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On the Stability of a Rotating Elastic Shaft

Supported by Journal Bearings*

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The theoretical stability characteristics of a rotating elastic shaft are investigated. For a symmetrical rotating shaft model, in which the shaft with a rotor at its midpoint is supported by journal bearings, the limit frequency at which oil whirl of conical mode starts (for both elastic and rigid shafts) can be analytically obtained in addition to the critical conditions which are absolutely stable against that mode of oil whip.

A stability chart for the translational mode based on infinitely short bearing approximations and Gumbel's boundary condition is also presented.

1. Introduction

There have been written many papers on "oil whip" of a rotating shaft supported by journal bearings and also various kinds of stability charts⁽¹⁾ have been presented for determining its whirl frequency, i.e. the limit frequency of self-excited vibrations which occur due to forces of the oil film in the bearing. Consequently, it seems that the accuracy of estimating the limit frequency has been considerably improved. However, as both the structure of a practical rotating machine and the mechanics of an oil film in a journal bearing are generally complicated, it may be considered that there remains a quantitative difference between the theoretically and experimentally obtained values of this limit frequency.⁽³⁾

In this paper, linearized approximate stability characteristics of a rotating shaft with a conical whirl mode are presented, taking the effect of moment of inertia of its rotor into account. Although a contribution⁽³⁾ dealing with these characteristics has already been reported, the present paper makes it clear that the limit frequency of both rigid and flexible shafts can be obtained analytically, for a symmetrical rotating system, in which the shaft with the rotor at its midpoint is supported by a journal bearing at both ends. In addition, a parallel mode stability chart, which is based on the assumption of infinitely short bearings, is presented and, by using the characteris-

tics of the oil film derived from this approximation, a conical mode stability chart obtained by numerical calculations is shown.

In this analysis, the same rotating system and assumptions as in reference⁽³⁾ are assumed.

2. Nomenclature⁽³⁾

The principal symbols in this paper are as follows:

- $() = d()/dt$
- $()' = d()/d\phi$
- $\phi = \omega t$
- ω : angular frequency of shaft
- ω_w : angular frequency of whirl of shaft
- M : rotor mass
- I_D, I_P : diametral and polar mass moments of inertia of rotor, respectively
- h : radial clearance of bearing
- h_o : mean radial clearance of bearing
- L : bearing length
- D : bearing diameter
- $R = D/2$
- $\beta = L/D$, which is different from β_{ij}
- l : shaft length
- k_s : elastic constant of shaft
- g : gravitational acceleration
- $U \approx R\omega$
- κ : eccentricity ratio of journal
- κ_o : eccentricity ratio of journal at equilibrium point
- $\Theta = 2h_o/l$
- $r_d = \sqrt{I_D/M}$
- $\sigma = r_d/(l/2), \quad v = I_P/I_D$
- $\omega_0 = \sqrt{g/h_o}$
- $\omega_o^* = \sqrt{k_s/M}$
- $\Omega = \omega/\omega_o$
- $\Omega_o = \omega/\omega_o^*$
- $Q = \sigma\Omega$
- Q_o (or ω_o): dimensionless limit whirl frequency

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$\Omega_c = (\omega_w / \omega)_c$: frequency ratio at limit frequency
 $\alpha = Mg / k_s h_o$
 μ : viscosity of lubricant
 k_{ij}^* (i, j=1, 2): stiffness coefficients of oil film
 c_{ij}^* (i, j=1, 2): damping coefficients of oil film
 γ_{ij} (i, j=1, 2): dimensionless stiffness coefficients of oil film
 β_{ij} (i, j=1, 2): dimensionless damping coefficients of oil film
 $\gamma_{ij} = k_{ij}^* h_o / (Mg/2)$
 $\beta_{ij} = c_{ij}^* h_o \omega / (Mg/2)$
 p : oil film pressure
 x, y : radial displacements
 ψ, φ : angular rotor displacements
 $\begin{bmatrix} x \\ y \end{bmatrix} = h_o \begin{bmatrix} \xi \\ \eta \end{bmatrix}$
 $\Phi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$
 $\Psi = \begin{bmatrix} \psi \\ \varphi \end{bmatrix}$
 $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
 $\bar{\gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$
 $\bar{\beta} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$
 Indexes i, j=1, 2: 1= x-direction, 2= y-direction

3. Characteristics of Oil Film

Referring to Fig.1, the Reynolds equation for this oil film may be written in the following form:

$$\frac{\partial}{\partial u} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial u} \right) + \frac{\partial}{\partial z} \left(\frac{h^3}{\mu} \frac{\partial p}{\partial z} \right) = 6u \frac{\partial h}{\partial u} + 12 \frac{\partial h}{\partial t} \quad (1)$$

Fig.1 Journal Bearing

Fig.2 Vibrating System

From Eq.(1), under an infinitely short bearing approximation and precisely the same assumptions as in reference (2), the characteristics of the oil film, i.e. linearized approximate stiffness γ_{ij} (i, j=1, 2) and damping coefficients β_{ij} (i, j=1, 2), are obtained as follows:

$$\begin{aligned}
 \gamma_{11} &= \frac{4 [2\pi^2 + (16 - \pi^2) \kappa_o^2]}{[\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \gamma_{12} &= \frac{\pi [-\pi^2 + 2\pi^2 \kappa_o^2 + (16 - \pi^2) \kappa_o^4]}{\kappa_o (1 - \kappa_o^2)^{1/2} [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \gamma_{21} &= \frac{\pi [\pi^2 + (32 + \pi^2) \kappa_o^2 + 2(16 - \pi^2) \kappa_o^4]}{\kappa_o (1 - \kappa_o^2)^{1/2} [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \gamma_{22} &= \frac{4 [\pi^2 + (32 + \pi^2) \kappa_o^2 + 2(16 - \pi^2) \kappa_o^4]}{(1 - \kappa_o^2) [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \beta_{11} &= \frac{2\pi (1 - \kappa_o^2)^{1/2} [\pi^2 + 2(\pi^2 - 8) \kappa_o^2]}{\kappa_o [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \beta_{12} &= \frac{8 [\pi^2 + 2(\pi^2 - 8) \kappa_o^2]}{[\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \beta_{22} &= \frac{2\pi [\pi^2 + 2(24 - \pi^2) \kappa_o^2 + \pi^2 \kappa_o^4]}{\kappa_o (1 - \kappa_o^2)^{1/2} [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}}, \\
 \beta_{21} &= \beta_{12}.
 \end{aligned}$$

..... (2)

$$\beta'_{22} = \frac{2\pi [\pi^2 + 2(16 - \pi^2) \kappa_o^2 + (16 + \pi^2) \kappa_o^4]}{\kappa_o (1 - \kappa_o^2)^{1/2} [\pi^2 + (16 - \pi^2) \kappa_o^2]^{3/2}} \quad (2-a)$$

in which Gümbel's condition is also assumed as the boundary condition. Coefficient β'_{22} , i.e. Eq.(2-a) is the result of reference (2), whereas β_{22} in Eq.(2) is what is computed by the authors under the same assumptions and boundary condition as in reference (2).

These coefficients are illustrated as an eccentricity ratio κ_o at an equilibrium point in Fig.3, where the broken line corresponds to β'_{22} of Eq.(2-a). Fig.3 shows that β_{22} is generally larger than β'_{22} , the difference between β_{22} and β'_{22} having an influence upon the stability conditions. The stability chart^{(1),(2)} of a translational whirl mode based on these coefficients is shown in Fig.4, where the right-hand side and the left-hand side of each curve correspond to the stable and the unstable running of the shaft, respectively. In Fig.4, furthermore, the solid lines correspond to Eq.(2) and the broken lines⁽²⁾ to both Eq.(2-a) and Eq.(2) except for β_{22} . Compared with the area covered by the broken lines, the solid lines indicate a larger stable region. Hitherto, the results based on the former (the broken lines) have usually been regarded as the

stability chart derived from an infinitely short bearing approximation.⁽¹⁾ However, so far as one adopts the infinitely short bearing approximation in this problem, the results due to the latter (the solid lines) may be considered to be valid as the stability chart for the translational mode.

Fig.3 Characteristic coefficients of Journal Bearing

Fig.4 Stability Chart (translational whirl mode)

4. Linearized Approximate Stability Conditions of Rotor-bearing Systems

Let us consider a symmetrical rotating shaft which has a rotor at its mid-point and is supported by a journal bearing at both ends, see Fig.2. Under various assumptions,^{(2),(3)} the equation of motion of this shaft, which is running in a conical whirl mode, can be written in the following form:⁽³⁾

$$\left. \begin{aligned} \psi'' + v T \psi' + \frac{1}{\alpha Q^2} (\psi - \theta \cdot \phi) &= 0, \\ \bar{\beta} \phi' + \bar{\gamma} \phi - \frac{1}{\alpha \theta} (\psi - \theta \cdot \phi) &= 0 \end{aligned} \right\} \quad (3)$$

The characteristic equation of Eq.(3) is as follows:

$$r_6 s^6 + r_5 s^5 + r_4 s^4 + r_3 s^3 + r_2 s^2 + r_1 s + r_0 = 0$$

where

$$\begin{aligned} r_6 &= \alpha^2 Q^4 A_3, & r_5 &= \alpha Q^4 (\alpha A_1 + A_5), \\ r_4 &= (\alpha^2 v^2 A_3 + \alpha^2 A_2 + \alpha A_4 + 1) Q^4 + 2\alpha Q^2 A_3, \\ r_3 &= \alpha v^2 Q^4 (\alpha A_1 + A_5) + Q^2 (2\alpha A_1 + A_5), \\ r_2 &= v^2 Q^4 (\alpha^2 A_2 + \alpha A_4 + 1) \\ &\quad + Q^2 (2\alpha A_2 + A_4 - v A_7) + A_3, \\ r_1 &= A_1 - v Q^2 A_6, & r_0 &= A_2, \end{aligned}$$

$$A_1 = \gamma_{11} \beta_{22} - \gamma_{12} \beta_{21} - \gamma_{21} \beta_{12} + \gamma_{22} \beta_{11},$$

$$A_2 = \gamma_{11} \gamma_{22} - \gamma_{12} \gamma_{21},$$

$$A_3 = \beta_{11} \beta_{22} - \beta_{12} \beta_{21},$$

$$A_4 = \gamma_{11} + \gamma_{22}, \quad A_5 = \beta_{11} + \beta_{22},$$

$$A_6 = \gamma_{12} - \gamma_{21}, \quad A_7 = \beta_{12} - \beta_{21}.$$

..... (4)

Applying the Routh-Hurwitz criteria to Eq.(4), the following equation is obtained eventually for the stable running in the equilibrium state of the journal:^{(3),(4),(7)}

$$\Delta = \begin{vmatrix} r_5 & r_3 & r_1 & 0 & 0 \\ r_6 & r_4 & r_2 & r_0 & 0 \\ 0 & r_5 & r_3 & r_1 & 0 \\ 0 & r_6 & r_4 & r_2 & r_0 \\ 0 & 0 & r_5 & r_3 & r_1 \end{vmatrix} > 0 \dots (5)$$

Eq.(5) is found (after a little calculation) to be

$$\Delta = T_1 Q^6 + T_2 Q^4 + T_3 Q^2 + T_4 > 0 \dots (6)$$

where

$$T_1 = -v^2 A_6 B_1 B_2 [(\alpha v A_1 + A_6)(B_2 - \alpha^2 v^2 A_3) + v B_1 (B_3 - \alpha B_4)],$$

$$\begin{aligned} T_2 &= -v B_1^2 [B_4 [-\alpha v (\alpha A_2 B_1 - A_1 B_2) + A_6 (B_3 - \alpha B_4)] - v B_2 (A_1 B_3 - \alpha A_2 A_5) \\ &\quad + \alpha A_2 [A_6 (2B_2 - \alpha^2 v^2 A_3) + \alpha^2 v^3 A_3 A_5]] - v B_1 [-\alpha v A_3 A_6 [-2\alpha v A_1 (3B_2 - \alpha B_4) \end{aligned}$$

$$\begin{aligned}
& + 2\alpha^2 v^2 A_3 (-\alpha v A_1 + A_6) + A_6 (2\alpha B_4 - 3B_3)] + \alpha A_1 [A_6 B_3 B_4 - v B_2 (A_1 B_2 - \alpha^2 v^2 A_1 A_3 \\
& - 2\alpha A_2 B_1)] - 2A_1 A_6 B_3^2] + \alpha^2 v^2 A_3 A_6 [(\alpha v A_3 A_6 + A_1 B_3) A_6 + \alpha v A_1^2 B_2], \\
T_3 = & B_1^2 [B_4 [\alpha v A_3 A_6 + A_1 (B_3 - \alpha (A_4 - v A_7))] + \alpha A_2 A_5] - v A_3 A_6 (B_3 + \alpha^2 A_2) \\
& + A_2 [B_2 (\alpha A_1 - A_5) - \alpha^2 (A_2 B_1 + v^2 A_3 A_5)] + B_1 [-\alpha^2 v A_1 A_3 [2B_4 (2A_6 + \alpha v A_1) - 3\alpha A_2 A_6] \\
& + A_1 B_3 [A_1 (\alpha B_4 - B_3) + 5\alpha v A_3 A_6] + 2\alpha A_1 B_2 (\alpha v^2 A_1 A_3 - A_2 B_1) + \alpha^2 v^2 A_1 A_3 (A_1 B_3 \\
& - 2\alpha v A_3 A_6 - 2\alpha A_2 A_5)] + \alpha^2 A_1 [-v A_3 A_6 [\alpha v A_3 A_6 + A_1 (2B_3 - \alpha B_4)] - A_1 B_2 (A_2 B_1 \\
& + \alpha v^2 A_1 A_3)], \\
T_4 = & A_1 A_3 [B_1^2 [1 + \alpha (\alpha (v^2 A_3 + 2A_2) + v A_7)] + \alpha B_1 [-\alpha A_1 (\alpha A_2 + 2v (A_7 + \alpha v A_3)) \\
& - 2(A_1 + \alpha A_2 A_5)] + \alpha^2 A_1 [-\alpha A_2 A_5 + A_1 (\alpha v (\alpha v A_3 + A_7) - \alpha^2 A_2 + 1)]], \\
B_1 = & \alpha A_1 + A_5, \quad B_2 = \alpha^2 A_2 + \alpha A_4 + 1, \quad B_3 = \alpha^2 v^2 A_3 + \alpha^2 A_2 + \alpha A_4 + 1, \\
B_4 = & 2\alpha A_2 + A_4 - v A_7.
\end{aligned} \tag{7}$$

Eq.(6) gives the stability conditions of an elastic shaft, running in the conical whirl mode. As Eq.(6) is a cubic equation with Q^2 as variable, the limit frequency of self-excited vibrations, Q_c (or ω_c), is quite easily obtained by utilizing the root formula of cubic equations. The stability conditions of some special cases are represented in the following.

4-1. Rigid Shaft Case ($I_p \neq 0$ and $\alpha = 0$)

In Eqs.(6) and (7), putting $\alpha = 0$ yields the stability conditions for a rigid shaft. That is:

$$\left. \begin{aligned}
S_1 Q^6 + S_2 Q^4 + S_3 Q^2 + S_4 &> 0 \\
\text{where} \\
S_1 &= -v^2 A_6 (A_6 + v A_5), \\
S_2 &= v [v A_5 (A_1 + A_6 A_7) \\
&\quad + A_6 (2A_1 - A_4 A_5)], \\
S_3 &= -[A_2 A_5^2 + A_1^2 - A_1 A_4 A_5 \\
&\quad + v A_5 (A_1 A_7 + A_3 A_6)], \\
S_4 &= A_1 A_3 A_5
\end{aligned} \right\} \dots (8)$$

Eq.(8) is also the same cubic equation in Q^2 as the stability condition of flexible

shaft, Eq.(6),⁽³⁾ so that its limit frequency can be easily given by the root formula of cubic equations.

4-2. $I_p = 0$ Case (i.e., $v = 0$)

Substituting $v = 0$ into Eqs.(6) and (7), the stability condition of this case is found to be

$$Q^2 < \frac{1}{(\alpha A_1 + A_5)} \frac{A_1 A_3 A_5^2}{(A_1^2 - A_1 A_4 A_5 + A_2 A_5^2)} \tag{9}$$

Eq.(9) is the same as the traditional stability expression^{(2),(3)} for a translational mode (where $\sigma = 1$), see Fig.5.

5. Whirl Frequency of Shafts

Substituting $s = j(\omega_w / \omega)$ (where $j = \sqrt{-1}$) into the characteristic equation Eq.(4) and utilizing the following:⁽⁵⁾

$$\left. \begin{aligned}
(\text{real part}) &= 0, \\
(\text{imaginary part}) &= 0
\end{aligned} \right\} \dots (10)$$

the ratio of the whirl frequency of the shaft ω_w to its rotating frequency ω , at the limit frequency for the generation of self-excited vibrations, is given in the following:

$$\begin{aligned}
\tilde{Q}_c^2 &= \left(\frac{\omega_w}{\omega} \right)_{c1}^2 \\
&= \frac{[\alpha v^2 B_1 Q_c^2 + (2\alpha A_1 + A_5) \pm \sqrt{\{\alpha v^2 B_1 Q_c^2 + A_5\}^2 + 4\alpha v Q_c^2 B_1 (\alpha v A_1 + A_6)}]}{2\alpha B_1 Q_c^2} \dots (11)
\end{aligned}$$

Eq.(11) gives two roots, but we only employ the root which satisfies all of the simultaneous equations Eq.(10). For the cases of both $I_p = 0$ (i.e., $v = 0$) and $\alpha = 0$ (i.e., a rigid shaft), the limit whirl

frequency ratio $\tilde{Q}_c$ can be gained in the same manner; i.e.,

$$\tilde{Q}_c^2 = A_1 / [Q_c^2 (\alpha A_1 + A_5)] \dots (12)$$

6. Absolute Stability Conditions against Self-excited Vibrations in Conical Modes

Let us consider the conditions (3) under which conical mode self-excited vibrations in the shaft-bearing system absolutely cannot occur. From Eq.(6) and Eq.(7), the limit value v_c at such critical conditions is found to be the root in the following equation: (3), (4)

$$\left. \begin{aligned} T_1 &= 0 \\ \text{or} \\ \alpha^2 A_1 A_5 v^3 + \alpha \{ A_7 B_1 - \alpha A_3 A_6 \} v^2 \\ &+ \{ \alpha^2 (A_1 A_4 - A_2 A_5) + 2\alpha A_1 + A_5 \} v \\ &+ A_6 B_2 = 0 \end{aligned} \right\} \dots (13)$$

In particular, this limit value v_c for rigid shafts is simply

$$v_c = -A_6 / A_5 \dots \dots \dots (14)$$

7. Numerical Results and Discussion

The numerical results are presented in the following. All results were obtained by employing stiffness and damping coefficients of the oil film based on the infinitely short bearing approximation (see Section 3) and a value of 0.5 for β , the ratio of the length to the diameter of the bearing.

7-1. Stability Charts for Conical Whirl Modes

The condition in Eq.(6) is represented in Fig.5, where the abscissa is κ_0 and

Fig.5 Stability Chart (conical whirl mode, $\beta = 0.5$)

the ordinate is Q_c . These graphs represent stability charts of the conical whirl mode. As in Fig.4, the right-hand side of each curve in Fig.5 corresponds to the stable state. It is clear that these results agree well qualitatively with those derived from the theory of oil film in bearings of finite length (see reference (3)). Further detailed considerations on these stability charts are omitted here but are available in reference (3).

7-2. Whirl Frequency of Shafts

Numerical results for the ratio of whirl frequency to rotating frequency of shafts at the limit frequency of self-excited vibrations are presented in Fig.6, which is derived from the condition in Eq. (11). As for the results of reference (3), the frequency ratio $\tilde{\omega}_c$ based on the infinitely short bearing approximation is not always equal to 0.5 and varies in quite a complicated manner as shown.

7-3. Absolute Stability Conditions against Generating Conical Whirl Mode

If stiffness and damping properties of the oil film are employed for its characteristics, when the shaft is rigid, the absolute stability condition Eq.(14) against generating conical whirl mode is found to be

$$v_c \equiv 0.5 \dots \dots \dots (15)$$

This is apparently different from the condition derived from the theory of finite length bearings; i.e., $v_c \leq 0.5$. Furthermore, even if the shaft is flexible, the critical value v_c can be obtained analytically without difficulty since Eq.(13)

Fig.6 Dimensionless whirl frequency at the limit frequency ($\beta = 0.5$)

is a cubic equation in v . The numerical results for an elastic shaft are given in Fig.7. These graphs confirm that the hypothesis of T. Someya (6) for flexible shafts is valid.

8. Conclusions

Considering linearized approximate stability conditions of a simple and symmetrical shaft-bearing system, in which the shaft with one disk at its midpoint is

Fig.7 Limit value v_c for absolute stability against conical whirl mode ($\beta = 0.5$)

supported by a cylindrical journal bearing at each end, the following are clarified:

(i) Both the limit frequency for generation of self-excited vibrations in the conical whirl mode and the critical condition for the absolutely stable running of shafts against the occurrence of those vibrations can be analytically obtained without difficulty, regardless of whether the shafts being considered are elastic or rigid.

(ii) Assuming the ratio of length to diameter of bearings to be equal to 0.5, stability charts derived from stiffness and damping properties based on the infinitely short bearing approximation are in good agreement with those based on the theory of bearings of finite length.

(iii) Adopting Gumbel's boundary condition, the stability shaft for the translational mode obtained from infinitely short bearing approximations is also shown.

References

- (1) Handbook of vibrations (in Japanese), Yhokendo Co. (1976), 935.
- (2) Funagawa, M. and Tatara, A., Stability Criterion of an Elastic Rotor in Journal Bearings, Trans. JSME (in Japanese), 30- 218 (1964), 1238.
- (3) Someya, T., On the Stability of a Rotor Supported by Journal Bearings, Trans. JSME (in Japanese), 42- 360 (1976), 2345.
- (4) Tatara, A., Dynamic Characteristics of Oil Film in Journal Bearings, Teaching material of 70-th information course of JSME (in Japanese), (1976), 27.
- (5) Gasch, R. and Pfützner, H., Rotor-dynamik Eine Einführung (translated by Miwa, S.), Springer-Verlag, (1978), 57.
- (6) page 2350 of reference (3).
- (7) Hori, Y., A Theory of Oil Whip, Trans. ASME, 26- 2, (1959), 189.