with broken line in Fig.2(c) contributing to development of a crack. That is, it gives the crack energy density to continuum. In the above discussion $\mathcal{E}$ was defined as a limit when ρ was changed as Eq.(2) from a notch such as in Fig.2(a) or (b). However, generally, we will be able to define $\mathcal{E}$ in an arbitrary process of change of the shape so long as the shape of the notch can be changed to the crack shown in Fig.1(c) as the limit, and a fixed limit value of $\mathcal{E}$ can be obtained independently of the process of change of the shape. This will be proved in the following section for the case that a constitutive equation is given by $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$.

Now let's think about W in Eq.(1). The integration is meant to be performed along the load path, and in case that plastic deformation is expected, then

$$\left. \begin{aligned} dW &= dW^e + dW^p = d\epsilon_{ij}^e + \sigma_{ij} d\epsilon_{ij}^p \\ W &= \int dW = \int (dW^e + dW^p) = W^e + W^p \end{aligned} \right\} (3)$$

Fig.2 Physical meaning of crack energy density $\mathcal{E}$

where $d\epsilon_{ij}^e$ and $d\epsilon_{ij}^p$ are elastic and plastic strain increments respectively, dW^e and dW^p are the corresponding strain energy increments, and W^e and W^p are elastic strain energy density and plastic strain energy density respectively. Thus the crack energy density $\mathcal{E}$ is generally represented by

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \int_{\Gamma_0} (W^e + W^p) dx_2 = \mathcal{E}^e + \mathcal{E}^p \dots\dots\dots (4)$$

where $\mathcal{E}^e$ and $\mathcal{E}^p$ are elastic crack energy density and plastic crack energy density, respectively. With regard to evaluation of crack energy density $\mathcal{E}$ with plastic deformation, in case that the plastic behavior follows the deformation theory, it can be evaluated by J -integral as will be shown in next section. Even in other cases where the plastic behavior follows the incremental theory, definition of crack energy density by Eqs. (2) or (4) is valid, apart from how to evaluate it actually, and it should be noted that the physical meaning is not lost even in such case that unloading will occur.

When we deal with the continuum, stress, strain and strain energy density W at crack tip generally have singularities and it may be noted that crack energy density by Eq. (2) or (4) is one expression characterizing the singular point itself at the crack tip. Also crack energy density has a meaning of strain energy area density in case of continuum model as shown above, and it is apparent that strain energy area density in continuum takes a finite value other than zero only at singular (specific) point such as a crack tip.

3. Relation between crack energy density and other fracture mechanics parameters

The aim of this section is to elucidate the relations between the crack energy density defined in the previous section and other well-known fracture mechanics parameters defined based on the continuum mechanics.

3.1 Relation to J -integral

No matter whether it is linear or non-linear, in cases where generally the constitutive equation is given by $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$ such as when continuum shows elastic behavior or plastic behavior following the deformation theory, then J -integral can be introduced as an integral independent of the integration path, that is,

$$J = \int_{\Gamma} (W dx_2 - T u_{i,1} ds) \dots\dots\dots (5)$$

where T_i is the traction force and u_i is the displacement on Γ in Fig. 1. Then, the crack energy density $\mathcal{E}$ is given as

$$\mathcal{E} = \lim_{\rho \rightarrow 0} \int_{\Gamma_0} W dx_2 = \lim_{\rho \rightarrow 0} J^n = \lim_{\rho \rightarrow 0} \int_{\Gamma} (W dx_2 - T u_{i,1} ds)^n = \int_{\Gamma} (W dx_2 - T u_{i,1} ds)^c = J^c \dots\dots\dots (6)$$

and thus the crack energy density $\mathcal{E}$ is given as the same value as J -integral for crack. Here the superscript n or c means notch or crack respectively and the former corresponds to the case in Fig. 1(a), (b) and the latter to that in Fig. 1(c). Fur-

ther, from the process of Eq. (6), it is clear that the equation $\mathcal{E} = J^c$ holds even for $\mathcal{E}$ defined through the process of change of an arbitrary notch shape as mentioned in the previous section, and this turns out to be a proof of independence of $\mathcal{E}$ from the shape change process of notch for the case $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$.

3.2 Relation to stress intensity factor

When we consider a linear elastic material, we can obtain the following equation applying a well-known singular solution for the vicinity of a crack tip,

$$J^e = \int_{\Gamma} (W dx_2 - T u_{i,1} ds)^e = \lim_{\epsilon \rightarrow 0} \int_{\Gamma} (W dx_2 - T u_{i,1} ds)^e = (1 - \nu^2 / E) (K_I^2 + K_{II}^2) + (1 / 2G) K_{III}^2 \dots\dots (7)$$

where E and G are Young's modulus and shear modulus respectively, ν is Poisson's ratio for plane strain and is zero for plane stress, and K_I , K_{II} or K_{III} is a stress intensity factor which corresponds to each mode, and therefore $\mathcal{E}$ is given as

$$\mathcal{E} = (1 - \nu^2 / E) (K_I^2 + K_{II}^2) + (1 / 2G) K_{III}^2 \dots\dots\dots (8)$$

Further, under the condition of so-called small scale yielding, this relation approximately holds good, then

$$\mathcal{E} \approx (1 - \nu^2 / E) (K_I^2 + K_{II}^2) + (1 / 2G) K_{III}^2 \dots\dots (9)$$

because the state of deformation sufficiently far from the crack tip is thought to be almost the same as the case with no plastic deformation and the value of J for a path sufficiently far from the crack tip is nearly equal to that for completely elastic case.

3.3 Relation to energy release rate

For linear elastic material there is a relation

$$\mathcal{G} = (1 - \nu^2 / E) (K_I^2 + K_{II}^2) + (1 / 2G) K_{III}^2 \dots\dots (10)$$

between energy release rate $\mathcal{G}$ and stress intensity factors, and thus compared with Eq. (8), $\mathcal{E}$ is related to $\mathcal{G}$ as $\mathcal{G}_R$

$$\mathcal{E} = \mathcal{G} \dots\dots\dots (11)$$

Furthermore, even in a non-linear case, since J^c in Eq. (6) is known to have a meaning of energy release rate for non-linear elastic material, the relation

$$\mathcal{E} = J_R^c \dots\dots\dots (12)$$

is obtained from Eq. (6) when the expression J_R^c is used as J^c specifically with meaning of the energy release rate. The relations in Eqs. (11) and (12) can be fully understood in terms of feeling too, when we consider that the crack energy density means the elastic strain energy stored at the crack tip in the plane containing the crack per unit area shown with broken line in Fig. 2(c) in case of elastic material, and also that energy release rate is the energy released per unit area when the crack grows in the

interplane direction. Although these two quantities have different physical definitions from each other, they can be considered to represent one state of a crack, as it were, viewed from inside and outside respectively. It may be considered that $\mathcal{E}$ turns to $\mathcal{G}$ or J_R as a crack grows. Though the energy release rate in the case where a plastic region exists in the vicinity of the crack tip has not been defined as a general parameter in fracture mechanics, it is an important quantity as a necessary condition for crack growth. Therefore, a new definition for it shall be proposed here for the case where plastic region exists as available energy $\mathcal{E}$ actually released at the crack tip per unit area when a crack grows. Under the existence of plastic region, $\mathcal{E}$ is expressed as the sum of that of elastic part $\mathcal{E}^e$ and not of plastic part $\mathcal{E}^p$. The energy actually released at the crack tip for growth of crack is $\mathcal{E}^e$ only, thus

$$\mathcal{G}_R = \mathcal{E}^e \dots\dots\dots(13)$$

Here it should be noted that even though the relation (9) holds for small scale yielding, the relation

$$\mathcal{G} \approx (1-\nu^2/E)(K_I^2 + K_{II}^2) + (1/2G)K_{III}^2 \dots(14)$$

apparently does not hold, although in the past fracture mechanics, this has been assumed to be true for small scale yielding.

3.4 Relation to compliance

Now consider a cracked specimen under load P as shown in Fig.3. In such a case the following relation is widely known to hold for linear elastic material,

$$\mathcal{G} = (1/2)P^2(d\lambda/dA) \dots\dots\dots(15)$$

where λ is compliance and A is the (net sectional) area of crack plane (length of the crack $\times$ thickness). From this relation and Eq.(11), the relation

$$\mathcal{E} = (1/2)P^2(d\lambda/dA) \dots\dots\dots(16)$$

Fig.3 Cracked specimen under tension

is obtained.

In the past, $\mathcal{G}$ has been often evaluated through the experimental evaluation of compliance λ based on Eq.(15). What should be noted, however, is that the value of $(1/2)P^2(d\lambda/dA)$ so evaluated does not give the energy release rate even in the approximate meaning under the circumstance where a load is imposed on an actual sample piece (where a small scale yielding occurs with no exception), although it certainly evaluates the energy release rate (and also the crack energy density at the same time since Eqs.(11) and (15) hold) in such a case where the sample piece satisfies the condition of linear elasticity ideally. In Fig.4 the load-displacement relations are shown for the net sectional areas of crack planes A and $A+dA$, where the difference between ideal elastic material and actual material (with small scale yielding) is exaggerated. In this case, the areas of the part with vertical lines and of the part with horizontal lines may always represent $\mathcal{E}dA$ of interest regardless of existence of yielding at the crack tip, if $\mathcal{E}dA$ is understood as a sum of $\mathcal{E}$ determined for each stage of the (net sectional) area between A and $A+dA$. However, it is clear that whole area of the part with vertical lines is releasable energy, on the other hand only a part (that is, $\mathcal{E}^e dA$) of the area with horizontal lines corresponds to the releasable energy, and since (the area with horizontal lines) $\approx$ (the area with vertical lines for small scale yielding case),

$$\mathcal{E} \approx (1/2)P^2(d\lambda/dA) \dots\dots\dots(17)$$

can be true, but

$$\mathcal{G} \approx (1/2)P^2(d\lambda/dA) \dots\dots\dots(18)$$

does not hold. It goes without saying that the above discussion corresponds to the discussion in section 3.3 saying that Eq. (14) does not hold, although the relation of Eq.(9) is true.

Now, as an evaluation method of J -integral, experimental determination of it from the load-displacement curve has been extensively performed, without clarification of its meaning as a physical quantity, with regard to J_{IC} fracture toughness test,

Fig.4 Load-displacement curves for linear elastic and small scale yielding cases

based on the fact that J -integral has the meaning of energy release rate for non-linear elastic material. From the above discussion, however, it is apparent that the area with horizontal lines of Fig.4 represents $\mathcal{E}dA$ even in the case with a large scale yielding, and it can be said that in fact it has been evaluation of the crack energy density with clear meaning as a physical quantity.

3.5 Relation to plastic stress intensity factor and plastic strain intensity factor

When the work hardening curve after yielding is expressed by N th power rule, the following relation is known to hold⁽¹⁾, according to the procedure similar to that for Eq.(7),

$$J^c = A_N K_t K_\sigma \dots \dots \dots (19)$$

Where A_N is a coefficient relation to the hardening index N and K_σ and K_t are plastic stress intensity factor and plastic strain intensity factor, respectively^{1,2,3}. And since Eq.(6) holds generally in deformation theory, the following relation also holds,

$$\mathcal{E} = A_N K_t K_\sigma \dots \dots \dots (20)$$

4. Crack energy density $\mathcal{E}$ for actual material and its validity

In actual material, the deformation close to a crack tip does not satisfy the requisites as a continuum. In this section, the crack energy density $\tilde{\mathcal{E}}$ ($\tilde{\mathcal{E}}$ is used in order to distinguish $\mathcal{E}$ for the continuum model) for such case will be considered and the reason why $\mathcal{E}$ defined in the continuum model is valid as a fracture parameter, (even for such case) in spite of the requisite as a continuum not being satisfied, is discussed. Further the relation between crack energy density and COD is also clarified. In this section only the deformation for opening mode (mode I) is considered.

Fig.5 Barenblatt model

4.1 Ideally elastic material

In the continuum model the crack tip is thought to be infinitely sharp. In actual material, however, this is impossible. Therefore Barenblatt's model⁽¹⁴⁾ which is thought to be closer to actual case is considered. That is, as shown in Fig.5, we substitute a case where a cracked material with a crack $a+\omega$ long is subjected to the external force and bonding force at the ω part of the inner surface of the crack for a state of a cracked material with a crack of length a subjected to the external force. Here, the width of ω is determined such as to eliminate the singularity at the tip of the $a+\omega$ crack, and $\sigma(\delta)$ is determined by assuming the amount of opening δ at the ω part of the $a+\omega$ crack to be the increase δ of the distance between atomic planes. Regarding the crack tip (note the length of the actual crack is a), $\sigma(\delta)$ increases at the beginning with an increase of the amount of opening $\phi [= \delta(0)]$ as the external force increases up to maximum value $\sigma_{\max}$ (refer to Fig.6), then starts to decrease, gets to zero at $\phi = \phi_c$, and comes to satisfy the condition for the crack tip to present a new surface, that is, for the crack to grow.

In this model the displacement of a point at the crack tip before deformation becomes polyvalent after deformation. This corresponds to the fact that stress and strain become infinite in case of the continuum. By the way, a quantity called COD is thought to be best defined as the maximum of the differences among the polyvalenced displacements generally in the case where a point at the crack tip before deformation turns polyvalent (so it is for the most of actual materials).

Now, regarding the crack energy density of this model, since the crack energy density has been defined as "the quantity expressed per unit area of the work done on the point before deformation of crack tip in crack plane by outside of the plane through deformation", the crack energy density $\tilde{\mathcal{E}}$ for the present case is given by

$$\tilde{\mathcal{E}} = \int_0^{\phi} \sigma(\delta) d\delta \dots \dots \dots (21)$$

Here, J -integral of this model is written as $\tilde{J}^c$ in order to distinguish it from J^c for the continuum model, and then the following relation for $\tilde{J}^c$ is obtained

Fig.6 Variation of bonding force between atomic planes

$$\begin{aligned}\tilde{J}^e &= \int_{r_0} (W dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds) = \int_{r_0} (-T_i \frac{\partial u_i}{\partial x_1} ds) = \int_{r_0} \sigma \frac{\partial u_2}{\partial x_1} dx_1 \\ &= \int_0^\omega \sigma \left\{ \frac{\partial u_2(x_1, 0^-)}{\partial x_1} - \frac{\partial u_2(x_1, 0^+)}{\partial x_1} \right\} dx_1 = - \int_0^\omega \sigma(\delta) \frac{\partial \delta}{\partial x_1} dx_1 = \int_0^\phi \sigma(\delta) d\delta \dots\dots\dots (22)\end{aligned}$$

Since in general ω is sufficiently small compared to the crack length a , a relation

$$\tilde{J}^e \approx J^e = \mathcal{E} = \left(\frac{1-\nu^2}{E} \right) K_I^2 \dots\dots\dots (23)$$

holds, and after all, from Eqs.(21), (22) and (23) we obtain a relation

$$\tilde{\mathcal{E}} = \tilde{J}^e \approx J^e = \mathcal{E} \dots\dots\dots (24)$$

This relation indicates that $\mathcal{E}$, which has been given as a quantity representing the property of the singularity itself at the crack tip in the linear elastic continuum model, appears to give $\tilde{\mathcal{E}}$ approximately which represents the state of a crack tip in a model closer to actual material, and because of this fact $\mathcal{E}$ defined in the continuum model is regarded to valid as a fracture parameter. Further, Eq.(21) appears to give the relation between $\tilde{\mathcal{E}}$ and COD, and thus these are found to be in one-to-one correspondence with each other.

4.2 Case with plastic region at crack tip

The Dugdale model⁽¹⁵⁾ is a special case of the Barenblatt model and corresponds to the case where the bonding force $\sigma(\delta)$ in Fig.5 is set at a constant value σ_Y (yielding stress). The crack energy density in this model is given as

$$\tilde{\mathcal{E}} = \frac{1}{2} \sigma_Y \phi^e + \int_0^{\phi^p} \sigma_Y d\delta = \frac{1}{2} \sigma_Y \phi^e + \sigma_Y \phi^p = \tilde{\mathcal{E}}^e + \tilde{\mathcal{E}}^p \dots\dots\dots (25)$$

where ϕ^e and ϕ^p are the elastic opening displacement and plastic opening displacement respectively, and $\phi = \phi^e + \phi^p$, thus in this case the crack energy density becomes the sum of the elastic fraction $\tilde{\mathcal{E}}^e$ and the plastic fraction $\tilde{\mathcal{E}}^p$. By the way, in this model ϕ^e becomes constant (thus $\tilde{\mathcal{E}}^e$ is constant as well) with an increase of external force since the ideal elasto-plastic material is assumed. It is apparent that the relation in Eq.(24) holds good as is in the Barenblatt model and hence it follows that the relation between $\tilde{\mathcal{E}}$ and COD is given by Eq.(25).

In the next, we consider a general case where a plastic region such as the part with oblique lines in Fig.7 is formed. In the case when it is supposed that the crack tip has been blunted as shown in Fig.7 and thus a point representing the crack tip before deformation has spread to two points indicated by large black points in the figure, then COD would be defined as the distance ϕ between the two points. Here let's consider a path Γ_0 shown with broken line in the figure for the state after deformation. We regard the part outside of Γ_0 and the part inside of it except neighborhood of the crack tip to be a continuum which follow the deformation theory, and make T_1 and T_2 represent the traction forces formed actually on Γ_0 . Then the crack energy density of the crack in Fig.(7) is given by the following equation,

$$\tilde{\mathcal{E}} = \int_0^\phi |T_2(\delta)| d\delta \approx \int_0^\phi \sigma_{22}(\delta) d\delta \dots\dots\dots (26)$$

Furthermore if we regard $\tilde{J}^e$ as J -integral in this model and J^e as J -integral in the continuum model in which whole body including the part of crack tip follows the deformation theory, and express the displacements upper side and lower side of Γ_0 , putting x_1 axis in between, by u_i^+ and u_i^- , respectively, then since $\partial u_i^- / \partial x_1 = \partial u_i^+ / \partial x_1$, the following equation is obtained,

$$\begin{aligned}\tilde{J}^e &= \int_{r_0} (W dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds) \approx \int_{r_0} (-T_i \frac{\partial u_i}{\partial x_1} ds) = \int_0^\omega |T_2| \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_2^+}{\partial x_1} \right) dx_1 \\ &= - \int_0^\omega |T_2(\delta)| \frac{\partial \delta}{\partial x_1} dx_1 = \int_0^\phi |T_2(\delta)| d\delta \dots\dots\dots (27)\end{aligned}$$

Here the relations $dx_2 \approx 0, dx_1 \approx ds$ on Γ_0 are used. Furthermore, at the place which is sufficiently far from the crack tip, it is likely that the deformation states are not much different between this model and the continuum model, and thus we get

$$\tilde{J}^e \approx J^e = \mathcal{E} \dots\dots\dots (28)$$

From these relations we find that the following relation holds good even for the general elasto-plastic state,

$$\tilde{\mathcal{E}} \approx \tilde{J}^e \approx J^e = \mathcal{E} = A_N K_I K_\sigma \dots\dots\dots (29)$$

When it satisfies small scale yielding, then of course the equation

$$\tilde{\mathcal{E}} \approx \mathcal{E} \approx (1-\nu^2/E) K_I^2 \dots\dots\dots (30)$$

holds good.

As is evident from above discussion, it can be thought that the essence of the fact that

$\tilde{\mathcal{E}}$ derived from the continuum model is valid even for the case with plastic region lies in that it approximately gives $\tilde{\mathcal{E}}$ (which is) for a more realistic model. Further it is found that the relation between $\tilde{\mathcal{E}}$ and COD is given by Eq.(26), and COD is a quantity that generally have one-to-one correspondence to $\tilde{\mathcal{E}}$.

5. On the interpretation of fracture mechanics parameters

It seems that the basic conception to fracture mechanics so far has been to discuss the fracture phenomena on the basis of the fact that the state of the crack tip vicinity (involving the crack tip) is determined uniquely by a single fracture mechanics parameter, setting aside the character of the crack tip itself and the physical background of the fracture mechanics parameters themselves in actual fracture phenomena. And it is surely true that this has been a motive force which developed the fracture mechanics into a technical methodology. As is apparent from the discussion in the previous section, however, the reason why fracture mechanics has been valid is not because the fracture mechanics parameters have been quantities giving information on the area involving the crack tip, but because they give information on the crack tip itself, and this viewpoint should be emphasized to come closer to the essence of fracture. After all, only $\tilde{\mathcal{E}}$ which is determined according to the form of actual fracture as shown in the previous section or COD which has one to one correspondence to $\tilde{\mathcal{E}}$ can have their physical background in actual fracture phenomena (of course the discussion here is restricted to the mode I (opening mode) fracture). All other parameters determined from the continuum model seem to have meaning only as quantities which make it possible to evaluate $\tilde{\mathcal{E}}$ or COD indirectly. That is $\mathcal{E}$, bears meaning as a quantity making it possible to evaluate $\tilde{\mathcal{E}}$ or COD indirectly and K , K_I or K_{II} is thought to be a quantity which represents the measure of $\tilde{\mathcal{E}}$ or COD of a crack-tip itself through $\mathcal{E}$. Thinking in this way, we may be able to make the restriction in definition of the small scale yielding looser than it has been. That is,

Fig.7 Plastic deformation at the vicinity of crack tip

so far we have been thinking the small scale yielding as the case where if the same K is given then always the same elasto-plastic deformation occurs, but thinking in the way above it appears that the important condition for the small scale yielding is that the same values of $\tilde{\mathcal{E}}$ or COD at the crack tip be given for the same value of K . This condition is satisfied if Eq.(9) holds good, even if the elasto-plastic deformations are not the same for the same values of K .

J -integral is the quantity which evaluates $\mathcal{E}$ directly if it follows the deformation theory, and it plays an important role because of its path independency as is apparent from the discussion in this paper. It has been playing the role of $\mathcal{E}$ as well, as a result, without clarification of its meaning as a physical quantity. It seems to be reasonable to define J -integral only as an integral having path independency, in order to avoid confusion in the establishment of fracture mechanics in future. By the way, the evaluation of J -integral so far performed through calculation has naturally been an evaluation of $\mathcal{E}$, and the experimental evaluation through the load-displacement curve as discussed a little in the section 3.4 is a direct evaluation of $\tilde{\mathcal{E}}$ but not $\mathcal{E}$, and thus these hold sufficient significance.

6. Conclusions

The concept of crack energy density as a fundamental parameter in fracture mechanics has been established and its relations to other fracture mechanics parameters have been clarified in order to make clear the position of it in the system of fracture mechanics. It is believed that introduction of the concept will help the construction of a clearer and more comprehensible system of fracture mechanics, and promise future development in this field.

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