Stability Analysis of Multi-span Rotor System
and its Application*

(Part III, Theory of multi-span rotor and its experiment)

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In this paper, the theory of the part I for the two-bearing rotor system is extended to a multi bearing rotor system by using the transfer matrix method. The extended theory is applied to the three-bearing rotor system and it is experimentally proved. Specially the effect of bearing pressure on the stability of the rotor system is experimented for the rotor system and compared with the theoretical result. Finally it is concluded from the above discussions that the proposed theory is useful for diagnosing the unstable rotor system and for improving the bearings.

1. Introduction

In a series of papers, the stability evaluation of a rotor supported by multi bearings and the effect of each bearing on the stability of the system are being investigated by using the energy concept in order to diagnose the stability of rotor bearing systems by evaluating each bearing energy and to design the optimum rotor system. In the first paper, the basic theory of the energy concept, the calculation method of the energy and the evaluation method using energy are proposed. In the second report experiments are performed in order to verify the first report and the sensitivity concept to the system stability is proposed in order to use the synthesis of the rotor system and to apply it to the diagnosis of a rotor bearing system.

In this report, the energy calculation method of each bearing is proposed for a multi bearing rotor system by using the transfer matrix method. Then the energies of the bearings are calculated for a three-bearing rotor system and also an experiment is performed on the system in order to compare with the theoretical result. That is, the bearing load of the center bearing is changed and the effect of bearing load on the system stability is theoretically and experimentally investigated.

2. Energy of Bearings

2-1 Eigenvalue and Eigenmode
For analysis purposes, the coordinate system (O-xyz) is fixed in space such that z-axis is horizontal and coincides with shaft center, x-axis is vertically downwards and y-axis is horizontal and origin of the shaft in xy-plane at the journal bearing is set at the static equilibrium point of rotor system. As shown in Fig.1, the rotor is divided and approximated to a uniform beam and a lumped mass in order to apply the transfer matrix method. Then the differential equation for the ith lumped mass is written as follows;

\[
\begin{align*}
\dot{m}_i \ddot{y}_i &= V_{ei} \dot{y}_i - V_{ei} \dot{y}_i - \beta_{ei} \dot{y}_i \\
- \beta_{ei} \ddot{y}_i &= \dot{b}_{ei} \ddot{y}_i - \dot{b}_{ei} \dot{y}_i - \beta_{ei} \ddot{y}_i \\
\dot{b}_{ei} \ddot{y}_i &= V_{ei} \dot{y}_i - V_{ei} \dot{y}_i - \dot{b}_{ei} \dot{y}_i \\
- \dot{b}_{ei} \ddot{y}_i &= \dot{b}_{ei} \ddot{y}_i - \dot{b}_{ei} \dot{y}_i - \dot{b}_{ei} \ddot{y}_i \\
\end{align*}
\]

where

\( I_n \) : transverse mass moment of inertia at rotor station
\( I_p \) : polar mass moment of inertia at rotor station
\( V_{ei}, V_{ei}^\prime \) : shear force of the ith element (subscripts L and R mean left and right sides of the element and subscripts x and y mean x and y directions, respectively).
\( M_{ei}^L, M_{ei}^R \) : bending moment of the ith element (subscripts L and R mean left and right sides of the element and subscripts x and y mean x and y directions, respectively).

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where \( q \) is number of divided points of the rotor. Assuming that both ends of the rotor are free, the boundary condition of the following differential equation is obtained:

\[
\begin{pmatrix}
I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\
I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\
I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I_{n-1} & I_{n-1} & I_{n-1} & \cdots & I_{n-1}
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_1 \\
\beta_1 \\
\vdots \\
\beta_1
\end{pmatrix} =
\begin{pmatrix}
D_1 \\
D_2 \\
D_3 \\
\vdots \\
D_q
\end{pmatrix}
\]

(8)

From this equation, the following characteristic equation is obtained:

\[
\det(D) = 0
\]

Eq. (9) is an algebraic equation with complex coefficients. Eigenvalue of the equation is obtained by numerical method that is the steepest descent method.

The displacement of \( x \) direction for eigenvalue \( \lambda \) is

\[
x = \Re(e^{\lambda t})
\]

(10)

and the displacement of \( y \) direction is similarly

\[
y = \Re(e^{\lambda t})
\]

where \( \omega \) is the damped eigenfrequency and \( \phi_x \) and \( \phi_y \) are phase angles of \( x \) and \( y \) directions, respectively.

At the bound of stability i.e. \( \gamma = 0 \), major axis and minor axis of the ellipse and angle between \( x \) axis and principal axis of the ellipse are described, respectively,

\[
a_x = \frac{1}{2}(x_{21}^2 + x_{31}^2 + y_{21}^2 + y_{31}^2) + \frac{1}{4}(x_{21}^4 + x_{31}^4 + y_{21}^4 + y_{31}^4 - 2x_{21}x_{31}y_{21}y_{31})
\]

(11)

\[
b_{xy} = \frac{1}{2} \tan^{-1} \left[ \frac{2(x_{21}x_{31} + y_{21}y_{31})}{(x_{21}^2 + x_{31}^2 + y_{21}^2 + y_{31}^2)} \right]
\]

(12)

If \( 0 \)-xy coordinate system is rotated by angle \( \theta_{n} \), motion of the rotor is described in the following form;

\[
x_{i} = a_{ij} \cos(\omega_{j}t + \theta_{n}) \quad \text{and} \quad y_{i} = b_{ij} \sin(\omega_{j}t + \theta_{n})
\]

(13)

2-2 Calculation of vibration energy

In the first and the second report, a method to evaluate the stability degree of the two rotor system is proposed by using the vibrational energy level and it is verified experimentally. Here the method is extended to an arbitrary type of rotor system, that is, multi-span rotor system.

Assuming that the damping force exists only at the bearings and using the transfer matrix method developed in the former section, eigenvalues at the arbitrary rotating speed \( \omega \) (3 = 1, 2, \ldots \), are obtained and \( j \)th eigenmode is obtained. When one of these eigenvalues has a positive real part, vibrational energy of each bearing for the eigenvalue is calculated by the following procedure. And the bearing which causes instability of the system can be detected.
The differential equation for the ith

divided point is written from Eq.(1) for x
direction;

\[ m_i \ddot{x}_{i} + V_{xi} \dot{x}_{i} + V_{ri} \dot{r}_{i} + P_{i} = 0 \]

\[ \frac{d \theta_{i}}{dt} + \omega_{i}^{2} \dot{\theta}_{i} + \frac{d \theta_{0}}{dt} \frac{d \theta_{0}}{dt} = M_{ri} \theta_{r} - M_{ri} \theta_{r} \]

\[ \theta_{i} = \theta_{0} + a_{i} \theta_{r} + b_{i} \theta_{r} \]

\[ x_{i} = x_{0} + a_{i} x_{r} + b_{i} x_{r} \]

where \( P_{i} \) is reaction force of a bearing.

The equation for the uniform beam is

\[ V_{xi} = V_{xi} \]

\[ \theta_{i} = \theta_{0} + a_{i} \theta_{r} + b_{i} \theta_{r} \]

\[ x_{i} = x_{0} + a_{i} x_{r} + b_{i} x_{r} \]

where \( a_{i} = l_{i}/E_{i}, b_{i} = l_{i}^{2}/E_{i}, d_{i} = l_{i}^{3}/E_{i} \)

Substituting \( M_{ri}, B_{ri}, V_{ri}, E_{ri} \) which are

obtained from Eq.(15) into Eq.(14), yields

\[ m_{i} \ddot{x}_{i} + \frac{2}{a_{i}} x_{i} + \frac{3}{b_{i}} x_{i} + \left( \frac{2}{a_{i}} + \frac{3}{b_{i}} \right) x_{i} + \left( \frac{3}{b_{i}} \right) \theta_{i} \]

\[ \frac{d \theta_{i}}{dt} + \omega_{i}^{2} \dot{\theta}_{i} + \frac{d \theta_{0}}{dt} \frac{d \theta_{0}}{dt} = 0 \]

\[ I_{ri} \theta_{i} + \omega_{i}^{2} \dot{\theta}_{i} + \frac{d \theta_{0}}{dt} \frac{d \theta_{0}}{dt} = 0 \]

\[ \rho_{i} \theta_{i} + \omega_{i}^{2} \dot{\theta}_{i} + \frac{d \theta_{0}}{dt} \frac{d \theta_{0}}{dt} = 0 \]

\[ \theta_{i} = \theta_{0} + \frac{a_{i}}{a_{i}} \theta_{r} + \frac{b_{i}}{b_{i}} \theta_{r} \]

\[ x_{i} = x_{0} + \frac{2}{a_{i}} x_{r} + \frac{3}{b_{i}} x_{r} + \theta_{i} \theta_{r} \]

where \( \omega_{i} \) is the angular frequency of oscillation.

Substituting Eq.(16) into Eq.(14), yields

\[ M_{i} x_{i} + B_{i} \ddot{x}_{i} + (K_{i} + K_{ex}) x_{i} = 0 \]

where mass matrix \( M \) is symmetric, damping matrix \( B \) is expressed as \( B = B_{r} + B_{ex} \)

where \( B_{r} \) corresponds to gyroscopic which is conserving force and \( B_{ex} \) to damping force due to bearing, and stiffness matrix \( K \) is expressed as \( K = K_{r} + K_{ex} \), where \( K_{r} \) is stiffness matrix of shaft which is symmetric and \( K_{ex} \) is stiffness matrix of bearing which is unsymmetric.

As the equation of motion for a multi

span rotor system is usually described by

Eq.(17), vibrational energy for one period is obtained by setting the periodic solution of Eq.(17) \( x = X_{e} \exp(\omega_{n} t) \) and by using the eigenmode corresponding to unstable mode, that is;

\[ E = \int \frac{1}{2} M_{i} \dot{x}_{i}^{2} + \frac{1}{2} B_{i} \ddot{x}_{i}^{2} + \frac{1}{2} K_{i} \theta_{i}^{2} + K_{ex} x_{i}^{2} \]

where energy terms with regard to \( M_{i} \), \( B_{i} \)

and \( K_{i} \) are conservative, so these energies become zero. Thus Eq.(18) becomes,

\[ E = \int \frac{1}{2} (2 \omega_{n}^{2} K_{ex} + \omega_{n}^{2} K_{ex}) x_{i}^{2} \]

The energy obtained by Eq.(19) corresponds to the terms of bearing coefficient, so these terms may be divided into terms of bearing coefficient;

\[ E = \int \frac{1}{2} (2 \omega_{n}^{2} K_{ex} + \omega_{n}^{2} K_{ex}) x_{i}^{2} \]

It is known from Eq.(20) that sum of

energy of each bearing becomes total

energy of the system. Vibrational mode of

the ith bearing is described from Eqs.(10)

and (11) as

\[ x_{i} = [1, \omega_{n} \sin(\omega_{n} t - \phi_{i})] \]

\[ y_{i} = [1, \omega_{n} \cos(\omega_{n} t - \phi_{i})] \]

Substituting Eq.(21) into Eq.(20), energy

for ith bearing is

\[ E_{i} = \frac{1}{2} (2 \omega_{n}^{2} + \omega_{n}^{2}) x_{i}^{2} \]

\[ - \frac{1}{2} \omega_{n}^{2} \sin(\phi_{i}) \pi \]

Therefore energy level of each bearing can be obtained by using Eq.(22) and the degree of bearing contribution to stability of the system is quantitatively obtained by comparing the energy levels of bearing. Then the bearing which causes instability of the system can be detected.

3. Experiment

Instability of a multi-span rotor system is not so well investigated experimentally, because the experiment is very difficult and its result is not coincident with the theoretical one. For example, measurement of alignment of the bearings is very difficult in a rotor system with more than three bearings. Therefore the influence of misalignment on the system stability is important. Here stability of a three-bearing rotor system is experimented for some different load conditions of the center bearing and the results are discussed.

The experimental apparatus and the measuring devices are same ones as used in the 2nd part of the serial work. The rotor provided in the three-bearing system is shown in Fig.2. Two experimental models are made by changing the bearing location of the center bearing. Fig.2
shows the model I and model II rotors and the center bearing location in the model II rotor is shifted 80 mm from the location of model II to the left side in order to obtain a lower first critical speed. As this model is statically indeterminate, it is difficult to evaluate the load of each bearing. In this experiment the strain near each bearing is measured by strain gage, and the static bearing load is calculated by using the strain. Surface pressure of the bearing is obtained for this load. Experimental conditions for the model I rotor are shown in Table I, and for the model II in Table II.

The aim of the experiment on the model I rotor is to obtain the conditions for occurrence of oil whip due to misalignment of the center bearing. Then occurrence of the oil whip is measured by changing the surface pressure of the center bearing. The experiment on model II rotor is aimed at discussing the occurrence of oil whip due to misalignment from the viewpoint of energy of each bearing. Then the surface pressure of the center bearing is changed and the energy levels of the bearings and stability degree of each bearing are discussed.

To produce different bearing conditions, two center bearing sizes are chosen, i.e. L/D (width of bearing)/(diameter of bearing) = 1.0 and 0.5. The energy of each bearing is calculated by using Eq. (22), the experimental values of both amplitude and phase difference of x and y directions and both whirling frequency and spinning frequency of the rotor at the beginning of the oil whip. The spring coefficients and the damping coefficients of the bearings are calculated by the assumption of infinitely short bearing theory, in which Sommerfeld number is calculated from the experimental data.

4. Discussions

The theoretical first critical speed and onset speed of oil whip are obtained by transfer matrix method. Numbers of divided nodes are 17 and 18 for model I and model II rotor, respectively. Bearing coefficients are calculated by using the infinitely short bearing assumption.

4-1 Eigenvalue and stability boundary

Eigenvalue and stability boundary are obtained by changing the height of the center bearing (b, 0.2) in the experiment and calculated by changing the bearing load depending on the change of the height of the center bearing in theory. The bearing load is calculated as represented in section 3. Table 3 shows the results with model I rotor, in which oil whip is not observed for experimented conditions A except No. 5 in the speed range of 5400 to 5400 rpm. The theoretical results of condition A, Nos. 2, 4, 6 and 8 are similar to the others, so the results are not given. Table 2 shows the results with model II rotor. The theoretical whirling frequencies of model I and II rotors shown in Tables 3 and 4 are in good agreement with experimental ones. In the experiment the on-set speed of stability of a rotor system with L/D = 1.0 at 1078 is lower that of one with L/D = 0.5 in the same load condition, but theoretically there are little differences. In the conditions in which oil whip has occurred, theoretical and experimental results are in good agreement within 10%, but in the conditions in which oil whip has not occurred, difference between the results is large. This is the reason why the stiffness and damping coefficients are simplified as linear and the moment force of the bearing is not considered. As examples, Figs. 3-5 show the eigenvalues and damping ratios for model I rotor conditions A and B, No. 5 and for model II rotor condition A, No. 2. Eigenvalue curves are almost the same as

<table>
<thead>
<tr>
<th>No.</th>
<th>BG. No.</th>
<th>Bearing load</th>
<th>Condition A (B = 1.0, L/D = 1.0)</th>
<th>Condition B (B = 1.0, L/D = 1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Bearing</td>
<td>Surface</td>
<td>viscosity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>load</td>
<td>pressure</td>
<td>(kg/cm²)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>30.8</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>30.0</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>30.2</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>30.4</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>30.6</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>30.8</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>31.0</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>31.2</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>31.4</td>
<td>10</td>
<td>3.00</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>31.6</td>
<td>10</td>
<td>3.00</td>
</tr>
</tbody>
</table>
those for a two-bearing system which are shown in the first report, but in the three-bearing system, an eigenvalue shown by dotted line is newly added. This eigenvalue does not influence the system stability, so an important eigenvalue which affects the stability is the same as in the case of the two-bearing system. The value of L/D and the bearing load of the center bearing do not affect the eigenvalue of the system.

4-2 Energy value and frequency response

In this section energy value and frequency response of each bearing are discussed for the model II rotor. In experiments for Nos. 1, 4, 5 of the condition A, oil whip does not occur, so energy values are not calculated.

As shown in the theory of section 2 when the energy is negative, the bigger the absolute value of the energy becomes, the more unstable the system becomes. Observing the experimental energies from this point of view, the bearing B0.1 becomes the cause of instability except for No.1 of condition B and No.2 of condition A. This fact agrees with the frequency response result. From the result of the experiments No.1 of condition B and No.2 of condition A, vibrational modes of BG.1 and BG.2 are almost the same, so it is very difficult to detect the unstable bearing. In this case, it is difficult to obtain a good agreement between experimental and theoretical results, but in the other cases a good agreement is obtained.

From the above discussion the bearing which makes the system unstable is the same one and the stability does not so much depend on the variation of alignment of the bearings in the three-bearing system.

Table 2: Experimental conditions for model II rotor

<table>
<thead>
<tr>
<th>No.</th>
<th>BG. Bearing load</th>
<th>Condition A</th>
<th>Experimental condition B</th>
<th>Condition A</th>
<th>Experimental condition B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[kg]</td>
<td>[kg/μm]</td>
<td>[kg/μm]</td>
<td>[kg]</td>
<td>[kg/μm]</td>
</tr>
<tr>
<td>1</td>
<td>26.8</td>
<td>1.79</td>
<td>1.44</td>
<td>8.70</td>
<td>1.44</td>
</tr>
<tr>
<td>2</td>
<td>110.9</td>
<td>10.9</td>
<td>10.9</td>
<td>6.00</td>
<td>6.00</td>
</tr>
<tr>
<td>3</td>
<td>13.5</td>
<td>13.5</td>
<td>13.5</td>
<td>13.5</td>
<td>13.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>29.2</td>
<td>29.2</td>
<td>29.2</td>
<td>29.2</td>
<td>29.2</td>
</tr>
<tr>
<td>5</td>
<td>29.4</td>
<td>29.4</td>
<td>29.4</td>
<td>29.4</td>
<td>29.4</td>
</tr>
</tbody>
</table>

Table 3: Experimental results for model I rotor

<table>
<thead>
<tr>
<th>Bearing load [kg]</th>
<th>Onset speed of stability (rpm)</th>
<th>damped eigenfrequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BG.1</td>
<td>BG.2</td>
<td>BG.3</td>
</tr>
<tr>
<td>1</td>
<td>28.8</td>
<td>30.0</td>
</tr>
<tr>
<td>2</td>
<td>32.5</td>
<td>30.4</td>
</tr>
<tr>
<td>3</td>
<td>16.3</td>
<td>30.6</td>
</tr>
<tr>
<td>4</td>
<td>54.1</td>
<td>40.7</td>
</tr>
<tr>
<td>5</td>
<td>47.5</td>
<td>51.3</td>
</tr>
<tr>
<td>6</td>
<td>51.2</td>
<td>56.3</td>
</tr>
<tr>
<td>7</td>
<td>54.9</td>
<td>61.5</td>
</tr>
</tbody>
</table>
5. Conclusions

From the above analysis and discussions, the following conclusions are drawn:
1) The stability of each bearing can be investigated from the energy method which is proposed in Ref. (1) and this paper.
2) For a given three-bearing rotor system, the bearing which causes instability of the system is invariably the same regardless of L/D and alignment of bearings.
3) Behavior of the eigenvalue of the three-bearing rotor system which dominates the system stability is the same as with the two-bearing rotor system.

Fig. 5 Damped natural frequency and damping ratio for model rotor II, No. 2 condition A

Fig. 6 Transient frequency analysis
### Table 1: Experimental results for model II rotor

<table>
<thead>
<tr>
<th>Condition</th>
<th>( \eta )</th>
<th>( G )</th>
<th>( \eta_{\nu} )</th>
<th>Energy</th>
<th>Condition</th>
<th>( \eta )</th>
<th>( G )</th>
<th>( \eta_{\nu} )</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.9</td>
<td>3.5</td>
<td>1.1</td>
<td>( \eta_{\nu} )</td>
<td>70</td>
<td>1.1</td>
<td>7.1</td>
<td>( \eta_{\nu} )</td>
<td>70</td>
</tr>
<tr>
<td>B</td>
<td>0.9</td>
<td>3.5</td>
<td>1.1</td>
<td>( \eta_{\nu} )</td>
<td>70</td>
<td>1.1</td>
<td>7.1</td>
<td>( \eta_{\nu} )</td>
<td>70</td>
</tr>
</tbody>
</table>

References