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**Green's Functions for Torsional Body Force Problems of
an Elastic Thick Infinite Plate and Their Application
(A Finite Solid Cylinder Twisted by Rigid Chucks)***

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The principal object of this paper are to show (i) a general method of solution for torsional body force problems of an elastic thick infinite plate, (ii) Green's functions for the torsional body force problems mentioned above and (iii) a numerical method of solution for elastic problems of a circular solid finite length cylinder twisted by a pair of axisymmetric rigid chucks.

Green's functions shown in the paper are defined as solutions to the elastic problem of an infinite thick plate subjected to torsional body forces acting on a circle in the interior of the plate. The numerical method for solid cylinders is an application of Green's functions shown in the paper. That is, solutions to the solid cylinder are constructed by Green's functions for torsional body force problems of the thick plate. Numerical examples for the problem of the solid cylinder are given.

1. Introduction

The principal object of this paper is to show both a method of solution and Green's functions for axisymmetric torsional body force problems of an elastic thick infinite plate and to consider the elastic problem of a finite solid cylinder subjected to torsion by a pair of rigid chucks, applying Green's functions shown in the paper. For this purpose, we use stress functions⁽¹⁾ for axisymmetric torsional body force problems of an elastic thick plate and a solution⁽²⁾ for the boundary value problems of an elastic thick plate subjected to torsional surface forces at the boundary.

Green's functions mentioned above are defined as a solution to the problem of an elastic thick plate subjected to torsional body forces acting on two circles in the interior of the thick plate and satisfying the boundary conditions of the stress free from applied surface forces. As an example of application of Green's functions shown in the paper, we propose a method of solution to the mixed boundary value problem of a finite solid cylinder mentioned above. A fundamental principle of the method of solution is to distribute the torsional body forces acting on two circles on a circular cylindrical surface in the thick plate with same elastic properties as those of the solid cylinder and to determine the intensities of the body forces distributed so as to satisfy the boundary conditions of the finite solid cylinder twisted by the

rigid chucks. In this case, it is easily seen from the principle of superposition that displacements and stresses of the solid cylinder are expressed by a linear combination of Green's functions mentioned above.

In actual state of a solid cylinder fixed by a chuck, there are opening parts between pawls of a chuck, but here these opening parts are neglected for simplification of the problem. The distributions of stresses and displacements of the solid cylinder are shown and the torques to twist the solid cylinder by the chucks are investigated.

2. An elastic thick plate with torsional body forces

In this Chapter, we shall show a method of solution to the problems of an elastic infinite thick plate subjected to torsional body forces $F_2(r, z)$ in an arbitrarily axisymmetric distribution. We use cylindrical coordinates (r, θ, z) as shown in Fig.1, where $2H$ represents the thickness of the thick plate. It is well known that solutions of a problem of an elastic body subjected to both surface forces and body forces can be expressed by superposition of solutions to a surface force problem over solutions to a body force problem. Therefore, we consider here a thick plate with body forces and no surface forces. In this case, the boundary conditions of the thick plate are expressed as follows:

$$z = \pm H; \quad \tau_{z\theta} = \tau_{rz} = \sigma_z = 0 \quad \dots (1)$$

where $\tau_{z\theta}$, τ_{rz} and σ_z are components of stresses and hereafter undefined notations are those used customarily.

To obtain a general method of solution to the body force problem of an elastic thick plate described above, we use stress functions⁽¹⁾ for axisymmetric body force problems of an elastic thick plate and a solution⁽²⁾ for the boundary value problems

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Fig. 1 An elastic thick plate subjected to torsional body forces acting on two circles of a radius a .

Fig. 2 A torsional body force acting on a circle of a radius a .

of an elastic thick plate subjected to only torsional surface forces. It has been shown in the previous paper⁽¹⁾ that the displacements $u_j(r, z)$ produced by torsional body forces $F_2(r, z)$ acting in the interior of a thick plate can be obtained from the formula

$$2\mu u_2 = 2(1-\nu) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2} \right) \phi_2, \quad u_1 = u_3 = 0 \quad \dots (2)$$

provided that the stress function $\phi_2(r, z)$ is calculated by the integral

$$\phi_2 = -\frac{1}{2H(1-\nu)} \sum_{m=0}^{\infty} \varepsilon_m \int_0^{\infty} \frac{\alpha J_1(\alpha r)}{(\alpha^2 + \beta^2)^2} (\bar{F}_{2c}(\alpha, \beta) \cos \beta z + \bar{F}_{2s}(\alpha, \beta) \sin \beta z) d\alpha \quad \dots (3)$$

where

$$\begin{Bmatrix} \bar{F}_{2c} \\ \bar{F}_{2s} \end{Bmatrix} = \int_{-H}^H \int_0^{\infty} r J_1(\alpha r) F_2(r, z) \begin{Bmatrix} \cos \beta z \\ \sin \beta z \end{Bmatrix} dr dz \quad \dots (4)$$

$\beta = m\pi/H$, $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \dots = 2$. u_j are displacements of r ($j=1$), θ ($j=2$) and z ($j=3$) directions. $J_1(\alpha r)$ is a Bessel function. μ and ν are the shear modulus of elasticity and Poisson's ratio respectively. Substitution of Eq. (3) into Eq. (2) yields

$$2\mu u_2 = \frac{1}{H} \sum_{m=0}^{\infty} \varepsilon_m \int_0^{\infty} \frac{\alpha J_1(\alpha r)}{\alpha^2 + \beta^2} (\bar{F}_{2c}(\alpha, \beta) \cos \beta z + \bar{F}_{2s}(\alpha, \beta) \sin \beta z) d\alpha \quad \dots (5)$$

By the way, the expressions of stresses obtained from the displacements u_j of Eq. (5) do not usually satisfy the boundary conditions (1). Therefore⁽³⁾, we have to prepare further solutions so that the boundary conditions (1) are satisfied when they are superposed on the solution of Eq. (5). That is, since the stresses $(\tau_{z\theta})_{z=\pm H}$ at the surfaces of the thick plate calculated from u_j of Eq. (5) are

$$\tau_{z\theta}|_{z=\pm H} = \frac{\pi}{H^2} \sum_{m=1}^{\infty} (-1)^m m \int_0^{\infty} \frac{\alpha J_1(\alpha r)}{\alpha^2 + \beta^2} \bar{F}_{2s}(\alpha, \beta) d\alpha \quad \dots (6)$$

it is necessary to apply such imaginative surface forces as cancel the stresses $(\tau_{z\theta})_{z=\pm H}$ of Eq. (6). Note that $u_1 = u_3 = 0$, $\sigma_r = \sigma_{\theta} = \sigma_z = \tau_{rz} = 0$ and the nonvanishing components of the stresses are given by

$$\tau_{r\theta} = \mu r \frac{\partial}{\partial r} \left(\frac{u_2}{r} \right), \quad \tau_{z\theta} = \mu \frac{\partial u_2}{\partial z}$$

because the present problem is one of axisymmetric torsional deformations.

As the solution for this purpose, we take the following one:

$$\left. \begin{aligned} 2\mu u_2 = \sum_{m=1}^{\infty} \int_0^{\infty} J_1(\alpha r) [\{ \bar{f}_1(\alpha) + \bar{f}_2(\alpha) \} \{ e^{-\alpha \{ (2m-1)H+z \}} + e^{-\alpha \{ (2m-1)H-z \}} \} \\ + (-1)^m \{ \bar{f}_1(\alpha) - \bar{f}_2(\alpha) \} \{ e^{-\alpha \{ (2m-1)H+z \}} - e^{-\alpha \{ (2m-1)H-z \}} \}] d\alpha \\ u_1 = u_3 = 0 \end{aligned} \right\} \quad \dots (7)$$

where $\bar{f}_i(\alpha)$ is a Hankel transform of $f_i(r)$

$$\bar{f}_i = \int_0^{\infty} r J_1(\alpha r) f_i(r) dr \quad (i=1, 2) \quad \dots (8)$$

The solution (7) represents the displacements of an elastic thick plate subjected to torsional surface forces $f_i(r)$ at the boundaries $z=\pm H$ and having no body forces in the interior of the thick plate. The formulas (7) and (8) hold in the conditions that $F_2(r, z)$ and $f_i(r)$ satisfy both Dirichle't conditions and the conditions at infinities

$$\lim_{r \rightarrow \infty} |F_2(r, z)|, \quad |f_i(r)| = O(r^{-t+1/2}) \quad \dots (9)$$

where $t > 1.0$.

Now, let us show a general method of solution for problems of an elastic thick plate subjected to torsional body forces $F_2(r, z)$ in the interior and satisfying the boundary conditions (1) at the surfaces. Assumption of $f_2 = -f_1 = f$ in the expressions of the stresses $(\tau_{z\theta})_{z=\pm H}$ obtained from u_2 of Eq. (7) yields

$$\tau_{z\theta}|_{z=\pm H} = - \sum_{m=1}^{\infty} (-1)^m \int_0^{\infty} \alpha J_1(\alpha r) \bar{f}(\alpha) \{e^{-2mH\alpha} + e^{-2(m-1)H\alpha}\} d\alpha \quad \dots (10)$$

Superposing Eq.(10) on Eq.(6) and determining the unknown function $\bar{f}(\alpha)$ in the expression superposed so as to satisfy the boundary conditions (1), we can find

$$\bar{f} = \frac{\pi}{H^2} \frac{m \bar{F}_{2s}(\alpha, \beta)}{\alpha^2 + \beta^2} \frac{1}{e^{-2mH\alpha} + e^{-2(m-1)H\alpha}} \quad \dots (11)$$

From above results, we see that the displacement u_2 of an elastic thick plate with torsional body forces $F_2(r, z)$ is expressed as follows:

$$\begin{aligned} 2u_2 = & \frac{1}{H} \int_0^{\infty} \frac{1}{\alpha} J_1(\alpha r) \bar{F}_{2c}(\alpha, 0) d\alpha + \frac{2}{H} \sum_{m=1}^{\infty} \int_0^{\infty} \frac{J_1(\alpha r)}{\alpha^2 + \beta^2} \left[\alpha \bar{F}_{2c}(\alpha, \beta) \cos \beta z + \bar{F}_{2s}(\alpha, \beta) \right. \\ & \left. \times \left\{ \alpha \sin \beta z + (-1)^m \beta \frac{e^{\alpha(H-z)} - e^{\alpha(H+z)}}{1 + e^{2\alpha H}} \right\} \right] d\alpha \quad \dots (12) \end{aligned}$$

As a special case of the expression (12), if we assume that the body forces $F_2(r, z)$ are an even function with respect to z , then Eq.(12) agrees with Eq.(5) because $\bar{F}_2(\alpha, \beta) = 0$ in this case. That is, Eq.(5) is a solution to the body force problem without a solution superposed when the body forces $F_2(r, z)$ are an even function in z .

3. Green's functions

Applying the results shown in the previous Chapter, here we shall show Green's functions for torsional body force problems of an elastic thick plate. Green's functions presented here are defined as a solution to the body force problem of an elastic thick plate satisfying the boundary conditions (1) and subjected to the axisymmetric torsional body force

$$F_2 = \frac{q}{2\pi r} \{\delta(z-h) \pm \delta(z+h)\} \delta(r-a) \quad \dots (13)$$

acting on two circles ($r=a, z=\pm H$) in the interior of the thick plate as shown in Figs.1 and 2. Here $\delta()$ is a Dirac delta function, q is the magnitude of a torsional body force and the notation $\pm$ means that we must take the sign $+$ when the two body forces act in the same direction (F_2 is an even function of z) and the sign $-$ when the two body forces act in opposite directions (F_2 is an odd function of z). Depending on the sign $\pm$ in Eq.(13), we identify the displacements due to the two body forces as $(u_2)_c$ and $(u_2)_s$ respectively.

If we can find the displacements $(u_2)_c$ and $(u_2)_s$ mentioned above, the displacements u_2^+ and u_2^- produced by a torsional body force acting on a circle ($r=a, z=+h$ or $r=a, z=-h$) in the interior of the thick plate with the thickness $2H$ are expressed as follows:

$$u_2^+ = \{(u_2)_c + (u_2)_s\}/2, \quad u_2^- = \{(u_2)_c - (u_2)_s\}/2 \quad \dots (14)$$

Let us show the displacements $(u_2)_c$ and $(u_2)_s$. Substitution of F_2 of Eq.(13) into Eq.(4) yields

$$(i) \text{ for the case of the sign } -: \quad \bar{F}_{2c}(\alpha, \beta) = 0, \quad \bar{F}_{2s}(\alpha, \beta) = \frac{q}{\pi} J_1(\alpha a) \sin \beta h \quad \dots (15)$$

$$(ii) \text{ for the case of the sign } +: \quad \bar{F}_{2s}(\alpha, \beta) = 0, \quad \bar{F}_{2c}(\alpha, \beta) = \frac{q}{\pi} J_1(\alpha a) \cos \beta h \quad \dots (16)$$

Substituting Eq.(15) into Eq.(12) and applying the formulas of the infinite series⁽⁴⁾

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m^2 + y^2} = \frac{\pi}{2y} \frac{\cosh y(\pi - x)}{\sinh y\pi} - \frac{1}{2y^2}, \quad \sum_{m=1}^{\infty} (-1)^{m-1} \frac{m \sin mx}{m^2 + y^2} = \frac{\pi}{2} \frac{\sinh yx}{\sinh y\pi} \quad \dots (17)$$

and the integral formula⁽⁵⁾

$$\int_0^{\infty} J_1(\alpha r) J_1(\alpha a) e^{-\alpha z} d\alpha = \frac{1}{\pi \sqrt{ra}} Q_{1/2} \left(\frac{r^2 + a^2 + z^2}{2ar} \right) \quad \dots (18)$$

we can get the displacement $(u_2)_s$ as follows:

$$(u_2)_s = \frac{q}{4\pi^2 \mu \sqrt{ra}} \sum_{m=1}^{\infty} \{Q_{1/2}(x_1) + Q_{1/2}(x_2) - Q_{1/2}(x_3) - Q_{1/2}(x_4)\} \quad \dots (19)$$

where $Q_k(x_i)$ is a Legendre function of the second kind of the order k and

$$x_i = \frac{r^2 + a^2 + z_i^2}{2ar} \quad (i=1 \sim 4) \quad \dots (20)$$

and z_i represent

$$\begin{aligned} z_1 &= z - 2(m-1)H + (-1)^m h, & z_2 &= z + 2mH - (-1)^m h \\ z_3 &= z - 2mH + (-1)^m h, & z_4 &= z + 2(m-1)H - (-1)^m h \end{aligned} \quad \dots (21)$$

Likewise, the displacement $(u_2)_c$ is obtained by substitution of Eq.(16) into Eq.(11) as follows:

$$(u_2)_c = \frac{q}{4\pi^2 \mu \sqrt{ra}} \sum_{m=1}^{\infty} \{Q_{1/2}(x_1) + Q_{1/2}(x_2) + Q_{1/2}(x_3) + Q_{1/2}(x_4)\} \quad \dots (22)$$

Applying the relation of Eq.(14) to Eqs.(19) and (22), we can find the displacements u produced by the two body forces ξ and ζ acting on a circle ($r=a_1, z=h_1$) and a circle ($r=a_2, z=-h_2$) as follows:

$$u_2 = \frac{1}{4\pi^2\mu} \sum_{m=1}^{\infty} \left[\frac{\xi}{\sqrt{ra_1}} \{Q_{1/2}(x_1') + Q_{1/2}(x_2')\} \pm \frac{\zeta}{\sqrt{ra_2}} \{Q_{1/2}(x_3') + Q_{1/2}(x_4')\} \right] \dots (23)$$

where x_1' and x_2' are expressed by substitution of $a=a_1$ and $h=h_1$ in x_1 and x_2 of Eq.(20), and x_3' and x_4' are expressed by substitution of $a=a_2$ and $h=h_2$ in x_3 and x_4 of Eq.(20) respectively.

It is easily seen that the displacements u_2' or u_2'' of Eq.(14) are obtained by assumptions of $\zeta=0$ or $\xi=0$ in Eq.(23). It is also noted that the stresses obtained from the displacements $(u_2)_c$ of Eq.(22) satisfy the boundary conditions

$$z=0, H: \sigma_z = \tau_{rz} = \tau_{z\theta} = 0 \dots (24)$$

This fact means that Eq.(22) may be also called a solution to the problem of an elastic thick plate ($0 \leq r < \infty, 0 \leq z \leq H$) subjected to a torsional body force acting on a circle ($r=a, z=h$) and satisfying the boundary conditions (24) if we consider only the region ($0 \leq r < \infty, 0 \leq z \leq H$).

Now, we shall reconsider the above results from a physical viewpoint of the theory of elasticity, although Green's functions shown in Eqs.(19), (22), (23) and so on are mathematically derived from the fundamental equations of elasticity as a solution satisfying the boundary conditions (1). It was shown in the previous paper⁽¹⁾ that the displacement u_2 produced by the torsional body force

$$F_2 = \frac{q}{2\pi r} \delta(r-a) \delta(z) \dots (25)$$

acting on a circle ($r=a, z=0$) as shown in Fig.2 in the interior of an elastic infinite solid was expressed by

$$u_2 = \frac{q}{4\pi^2\mu\sqrt{ra}} Q_{1/2} \left(\frac{r^2 + a^2 + z^2}{2ar} \right) \dots (26)$$

$$u_1 = u_3 = 0$$

Now, if we distribute the torsional body force F_2 of Fig.2 periodically as shown in Fig.3, then the stresses acting on the planes AA', BB' and so on in Fig.3 turn out to be zero because these planes are symmetrical with respect to the torsional body forces distributed. (This fact holds only in the case of torsional body forces).⁽²⁾ Therefore, if we are to consider only the region ($0 \leq r < \infty, |z| \leq H$) which is to be an infinite thick plate bounded by the two planes AA' and BB' in the infinite solid of Fig.3, then the region of an infinite thick plate satisfies the boundary conditions (1) and is subjected to the torsional body forces F_2 of Eq.(13) in the interior of the region.

In Fig.3, torsional body forces are distributed on the planes

- (i) $z=2(m-1)H - (-1)^m h, (+\theta)$
 - (ii) $z=-\{2mH - (-1)^m h\}, (+\theta)$
 - (iii) $z=2mH - (-1)^m h, (-\theta)$
 - (iv) $z=-\{2(m-1)H - (-1)^m h\}, (-\theta)$
- ... (27)

Fig.3 Torsional body forces periodically distributed

where $m=1,2,3,\dots$. Therefore, the displacements u_2'' produced by these body forces shown in Fig.3 can be obtained by the superposition

$$u_2'' = \sum_{m=1}^{\infty} \{u_2(r, z - [2(m-1)H - (-1)^m h]) + u_2(r, z + [2mH - (-1)^m h]) - u_2(r, z - [2mH - (-1)^m h]) - u_2(r, z + [2(m-1)H - (-1)^m h])\} \dots (28)$$

where $u_2(r, z)$ represents u_2 of Eq.(26). It is easily verified that u_2'' of Eq.(28) agrees with $(u_2)_s$ of Eq.(19). It is also seen that $(u_2)_c$ of Eq.(22) can be obtained if we assume that all the torsional body forces shown in Fig.3 act in the same direction.

4. A solid cylinder twisted by a pair of chucks

Applying Green's functions shown in the previous Chapter, we shall show a method of solution for a circular solid cylinder twisted by a pair of axisymmetric rigid chucks with the same inner radius C as the outer radius of the solid cylinder as shown in Fig.4, where cylindrical coordinates (r, θ, z) are used and $2H$ is the length of the cylinder. If we assume that the chucks are rotated through an angle ω about the z -axis and perfectly bonded to the solid cylinder, then the boundary conditions of the problem

Fig. 4 A finite solid cylinder twisted by rigid chucks

are expressed as follows:

$$\left. \begin{aligned} (i) \quad u_2|_{r=c} &= \pm \omega c \quad (H-b \leq |z| \leq H) \\ (ii) \quad \tau_{r\theta}|_{r=c} &= 0 \quad (|z| < H-b) \\ (iii) \quad \tau_{z\theta}|_{z=\pm H} &= 0 \quad (0 \leq r \leq c) \\ u_1 &= u_3 = 0, \quad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0 \end{aligned} \right\} \quad (29)$$

where b represents the contact length between the chucks and the solid cylinder.

The displacement $u_2(r, z, a, h, q)$ produced by the torsional body force

$$F_2 = \frac{q}{2\pi r} \{ \delta(z-h) - \delta(z+h) \} \delta(r-a) \quad (30)$$

acting on the two circles ($r=a, z=\pm h$) in opposite directions in the interior of an elastic thick plate with a thickness $2H$ as shown in Fig. 1, is expressed by $(u_2)_s$ of Eq. (19). The displacement U_2 produced by N torsional body forces ($q=q_n, n=1, 2, 3, \dots, N$) distributed on N circles ($a=a_n, h=h_n$) in the interior of the thick plate can be expressed by superposition of $u_2(r, z, a, h, q)$ mentioned above as follows:

$$\left. \begin{aligned} u_2 &= \frac{1}{4\pi^2 \mu \sqrt{ra}} \sum_{n=1}^N \sum_{m=1}^{\infty} q_n \{ Q_{1/2}(x_1) + Q_{1/2}(x_2) - Q_{1/2}(x_3) - Q_{1/2}(x_4) \}_{h=h_n} \\ \tau_{r\theta} &= \frac{1}{8\pi^2 r \sqrt{ra}} \sum_{n=1}^N \sum_{m=1}^{\infty} q_n \left\{ \sum_{i=1}^4 P \left[\frac{r/a - x_i}{x_i^2 - 1} \{ x_i Q_{1/2}(x_i) - Q_{-1/2}(x_i) \} - 3Q_{1/2}(x_i) \right] \right\}_{h=h_n} \\ \tau_{z\theta} &= \frac{1}{8\pi^2 r a \sqrt{ra}} \sum_{n=1}^N \sum_{m=1}^{\infty} q_n \left[\sum_{i=1}^4 P \frac{z_i}{x_i^2 - 1} \{ x_i Q_{1/2}(x_i) - Q_{-1/2}(x_i) \} \right]_{h=h_n} \\ u_1 &= u_3 = 0, \quad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0 \end{aligned} \right\} \quad \dots (32)$$

From the definition of Green's functions, the expressions (32) satisfy always the boundary conditions (iii) of Eq. (29) regardless of the values of q_n . Accordingly, if we determine the values of q_n so as to satisfy the boundary conditions (i) and (ii) of Eq. (29), namely

$$\left. \begin{aligned} (i) \quad & \text{for } H-b \leq z \leq H; \quad \{u_2 \text{ of Eq. (32)}\}_{r=c} = \omega c, \\ (ii) \quad & \text{for } H-b > z \geq 0; \quad \{\tau_{r\theta} \text{ of Eq. (32)}\}_{r=c} = 0 \end{aligned} \right\} \quad \dots (33)$$

then all the boundary conditions (29) are satisfied. We consider N points ($r=C, z=z_j, j=1, 2, 3, \dots, N$) on the surface of the solid cylinder and choose the values of q_n so as to satisfy the boundary conditions (33) at these selected points (circles) since it is difficult to satisfy Eq. (33) at all the points on the surface of the solid cylinder. Solutions obtained from the method described above satisfy completely both the fundamental equations of elasticity and the boundary conditions (iii) of Eq. (29) and satisfy not rigorously but approximately the boundary conditions (i) and (ii) of Eq. (29). However, it seems that any desired degree of accuracy may be obtained by increasing the number N of the body forces distrib-

Fig. 5 A solid cylinder of a radius C and a cylindrical surface of a radius a distributed by body forces

$$U_2 = \sum_{n=1}^N u_2(r, z, a=a_n, h=h_n, q_n) \quad (31)$$

Note that the stresses obtained from U_2 of Eq. (31) satisfy the boundary conditions (1). Therefore, if we distribute N torsional body forces of Eq. (30) on a cylindrical surface ($a/C = 1.0 - \epsilon, \epsilon \ll 1.0, h=h_n, 0 < h_n < H$) around the cylindrical surface ($r=C$) which is to be the surface of the solid cylinder as shown in Fig. 5 and if we determine the intensities q_n of the body forces thus distributed so as to satisfy the boundary conditions (i) and (ii) of Eq. (29), then a solution to the problem of a finite solid cylinder twisted by a pair of rigid chucks as shown in Fig. 4 is obtained. Note that there are no body forces in the solid cylinder because of body forces being distributed outside of the region which is to become the solid cylinder.

From the symmetry of the problem with respect to z , we may consider only the domain $0 < z < H$ of the solid cylinder. Based on the principle of the method of solution described above, let us express a practical procedure of calculation to the present problem. It is easily seen from the relations of Eqs. (31), (19) and so on that the stresses and displacement produced by N body forces of Eq. (30) distributed on N circles ($r=a, z=h_n$) in the interior of a thick plate are expressed as follows:

uted.^{(6),(7)} It follows from the above considerations that we can construct a set of simultaneous linear algebraic equations with N unknowns q_n . Solving the linear equations and substituting their solutions into Eq.(32), we can find the stresses and the displacements at any point of the solid cylinder.

The torque T necessary to twist the rigid chucks is given by

$$T = 2\pi c^2 \int_{H-b}^H (\tau_{r\theta})_{r=c} dz = 2\pi \int_0^c r^2 (\tau_{z\theta})_{z=0} dr. \quad \dots (34)$$

Now, if we suppose that body forces are distributed not on N points but continuously on the cylindrical surface ($r=a$), then Eq.(32) becomes

$$\left. \begin{aligned} u_2 &= \frac{1}{4\pi^2 \mu \sqrt{ra}} \sum_{m=1}^{\infty} \int_0^H q(h) \{Q_{1/2}(x_1) + Q_{1/2}(x_2) - Q_{1/2}(x_3) - Q_{1/2}(x_4)\} dh \\ \tau_{r\theta} &= \frac{1}{8\pi^2 r \sqrt{ra}} \sum_{m=1}^{\infty} \int_0^H q(h) \left\{ \sum_{i=1}^4 P \left[\frac{r/a - x_i}{x_i^2 - 1} \{x_i Q_{1/2}(x_i) - Q_{-1/2}(x_i)\} - 3Q_{1/2}(x_i) \right] \right\} dh \end{aligned} \right\} \dots (35)$$

where the expression for $\tau_{z\theta}$ is omitted and we must take $P=1$ when $i=1,2$ and $P=-1$ when $i=3,4$. In this case, the conditions corresponding to Eq.(33) are expressed by

- (i) for $H-b \leq z \leq H$; $\{u_2 \text{ of Eq.(35)}\}_{r=c} = \omega C$
(ii) for $H-b > z \geq 0$; $\{\tau_{r\theta} \text{ of Eq.(35)}\}_{r=c} = 0 \quad \dots (36)$

If we can find the unknown function $q(h)$ which is the solution to the dual integral equations (36), then the stresses and displacements at any points (r, z) of the solid cylinder are obtained from Eq.(35).

5. Numerical calculations

Using the method of solution described in Chapter 4, numerical calculations were performed for the solid cylinder with the size $C/H=0.5$ and with the widths $b/H=0.2$ and 0.5 of the chucks. In the numerical calculations, we used the values $N=120$, $m=40$ and $\epsilon=0.05$. As for h_n and z_j where body forces were distributed and the boundary conditions were to be satisfied, we took N points obtained by division of the thickness H at equal intervals. The results are shown in Figs.6 ~ 9. In these figures, the dotted lines represent the results for $b/H=0.5$ and the solid lines represent the results for $b/H=0.2$.

Figure 6 shows the distributions of the displacements u_2 . It is observed that the displacements u_2 are nearly in proportion to z if the points considered are a small distance away from the chuck in the axial direction and the boundary conditions at the portion of the chuck ($r/C=1.0$, $b/H \leq z/H \leq 1.0$) are well satisfied.

Figure 7 shows the distribution of the stresses $\tau_{z\theta}$. The stresses have a singularity at the contact point with the top corner ($r=C$, $z=H-b$) of the chuck. In the portion ($z < H-b$) of the cylinder with a surface free from applied surface forces, the

Fig.6 Distributions of displacements u_2

stresses $\tau_{z\theta}$ are nearly uniform-distributed in the z -direction and their maximum values appear at the surface ($r=C$). In the portion ($z > H-b$) caught by the chuck, the stresses $\tau_{z\theta}$ have very small values. It is seen that the boundary conditions (iii) of Eq.(29) are satisfied because of $\tau_{z\theta}=0$ at the end face ($z/H=1.0$, $0 \leq r \leq C$) of the solid cylinder.

Figure 8 shows the distribution of the stresses $\tau_{r\theta}$. The stresses have a similar singularity at the contact point ($r=C$, $z=H-b$) with the top corner of a chuck to that of the stresses $\tau_{z\theta}$. However, the stresses decrease rapidly when a point considered goes away from the top corner of the chuck in the r - or z -direction.

Figure 9 shows the relations among the torque T to twist the chucks, the width b/H of the chuck and the slender ratio C/H of the solid cylinder. The torque T increases rapidly as the width b/H of the chuck is

Fig.7 Distributions of stresses $\tau_{z\theta}$

larger ($b/H \rightarrow 1.0$). In the condition of the width $b/H = \text{constant}$, the torque T is greater as the slender ratio C/H is greater. The marks of a circle in Fig.9 show the values obtained from the simple theory of torsion of elasticity

$$u_2 = \frac{z}{H} \omega r, \quad \tau_{z\theta} = \mu \frac{\omega r}{H}, \quad \tau_{r\theta} = 0 \quad \dots (37)$$

It is seen that the torque T approaches the values of the simple theory (37) when $b/H \rightarrow 0$ and $C/H \rightarrow 0$.

Further, we can see that the distribution of the stresses $\tau_{z\theta}$ of Fig.7 is analogous to that of torsions of a semi-infinite solid cylinder with a circular disk shown by H. Saito.^{(8),(9)}

6. Concluding remarks

The principal results of the paper are summarized as follows:

(1) A general method of solution for torsional body force problems of an elastic thick plate has been shown. Applying the general method of solution, Green's functions for the torsional body force problems of an elastic thick plate have been obtained. Furthermore, the physical meaning of Green's functions shown has been also investigated.

(2) Applying Green's functions mentioned above (1), a numerical method of solution for elastic problems of a circular solid finite length cylinder twisted by a pair of axisymmetric rigid chucks has been shown.

(3) Performing the numerical calculations for the problem described above (2), the distributions of stresses and displacements of the solid cylinder have been considered. The relations among the torque T , the width b/H of the chuck and the slender ratio C/H of the solid cylinder have been shown. The torque has been referred to results of the simple torsion theory of elasticity.

(4) Green's functions shown in the paper are applicable to other torsion problems⁽¹⁰⁾ than the problem treated here.

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Fig.7 Distributions of stresses $\tau_{r\theta}$

Fig.9 Torques T to twist rigid chucks

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