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Vortex Breakdown Phenomena in a Circular Pipe *

(2nd Report, Flow Modes of Unsteady Type Breakdowns)

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It has been reported by many other authors that the vortex breakdown phenomena may occur in several distinct types. The aim of this study is to obtain a reasonable and consistent explanation for the various flow patterns of breakdowns.

In the preceding report, we have pointed out that the breakdown phenomena depend greatly upon the occurrence of internal waves within a swirling flow field, and also that the type of breakdown so far reported corresponds to an azimuthal mode of the wave disturbance.

The preceding paper, however, treated only steady (time-independent) types of breakdowns. The present paper concerns unsteady (or periodic with time) types of breakdowns. It is found that the time-dependency of breakdown has a close relation to the phase velocity of the wave disturbance.

1. Introduction

The vortex breakdown phenomena, which manifest abrupt changes along the swirling axis, occur in various forms, and give complicated flow fields. Although there are many theoretical and experimental studies on the phenomena, characteristics of breakdown have not been clarified enough. In the preceding paper^(1,2), we proposed a qualitative explanation of the vortex breakdown occurring in a pipe. There, it was found that the breakdown phenomena should depend greatly upon the internal wave (wave disturbance), the occurrence of which induces an abrupt change in a swirling flow field. It was also pointed out that various types of breakdowns, such as axisymmetric, spiral and double helix types, correspond to the azimuthal modes of the wave disturbances. The preceding paper, however, treated only steady (or stationary) types of breakdowns. Sarpkaya⁽³⁾, Leibovich⁽⁴⁾ and many others reported in their experimental studies that among the breakdowns visualized with dye filaments, there are such unsteady types that the streaklines rotate periodically with time about the swirling axis (the pipe axis). The present paper concerns these unsteady types of breakdowns, and shows that these types have a close relation to the phase velocity with which the wave disturbance

propagates in the swirling flow within a pipe.

2. Wave disturbance in a swirling flow within a pipe

It has been known, as mentioned in the preceding paper, that a wave disturbance (internal wave) can occur in a swirling flow which has an axial velocity. In preparation for the following discussions, expressions of the disturbance are again introduced briefly, and then the properties of the disturbance are presented.

2.1 Expressions of disturbance

In the analysis, the following assumptions are made in the same way as in the preceding study: (1) Fluid is incompressible and inviscid. (2) The primary flow before the occurrence of disturbance (to be referred to as mean flow) is a cylindrical flow whose azimuthal and axial velocity distributions are $\bar{V}(r)$ and $\bar{W}(r)$, respectively, both of which are arbitrary functions of the radius r . (3) The velocity variations of disturbance are small as compared with $\bar{V}(r)$ and $\bar{W}(r)$. According to assumption (1), the basic equations of flow are now Euler equations and a continuity equation. A stationary cylindrical coordinate system is used, as shown in Fig. 1. From assumption (2), the mean flow is expressed as

$$\begin{aligned} \bar{u} &= 0, \quad \bar{v} = \bar{V}(r), \quad \bar{w} = \bar{W}(r) \\ \bar{p}/\rho &= (\bar{p}_0/\rho) + \int_0^r \{\bar{V}(r)^2/r\} dr \end{aligned} \quad \dots\dots\dots (1)$$

Fig. 1 Coordinate system

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where u , v and w are the velocity components of r , θ and z ; p pressure; $\bar{p}$ pressure at the pipe axis; ρ density of fluid (the over bar denotes the mean flow). The flow field can be expressed in a sum of the mean flow and disturbance components as follows.

$$\left. \begin{aligned} u &= \bar{u}(r, \theta, z, t), \quad v = \bar{V}(r) + \tilde{v}(r, \theta, z, t), \quad w = \bar{W}(r) + \tilde{w}(r, \theta, z, t) \\ p &= \bar{p}_0 + \rho \int_0^r (\bar{V}^2/r) dr + \tilde{p}(r, \theta, z, t) \end{aligned} \right\} \dots\dots\dots (2)$$

where the tilde $\sim$ indicates the disturbance component and t indicates time. Applying Eq. (2) to the basic equation and using assumption (3) yields the following linear equations with respect to the disturbance components:

$$\left. \begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \frac{\bar{V}(r)}{r} \frac{\partial \tilde{u}}{\partial \theta} + \bar{W}(r) \frac{\partial \tilde{u}}{\partial z} - \frac{2\bar{V}(r)}{r} \tilde{v} &= -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial r} \\ \frac{\partial \tilde{v}}{\partial t} + \left(\frac{d\bar{V}(r)}{dr} + \frac{\bar{V}(r)}{r} \right) \tilde{u} + \frac{\bar{V}(r)}{r} \frac{\partial \tilde{v}}{\partial \theta} + \bar{W}(r) \frac{\partial \tilde{v}}{\partial z} &= -\frac{1}{\rho} \frac{1}{r} \frac{\partial \tilde{p}}{\partial \theta} \\ \frac{\partial \tilde{w}}{\partial t} + \frac{d\bar{W}(r)}{dr} \tilde{u} + \frac{\bar{V}(r)}{r} \frac{\partial \tilde{w}}{\partial \theta} + \bar{W}(r) \frac{\partial \tilde{w}}{\partial z} &= -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z} \\ \frac{\partial \tilde{u}}{\partial r} + \frac{\tilde{u}}{r} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial z} &= 0 \end{aligned} \right\} \dots\dots\dots (3)$$

These equations are linear with respect to $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$, but it is still difficult to obtain the analytical solution because $\bar{V}(r)$ and $\bar{W}(r)$, involved in the coefficients, are assumed to be arbitrary functions of r (c.f., Chap. 5). Hence, another assumption concerning the mean flow is added: (2') The axial velocity of mean flow is uniform, and the azimuthal velocity is that of rigid rotation. That is,

$$\bar{W}(r) = W_0 = \text{constant}, \quad \bar{V}(r) = \Omega r \dots\dots\dots (4)$$

Under assumption (2'), Eq. (3) can be solved analytically. The solution, which describes a neutrally stable wave disturbance, is written in the following form:

$$\left. \begin{aligned} \tilde{u} &= \frac{Am}{\sigma} \left[J_{s-1}(\sigma r) - \frac{2\Omega + (n-s\Omega - mW_0)}{n-s\Omega - mW_0} \left(\frac{s}{\sigma r} \right) J_s(\sigma r) \right] \sin(nt - s\theta - mz) \\ \tilde{v} &= \frac{Am}{\sigma} \frac{1}{n-s\Omega - mW_0} \left[2\Omega J_{s-1}(\sigma r) + \frac{s}{\sigma r} (2\Omega + (n-s\Omega - mW_0)) J_s(\sigma r) \right] \cos(nt - s\theta - mz) \\ \tilde{w} &= A J_s(\sigma r) \cos(nt - s\theta - mz), \quad \tilde{p} = \frac{A\rho}{m} (n-s\Omega - mW_0) J_s(\sigma r) \cos(nt - s\theta - mz) \end{aligned} \right\} \dots\dots\dots (5)$$

where A is an arbitrary constant of the amplitude of disturbance, n an angular velocity, m a real number, s zero or a positive integer, J_s the s -order Bessel function of the first kind; and σ is defined as;

$$\sigma = [m^2(4\Omega^2 - (n-s\Omega - mW_0)^2) / (n-s\Omega - mW_0)^2]^{1/2} \dots\dots\dots (6)$$

The eigenvalues are determined to solve the following equation:

$$J_{s-1}(\sigma a) / J_s(\sigma a) = (s/\sigma a) \{1 \pm \sqrt{1 + (\sigma/m)^2}\} \dots\dots\dots (7)$$

which is derived from the boundary condition at the pipe wall ($\tilde{u} = 0$ at $r = a$, a : pipe radius). The above equation has an infinite number of positive eigenvalues, whether the double sign in the right hand side of Eq. (7) is positive or negative. It is easily seen that a higher order of the eigenvalue corresponds to a higher mode of the disturbance in the r direction. However, since oscillatory phenomena usually occur in comparatively lower modes, the following discussion is limited to the case of the least order eigenvalue.

2.2 Properties of the wave disturbance

The properties of the wave disturbance represented by Eq. (5) are examined in this section. As seen from Eq. (5), when the integer parameter s is assumed to be zero, the disturbance becomes axisymmetric and propagates only in the axial direction (the z direction). On the other hand, when $s \neq 0$, the disturbance propagates both in the axial and azimuthal directions. Furthermore, it is found in the preceding paper that the parameter s specifies the azimuthal mode of the disturbance so that $s = 0, 1$ and 2 correspond to the axisymmetric, spiral and double helix types, respectively.

In the discussion below, the following nondimensional quantities are used:

$$\left. \begin{aligned} R_0 (\text{Rossby number}) &= W_0 / (a\Omega), \quad N = n/\Omega, \quad M = am \\ \sigma' = \sigma a, \quad T = \Omega t, \quad A' = A / (a\Omega), \quad R = r/a, \quad Z = z/a \end{aligned} \right\} \dots\dots\dots (8)$$

The angular frequency of disturbance can be derived from Eq. (6), that is,

$$N_z = s + MR_0 \pm 2\sqrt{M^2 / (M^2 + \sigma'^2)} \dots\dots\dots (9)$$

and the axial velocity (N/M) is written as

$$N_z / M = (s/M) + R_0 \pm 2\sqrt{1 / (M^2 + \sigma'^2)} \dots\dots\dots (10)$$

where the double sign in Eqs. (9) and (10) corresponds to that of Eq. (7). The wave disturbance given by Eq. (5) indicates a forward or backward progressive wave when the value of N/M is positive or negative, respectively. When $N/M = 0$ it becomes a standing wave which is fixed in the space. As reported in the preceding paper,^{1,0)} such a standing wave disturbance explains a stationary type of observed vortex breakdown.

Figure 2 plots the axial phase velocity N/M versus the wave length L_z ($L_z = 2\pi/M$, to be negative for $M < 0$) for $s = 0, 1$ and 2 . In the figure the Rossby number R_0 is taken as unity. The corresponding curves for any value of R_0 can be obtained by vertically shifting the curves for $R_0 = 1$, since as seen from Eq. (10) the intercepting point with the ordinate axis is at $N/M = R_0$, and the value of N/M is proportional to R_0 . Figures (a) and (b) give the corresponding curves for the positive and negative signs of the double sign in Eq. (10). The intersecting points of the curves and the abscissa ($N = 0$) determine the wave length of the standing wave mentioned above. These figures, calculated from Eqs. (7) and (10), represent the following properties of the wave disturbance occurring in a swirling flow; (i) For any value of the parameter s , the axial phase velocity of the mean flow in the limit of infinitesimal wave length. (ii) As the axial velocity of the mean flow, W_0 , becomes large, the range for the existence of a forward progressive wave becomes large and the absolute wave length of the standing wave also becomes large. (iii) In the case of $s = 0$, for the range of $R_0 > 0.522$, the phase velocity is always positive so that the disturbance with any wave length is a forward progressive wave. For the range of $R_0 < 0.522$, there exists such a wave length for the disturbance to be a standing wave. (iv) In the case of $s = 1$ or 2 , there are two standing waves with different wave lengths for any Rossby number. Furthermore, for $s = 1$ mode, two more standing waves can exist when $R_0 < 0.18$. (v) When $R_0 = 0$, that is, when the axial velocity of mean flow is

(a)

(b)

Fig. 2 Axial phase velocity of disturbance versus the wave length

absent, all types of disturbance with any wave length are to be unsteady, except a disturbance of $s = 1$ with a certain wave length. (vi) Only when $W_0 = 0$ can there exist a pair of forward and backward progressive waves whose wave length and absolute phase velocity are both equal so that a stationary wave, expressed as a sum of them, can take place in a swirling flow. (The term 'stationary' is used here for waves which have a node and a loop in the axial direction.)

As mentioned above, the disturbance occurring in a swirling flow exhibits different characteristics depending on the values of parameter R_0 , s and M .

3. Flow mode of unsteady disturbance and vortex breakdown

The flow modes of the disturbance given by Eqs. (2) and (5), are here examined, and a close relation between the disturbance and the vortex breakdown is clarified. As mentioned before, the purpose of this study is not to discuss the occurrence mechanism of breakdown or give a precise model including the nonlinear effect, but to explain the various flow patterns of breakdown through qualitative discussions. Therefore we assume that the disturbance is neutrally stable as given by Eq. (5) [however, in fact, Eq. (3) involves solutions other than the neutrally stable one], and the figures below are calculated by taking the amount of the disturbance as the order of unity in spite of the assumption of a small disturbance. Furthermore, most of the figures plot the calculated streaklines for comparison of the corresponding experimental results visualized with dye.

3.1 Flow mode of axisymmetric ($s = 0$) disturbance and bubble type of breakdown

Next we examine the unsteady flow perturbed by the disturbance of $s = 0$ with a finite phase velocity ($N/M \neq 0$). Figure 3 shows the calculated streaklines for the case of a forward progressive wave of the $s = 0$ disturbance ($N/M > 0$). The streaklines are calculated by computer, integrating Eqs. (2) and (5) numerically, and the results are obtained as the plotter output. Each figure describes six streaklines which surround the pipe axis with a small circle at the upstream end (here, $Z = 0$, and the disturbance is assumed to exist in the region of $Z > 0$). Figures (a), (b) and (c) show the streaklines at time $T = 0, 1/4\tau$ and $3/4\tau$, respectively ($\tau = 2\pi/|N|$). As seen from these figures, the axisymmetric vortex bubble, with a velocity of $|N|/|M|$. It should be noted that Fig. (c) shows discontinuous streaklines. In the experiment of flow visualization with dye, this discontinuity corresponds to the disappearance of the dye filament. In Fig. (c), discontinuity exists at points P, Q and R (each denoted by six dots). The streaklines, which come from the vicinity of the pipe axis at the upstream end, pass just outside of the first bubble, and after forming a part of the next bubble, they jump back to the inside

of the first one ($P \rightarrow Q$), circulating several times there to jump again from the position R to an area far downstream ($Z \gg 5$). This means that the fluid particles inside the bubble pass through the upstream end earlier than the particles just outside the bubble, which may give an interesting suggestion concerning the growth of the bubble.

So far we have treated only the case of a forward progressive wave. It is noted that for the case of a backward progressive wave, the flow mode is almost same as that of the forward progressive one, except that the bubble moves upstream with time. We now discuss the relation between the theoretical flow mode mentioned above and the bubble type of breakdown. Under fixed conditions of the mean flow, most of the bubble type of breakdowns so far observed are nearly stationary phenomena, although they are often accompanied by small fluctuations. (It was found in the preceding paper that the stationary breakdown can be explained by the standing wave of the $s = 0$ type which has no phase velocity.) However, Sarpkaya observed in his experiment that the existing vortex bubble moves upstream or downstream along the swirling axis, when the mean flow is accelerated or decelerated by rapidly changing the exit hole at the end of the test tube. Sarpkaya calls this a 'travelling vortex bubble breakdown'. Although the result of Fig. 3 is derived under the assumption of a fixed mean flow, it explains the moving vortex bubble, and Sarpkaya's travelling vortex breakdown corresponds to the streak mode under the influence of the wave disturbance propagating with a phase velocity.

$s = 0$, $R_0 = 0.5$, $N = -0.024$, $M = -2.62$, $A' = 1.0$, $N/M > 0$:

a forward progressive wave

Fig. 3 Unsteady type of axisymmetric breakdown (streaklines)

(According to Sarpkaya's experimental results, when the mean flow establishes a new steady state, the vortex bubble reaches a new fixed position, and becomes a stationary breakdown. But the mechanism of the halting of the travelling bubble has not been clarified.) In Sarpkaya's experiment the coexistence of two vortex bubbles is observed for the transient of mean flow. These facts also suggest that it is possible for the disturbance of the progressive wave to occur in an actual swirling flow, where the corresponding streaklines account for the moving vortex bubble.

3.2 Flow mode with nonaxisymmetric disturbance ($s = 1, 2$)

In this section the relation between the wave disturbance of the $s = 1$ mode and the unsteady vortex breakdown of the spiral type is discussed, and then the flow mode of the unsteady disturbance for $s = 2$ is treated briefly.

3.2.1 Spiral type of vortex breakdown

Leibovich found a stationary spiral vortex breakdown (his type 6) where the dye filament formed a gentle spiral curve. However, typical spiral type of breakdown, which has been reported so far, is such an unsteady phenomenon that the spiral streakline of dye filament rotates periodically about the swirling axis.

Figure 4 gives a result of calculated streaklines of the flow field with a backward progressive wave of the $s = 1$ mode. Figure (a) plots the streaklines at time $T = 0$, which shows that six streaklines, passing through the vicinity of the axis ($R = 0.01$) at the upstream end ($Z = 0$), move together into a spiral form. Figure (a)' is a view of Fig. (a) from downstream side, and Figs. (b) through (e) indicate the successive streaklines at intervals of $1/8$ period. The arrow labelled Ω in the figure denotes the rotational direction of mean flow, and the arrows along the streaklines indicate their downstream direction. These figures show that the streaklines rotate about the axis with a period of $\tau = 2\pi/N$, and that for this case (the parameter values are given under the figures), the sense of the helix and its rotational direction are the same as the swirling mean flow. Figure 5 shows a calculated result of a forward progressive wave ($N/M > 0$) with $s = 1$. Figures (a) and (b) plot the streaklines at times $T = 0$ and $(3/8)\tau$, respectively, and Fig. (c) plots the corresponding pathlines, i.e., the trajectories of fluid particles which depart from the upstream end at $T = 0$. For this case the sense of the helical streakline is in the opposite direction to the mean flow. However, as seen from Fig. (c), the fluid particles themselves move downstream rotating in the same direction as the mean flow. The relation between the sense of the helical streakline and its rotational direction is summarized in Table 1, where the sign of the sense and the rotational direction are taken to be positive when they are in the same direction as the mean flow rotation. The sense has an opposite sign to the value

of M/s (m/s), and the rotational direction has the same sign as the value of N/s (n/s). It should be noted that the rotational period of the streakline is given by $\tau = 2\pi s / |N|$.

We now compare the flow mode mentioned above with the observed vortex breakdown. Breakdown where the helix form of the streakline rotates periodically about the axis have been reported in detail by Sarpkaya and Leibovich under the labels of spiral type (Sarpkaya) and type 2 (Leibovich). According to their observation of the vortex breakdown phenomenon in a pipe, the central streakline of dye filament deflects rapidly at an axial position and forms a helix curve, and the sense of the helix and its rotational direction are the same as the mean flow. These experimental results agree with the flow mode for the disturbance of a progressive wave mentioned before, and

Fig. 4 Streaklines of spiral type of breakdown

Fig. 5 Spiral type of breakdown for a forward progressive wave

the sense and rotational directions correspond to case No.2 of Table 1 (a backward progressive wave). The conclusion is that the spiral type of breakdown reported by Sarpkaya (called type 2 by Leibovich) should be such a phenomenon that a progressive wave disturbance of the mode $s=1$ occurs in a swirling flow field.

In the experiments, the spiral type of vortex breakdown is usually accompanied by abrupt kinks, and persists for one or two turns before breaking up into a large-scale turbulence. In order to explain such kinks, the flow mode with the wave disturbance is further examined. Figure 5 shows the calculated streaklines for a backward progressive disturbance of the $s=1$ mode. According to the order of Figs. (a) to (c), the phase velocity N/M , with which the wave propagates upstream, is evaluated as larger value, while the Rossby number and the amplitude of disturbance are kept constant. When the phase velocity is relatively small compared to the axial velocity of the mean flow, as in Fig. (a), the streaklines form a gentle helix. But when the phase velocity is large, as in Figs. (b) and (c), the streaklines form abrupt kinks. From Fig. 6 and other calculated results for various conditions, it can be said that the kink type of breakdown occurs when the $s=1$

Table 1 Relation of the spiral form of streaklines with the parameters of the $s=1$ disturbance

No.	n/s	n/m	m/s	Sense of helix	Direction of rotation
1	+	+	+	(-)	(+)
2	+	-	-	(+)	(+)
3	-	+	-	(+)	(-)
4	0 (Steady flow)	0	-	(+)	Fixed streakline

Fig. 6 Kink type of breakdown

mode of the disturbance propagates with a large velocity compared to the mean flow axial velocity. Although Fig. 6 shows periodic kinks along the axis, it is considered that only a few kinks can be observed in an actual swirling flow because the flow is largely disturbed behind the kink due to an abrupt change of the flow state. Looking at the sense of the helical streakline, it is noted that according to Lambourne's report on the vortex breakdown on delta wings, the sense happens to be in the opposite direction to the mean flow, which corresponds to case No. 1 or No. 3 in Table 1.

3.2.2 Double helix type of breakdown

Figure 7 plots the calculated streaklines for a forward progressive wave of the $s=2$ mode. It describes 12 streaklines which surround the axis with a small circle ($R=0.01$) at the upstream end, expand along the axis into a sheet, both of which then form two helical lines of the so-called double helix type. (In the figure a streakline which departs from the radial position of $R=0.4$ is also presented.) The double helix type of breakdown, which has been observed so far, is always a stationary one (which corresponds to the standing wave of the $s=2$ mode as mentioned in the preceding paper). However, for the case of the progressive wave disturbance, the corresponding streaklines rotate about the axis with a period of $\tau = 2\pi s/|W|$. The reason why such a mode of breakdown does not occur in the experiment is inferred to be that the boundary condition of the pipe ends (i.e., the flow valve, etc.) counteracts the occurrence of an unsteady wave of the $s=2$ mode, since the double helix type of breakdown takes place in a relatively long space along the axis.

From the above results, together with the previous results on stationary vortex breakdowns, almost all the types of breakdown that have been reported by Sarpkaya and Leibovich are explained consistently in relation to the wave disturbance which occurs in a swirling flow. It should be noted that type 3 of Leibovich, which we mentioned for the first time here (altogether Leibovich has identified

six types, type 3 being where the central filament decelerates rapidly and moves abruptly off the axis to form a sharp kink whose filament oscillates from side to side in a preferential plane) is considered from the viewpoint of this study to be the flow state in which both the standing wave of the $s=2$ mode and the progressive wave of the $s=1$ mode exist at the same time.

4. Wave disturbance in experimental swirling mean flow and the breakdown

So far, the mean flow has been assumed to have an axially uniform and rigid rotational velocity. Next we apply the experimental equation given by Leibovich to the velocity profile of mean flow, after which we will point out that an approach similar to that above is also valid in explaining the vortex breakdown in an actual swirling flow.

Leibovich measured the velocity of a swirling flow where the breakdown phenomenon took place, and introduced the following experimental equations.

$$\left. \begin{aligned} \bar{V}(R) &= \frac{K}{R}(1 - e^{-qR^2}) \\ \bar{W}(R) &= W_1 + W_2 e^{-qR^2} \end{aligned} \right\} \dots\dots\dots (11)$$

where K , q , W_1 and W_2 are positive constants (nondimensional).

Next, we will describe a method to derive the wave disturbance when the $\bar{V}(R)$ and $\bar{W}(R)$ of the mean flow are functions of R . Equation (3), which governs the disturbance of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$, is linear with the real coefficients. Hence, a complex solution is first obtained and a real solution can be derived as the sum of the conjugate pair. Now, the disturbance of $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$ are put in the following form:

$$\left. \begin{aligned} \tilde{u} &= U(r) e^{-i(s\theta + mz) + \lambda t} \\ \tilde{v} &= V(r) e^{-i(s\theta + mz) + \lambda t} \\ \tilde{w} &= W(r) e^{-i(s\theta + mz) + \lambda t} \\ \tilde{p} &= P(r) e^{-i(s\theta + mz) + \lambda t} \end{aligned} \right\} \dots\dots\dots (12)$$

where $U(r)$, $V(r)$, $W(r)$ and $P(r)$ are complex functions of r , and λ is a complex number. Substituting Eq. (12) into Eq. (3) introduces the following equation with respect to only $U(r)$:

$$\begin{aligned} & \frac{d^2 U}{dr^2} + \frac{3s^2 + r^2 m^2}{s^2 + r^2 m^2} \frac{1}{r} \frac{dU}{dr} - U \frac{1}{r^2} \left\{ \frac{r^2 m^2 - s^2}{s^2 + r^2 m^2} + r^2 m^2 + s^2 \right\} \\ &= \frac{1}{\lambda - is(\bar{V}/r) - im\bar{W}} \frac{(-is)}{r^2} \left\{ \left(r \frac{d^2 \bar{V}}{dr^2} + \frac{d\bar{V}}{dr} - \frac{\bar{V}}{r} \right) \right. \\ & \quad - \frac{2r^2 m^2}{s^2 + r^2 m^2} \left(\frac{d\bar{V}}{dr} + \frac{\bar{V}}{r} - \frac{s}{rm} \frac{d\bar{W}}{dr} \right) - \frac{r^2 m}{s} \left(\frac{1}{r} \frac{d\bar{W}}{dr} - \frac{d^2 \bar{W}}{dr^2} \right) \Big\} U \\ & \quad + \frac{1}{\{\lambda - is(\bar{V}/r) - im\bar{W}\}^2} \frac{\bar{V}}{r} \left(\frac{d\bar{V}}{dr} + \frac{\bar{V}}{r} - \frac{s}{rm} \frac{d\bar{W}}{dr} \right) 2m^2 U \dots\dots\dots (13) \end{aligned}$$

$s=2$, $R=1.5$, $M=-1.75$, $N=-0.042$, $A'=1.0$
 $\sigma'=5.81$, $N/M > 0$: a forward progressive wave

Fig. 7 Double helix type of breakdown

The boundary condition of the pipe wall [i.e., $U(r)=0$ at $r=\infty$] specifies the eigenvalue λ and the eigenfunction $U(r)$. Since it is difficult to analytically solve the above equation, where $\bar{V}$ and $\bar{W}$ appear as in Eq. (11), we instead use a computer to obtain the numerical solutions of λ and $U(r)$.

In the following we mention a calculated result for the axisymmetric vortex breakdown ($s=0$ mode). Figure 8 plots the velocity profile of mean flow given by Eq.(11). Figure 9 describes some calculated streaklines for the axisymmetric standing wave ($s=0, \lambda=0$) which exists in a flow field such as one shown in Fig.8. In this standing wave case, the flow becomes steady so that the streaklines coincide with streamlines or pathlines. The flow mode as shown in Fig.9 agrees with the stationary bubble type vortex breakdown. This result indicates that a wave disturbance such as one which explains the flow mode of breakdown, can exist even for an actual form of the mean flow velocity. In addition, although Fig.9 shows a steady flow due to the standing wave (without phase velocity), the progressive wave can also exist for the experimentally constructed mean flow. Table 2 indicates the angular frequency N and the axial phase velocity N/M of the wave disturbance ($s=0$ mode) with the Rossby number $R_0=0.19$ for the analytical and the actual mean flows. As seen from the table, the

$W_1=0.086, W_2=0.104, q=-11.84, K=1.0$
Fig. 8 Velocity profile of mean flow represented by Eq.(11)

$s=0, R_0=0.19, \lambda=0.0, M=-2.0$
Fig. 9 A calculated result of axisymmetric vortex breakdown for the mean flow given in Fig.8

wave disturbance can be a forward or backward progressive wave or a standing wave, depending on the values of the axial-wave number M (or the wave length $2\pi/M$), which agrees qualitatively with the result of Fig.2 based on an axially uniform and rigid rotational mean flow. Figure 9 corresponds to the result for the disturbance (standing wave) denoted by * in Table 2. If $M=-6$ is chosen, for example, the disturbance has a positive phase velocity, and the corresponding streaklines represent a moving bubble in the same manner as Fig.3. The effect of the velocity profile of the cylindrical mean flow appears in the eigenfunctions of $U(R), V(R), W(R)$ and $P(R)$. A comparison is given in Fig.10 for the eigenfunction $U(R)$ in both cases of the analytical solution [$U(R)=k J_1(\sigma'R)$; k : a constant] for rigid rotational uniform flow versus the numerical one for an actual swirling flow. The analytical eigenfunction $U(R)$ does not depend on the values of the Rossby number R_0 and the wave number M so that $U(R)=k \times J_1(\sigma'R)$ always holds. However, the eigenfunction $U(R)$ for an actual swirling flow depends on the values of R_0 and M , though it takes a similar form qualitatively. The deviation of $U(R)$ from the analytical one becomes large at a position far off axis, and small in the vicinity of the axis. From the fact that the vortex breakdown suffers a drastic change near the pipe axis, it can be said that the results from the analytical solution do explain qualitatively the breakdown phenomenon occurring in an actual swirling flow. Furthermore, it is confirmed that Eq.(13) involves the solution for the

1: rigid rotating mean flow with uniform axial velocity [$J_1(\sigma'R)$], 2,3: mean flow given by Eq.(11) (2: $M=-3$, 3: $M=-12$)
Fig. 10 Comparison of eigenfunction $U(R)$

Table 2 Angular frequency N and phase velocity N/M for both cases of a rigidly rotating mean flow with uniform axial velocity and the one given by Eq.(11)

$M(2\pi/M)$		3(2.09)	1(6.28)	-1(-6.28)	-3(-2.09)	-6(-1.05)	-12(-0.50)
(1)	N	1.80	0.69	0.32	0.68	0.55	-0.36
	N/M	0.60	0.69	-0.32	-0.22	-0.092	0.030
		+	+	-	-	-	+
(2)	N	0.97	0.34	0.014	[0.0]*	-0.15	-0.71
	N/M	0.32	0.34	-0.014	0.0	0.024	0.059
		+	+	-	0	+	+

(1) rigidly rotating flow with uniform axial velocity

(2) mean flow given by Eq.(11) (with same parameters as Fig.9)

$R_0=0.19$ (defined by the value at the vicinity of pipe axis)

+: a forward progressive wave, -: a backward progressive wave, 0: a standing wave

nonaxisymmetric type of wave disturbance ($s \neq 0$).

5. Conclusions

In this paper, assuming the mean flow to be axially uniform and rigid rotational, it is found that the occurrence of a progressive wave disturbance corresponds well to the unsteady flow mode of the observed breakdown in a swirling flow. It is also found that the analytical results concerning the flow mode of breakdown can be applied to an actual swirling flow field.

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