Heat Transfer by Natural Convection in a Shallow Rectangular Cavity with Two Opposing Vertical Walls at Different Temperatures*

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The present paper deals numerically with natural convective heat transfer in a confined rectangular cavity of a relatively low geometric aspect-ratio H/W (H, W : height and width of cavity) with different vertical wall temperatures. The numerical calculations using the Up-Wind finite difference method are carried out for a confined rectangular cavity having $H/W = 0.03 \sim 1$, Prandtl number $Pr = 1 \sim 10^3$ and Rayleigh number $Ra = 10^2 \sim 10^6$. The results obtained indicate that the geometrical aspect-ratio H/W significantly affects the rate of heat transfer through the fluid layer. Moreover, useful and convenient correlations of the heat transfer through the fluid layer are derived from the results.

Key Words: Convective Heat Transfer, Thermal Engineering, Numerical Analysis

1. Introduction

A multitude of papers (1)~(5) are available on the problem of natural convective heat transfer in a rectangular cavity having a relatively high aspect-ratio ($H/W \geq 1$), in which the opposing vertical walls have been kept at different temperatures. On the other hand, only a few studies on the heat transfer in the rectangular cavity having a relatively low aspect-ratio ($H/W < 1$) have been reported. These studies are of importance in evaluation of the heat transfer coefficient in building with a low geometrical dimension ratio of height to width in which one vertical wall is heated by solar energy, and in thermal insulation of a shallow duct or pressure vessel. For example, Hirata et al. (6) reported the experimental and numerical studies on this problem with a low aspect-ratio of $0.42 \leq H/W \leq 1$. Their numerical results could not agree with the experimental ones due to the difference of the horizontal boundaries between the experiment and numerical analysis (incomplete insulation of both the upper and bottom walls in the experiments). Cormack et al. (7)~(9) mentioned theoretically and experimentally the behavior of a natural convective flow in a shallow cavity with differentially heated end walls. But their results could not refer to the effects of the aspect-ratio and physical properties of the fluid on the heat transfer in the fluid layer.

The purpose of this study is to clarify numerically the effects of the dimension of cavity (aspect-ratio H/W) and the physical properties of the fluid (Prandtl number Pr) on the heat transfer in the confined rect-

angular cavity with a relatively low aspect-ratio in a wide range of $0.03 \leq H/W \leq 1$, in which the opposing vertical walls have each uniform (but different) temperature, while the upper and bottom walls are thermally insulated.

2. Nomenclature

- Gr : Grashof number, $g\beta\Delta TH^3/\nu^2$
 - g : acceleration of gravity
 - H : height of rectangular cavity
 - h : average heat transfer coefficient on cold or hot wall, $q/(T_h' - T_c')$
 - H/W : geometrical aspect-ratio
 - Nu : average Nusselt number, $hH/\lambda = \int_0^1 \frac{\partial T}{\partial Y} \big|_{Y=0, W/H} dX$
 - $Nu_{h,X}$: local Nusselt number on hot wall, $\frac{\partial T}{\partial Y} \big|_{Y=0}$
 - Pr : Prandtl number, ν/α
 - q : average heat flux
 - Ra : Rayleigh number, $Gr \times Pr$
 - T' : temperature
 - T : non-dimensional temperature, $(T' - T_c')/(T_h' - T_c')$
 - ΔT : temperature difference between hot and cold walls, $(T_h' - T_c')$
 - U : velocity component in X' direction
 - U' : non-dimensional velocity component in X direction
 - V : velocity component in Y' direction
 - V' : non-dimensional velocity component in Y direction
 - W : width of rectangular cavity
 - X', Y' : co-ordinates
 - X, Y : non-dimensional co-ordinates
 - α : thermal diffusivity
 - β : coefficient of volumetric expansion
 - λ : thermal conductivity
 - ν : kinematic viscosity
 - ψ : stream function
 - Ψ : non-dimensional stream function
 - ω : vorticity
 - Ω : non-dimensional vorticity
- Subscripts

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c : refers to cold wall
 h : refers to hot wall.

3. Analysis

The present numerical analysis has been carried out by considering an enclosed two-dimensional rectangular cavity with a low aspect-ratio ($H/W \leq 1$), which is filled with a fluid. The two opposing vertical walls of the cavity are maintained at uniform (but different) temperatures T_h and T_c , while the upper and bottom walls are thermally insulated. The physical model is shown in Fig.1. The numerical solution can be obtained by solving the steady state finite difference equations. The usual Boussinesq approximation as a physical model is utilized. By introducing the stream function ψ ($U' = \frac{\partial \psi}{\partial Y}$, $V' = -\frac{\partial \psi}{\partial X}$), the vorticity ω ($\omega = \frac{\partial V'}{\partial X} - \frac{\partial U'}{\partial Y}$) and the following non-dimensional variables into the appropriate governing equations, namely the conservation of mass, momentum and energy balance equations,

$$\begin{aligned} X &= X'/H, \quad Y = Y'/H, \quad U = U'H/\sqrt{Gr}, \\ V &= V'H/\sqrt{Gr}, \quad \Psi = \psi/\sqrt{Gr}, \quad \Omega = \omega H^2/\sqrt{Gr}, \\ T &= (T' - T'_c)/(T'_h - T'_c), \quad Pr = \nu/\alpha, \\ Gr &= g\beta\Delta TH^3/\nu^2 \end{aligned} \quad (1)$$

one may obtain the following non-dimensional governing equations to be analyzed:

$$\frac{\partial}{\partial X} \left(\frac{\partial \Psi}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\frac{\partial \Psi}{\partial Y} \right) + \Omega = 0 \quad (2)$$

$$\frac{\partial}{\partial X} \left(\Omega \frac{\partial \Psi}{\partial Y} \right) + \frac{\partial}{\partial Y} \left(\Omega \frac{\partial \Psi}{\partial X} \right) - \frac{1}{\sqrt{Gr}} \left[\frac{\partial}{\partial X} \left(\frac{\partial \Omega}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\frac{\partial \Omega}{\partial Y} \right) \right] -$$

$$\frac{\partial T}{\partial Y} = 0 \quad (3)$$

$$Pr \left[\frac{\partial}{\partial X} \left(T \frac{\partial \Psi}{\partial Y} \right) - \frac{\partial}{\partial Y} \left(T \frac{\partial \Psi}{\partial X} \right) \right] - \frac{1}{\sqrt{Gr}} \left[\frac{\partial}{\partial X} \left(\frac{\partial T}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\frac{\partial T}{\partial Y} \right) \right] = 0 \quad (4)$$

3.2 Boundary conditions

The no-slip condition for the velocity as well as the constancy of the temperature on the hot or cold vertical wall requires the following vertical boundary conditions:

$$Y = 0, 0 \leq X \leq 1; \quad \frac{\partial \Psi}{\partial Y} = \frac{\partial \Psi}{\partial X} = \Psi = 0, \quad T = 1 \quad (5)$$

$$Y = W/H, 0 \leq X \leq 1; \quad \frac{\partial \Psi}{\partial Y} = \frac{\partial \Psi}{\partial X} = \Psi = 0, \quad T = 0 \quad (6)$$

Fig.1 The physical model of the cavity.

Fig.2 Illustration of a portion of the finite difference grid.

Assuming that the upper and bottom walls are thermally insulated, one can obtain the following horizontal boundary conditions:

$$X = 0, 0 < Y < W/H; \quad \frac{\partial \Psi}{\partial X} = \frac{\partial \Psi}{\partial Y} = \Psi = 0, \quad \frac{\partial T}{\partial X} = 0 \quad (7)$$

$$X = 1, 0 < Y < W/H; \quad \frac{\partial \Psi}{\partial X} = \frac{\partial \Psi}{\partial Y} = \Psi = 0, \quad \frac{\partial T}{\partial X} = 0 \quad (8)$$

3.3 Finite difference formulation

It will be convenient to write the governing equations in a form which differs slightly from the original version, namely,

$$a \left[\frac{\partial}{\partial X} \left(\Xi \frac{\partial \Psi}{\partial Y} \right) - \frac{\partial}{\partial Y} \left(\Xi \frac{\partial \Psi}{\partial X} \right) \right] - c \left[\frac{\partial}{\partial X} \left(b \frac{\partial \Xi}{\partial X} \right) + \frac{\partial}{\partial Y} \left(b \frac{\partial \Xi}{\partial Y} \right) \right] + d = 0 \quad (9)$$

where Ξ is a general dependent variable representing Ψ , Ω or T . The functions a to d are listed in Table 1. The system of governing equations (2) ~ (4) is thus completely indicated by Eq.(9).

On account of a mathematical complexity to solve Eq.(9), the Up-Wind finite difference technique is applied in the present study. The derivation of the finite difference scheme of Eq.(9) was described in great detail in the previous work⁽¹⁰⁾ One can derive the interested flow field into a number of rectangular cells with a space of k and l in the X -direction and Y -direction, respectively, as depicted in Fig.2. Here, $k = 1/M$ and $l = (W/H)/N$. The values of the function a at a nodal point of grid $[(i-1)k, (j-1)l]$ are identically indicated by (i,j) . i and j are integers in the range of $1 \leq i \leq M+1$ and $1 \leq j \leq N+1$, respectively. With this notation, the finite difference form of Eq.(9) gives the value of Ξ at a nodal point of grid $P(i,j)$ in terms of the values of Ξ at the neighbouring eight nodal points of grids, e.g., $N(i,j+1)$, $S(i,j-1)$, $E(i+1,j)$, $W(i-1,j)$, $NE(i+1,j+1)$, $SE(i+1,j-1)$, $NW(i-1,j+1)$ and $SW(i-1,j-1)$ as described in the literature. The temperatures at four corners of the rectangular cavity are set at the temperatures of

Ξ	a	b	c	d
Ψ	0	1	1	$-\Omega$
Ω	1	1	$1/\sqrt{Gr}$	$-\frac{\partial T}{\partial Y}$
T	Pr	1	$1/\sqrt{Gr}$	0

Table 1 The functions a, b, c and d associated with Eq.(9)

Fig.3 Comparison between the present numerical results and the previous ones for $H/W = 1$ and $Pr = 1$.

the isothermal vertical walls, that is,
 $T(1,1) = T(i+1,1) = 1$, $T(1,j+1) = T(i+1,j+1) = 0$.

In the iterative procedure used to solve Eq.(10), a relaxation factor δ_p is adopted to stabilize the numerical computations. It is defined by

$$\bar{\varepsilon}_p^{(n)} = (1-\delta_p)\bar{\varepsilon}_p^{(n-1)} + \delta_p \bar{\varepsilon}_p^{(n)*} \quad (10)$$

where the superscripts (n) and $(n-1)$ indicate the values at the (n) th and the $(n-1)$ th iterations, respectively, and the asterisk indicates the most recently correlated value. The relaxation parameter δ_p adopted is similar to that of Hennecke et al.⁽¹¹⁾ Each grid size $(k$ or $l)$, which is related to the numerical convergence of Ψ , Ω and T , is examined to ensure the desired accuracy of the numerical results. The number of grids in the vertical direction $[X]$ is 21, while the number of grids in the horizontal direction $[Y]$ is decided corresponding to the aspect-ratio H/W . The line iterative method is repeated until the following prescribed error is corrected.

$$\varepsilon = \frac{|\bar{\varepsilon}_{i,j}^{(n)} - \bar{\varepsilon}_{i,j}^{(n-1)}|}{|\bar{\varepsilon}_{i,j}^{(n)}|} < 10^{-4} \quad (11)$$

4. Results and discussions

In order to demonstrate the usefulness of the present numerical analysis, a comparison between the present numerical results and the previous ones is made in Fig.3 for aspect-ratio $H/W = 1$ and Prandtl number $Pr = 1$. The solid circle used in all figures denotes the calculated value in the present study. It would be clear that the present numerical results in the range of $Ra > 5 \times 10^3$ have approximately the same tendency as those obtained by Batchelor⁽¹⁾ and Emery et al.⁽²⁾, that is, the Nusselt number Nu increases monotonically with an increasing

Fig.4 Comparison between the present numerical results and the previous experimental ones.

Fig.5 Relationship between Nu and Ra .

Rayleigh number Ra . Moreover, it is interesting to note that the numerical results of this study are quite similar to MacGregor's ones⁽³⁾ that is, Nu does not increase linearly with an increasing Ra on the double logarithmic graph (in general, Nu would be on a straight line of $1/4$ slope against Ra in the laminar natural convective flow region), but both curves obtained possess an inflection point at small Ra . It might be said that an asymptotic flow region, in which the intensity of natural convection is very weak and the thermal conduction predominates, appears in the range of $Ra < 5 \times 10^3$ for $H/W = 1$ and $Pr = 1$ in the present results.

Fig.4 presents a comparison between the present results and the previous experimental ones reported by Hirata et al.⁽⁶⁾ The results indicated in the upper part of Fig.4 for $H/W = 1$ (their experiments were performed for $H/W = 0.92$) show a good agreement between the present numerical results and the experimental ones. However, in the lower part of Fig.4 for $H/W = 0.5$, their experimental results lie below the solid-line proposed by the present numerical results. This deviation could be caused by not only the difference in aspect-ratio H/W between the present analysis ($H/W = 0.5$) and the previous experiment ($H/W = 0.42$), but also the difference in the boundary conditions between the analysis and the experiment. In other words, the assumptions in the present analysis, in which the upper and bottom walls are thermally insulated or the surface temperatures of the hot and cold vertical walls are kept uniform, might not be satisfied in the previous experiments.

Fig.5 shows the relationship between Nu and Ra for various aspect-ratios H/W . In this figure, it can be seen that the conductive heat transfer, in which Nu is constant with respect to Ra , predominates in the fluid layer for small Ra . Here, the reader should keep in mind that the value of Nu may be below unity if Nu is defined with the height of the cavity as a representative length. Moreover, as Ra increases, the mode of heat transfer shifts gradually to an asymptotic flow heat transfer in which Nu is proportional to the $1/2$ power of Ra , finally the

Fig.6 Distribution of local Nusselt numbers $Nu_h x$ on the hot wall for $Pr = 480$ and $Ra = 10^5$.

Fig.7 Non-dimensional temperature distribution in the Y-direction for $Pr = 480$ and $Ra = 10^5$

mode of heat transfer becomes a laminar convective heat transfer in which Nu is proportional to the $1/4$ power of Ra . It should also be noted that the critical Rayleigh number for the onset of an asymptotic heat transfer or a laminar heat transfer increases as H/W decreases. These behaviors might be explained from the fact that as H/W decreases, that is, as the width of the cavity increases under a fixed height, the cumulative effect of locally small viscous forces over a sufficiently long width of the cavity.

From the results of calculated flow patterns in the present study, a unicellular flow circulated in a clockwise direction was observed in the range of $H/W = 0.03 \sim 1.0$. Fig.6 shows the variation of local Nusselt number on the hot wall $Nu_{h,x}$ with respect to X direction of various H/W . It is clear that the value of $Nu_{h,x}$ becomes large in the lower part of the hot wall (corresponding to large X) due to a collision of the cold fluid circulated from the cold wall with the hot wall, and $Nu_{h,x}$ decreases toward the upper part of the hot wall due to the boundary layer flow ascending in the vicinity of the vertical hot wall. It is of interest that the decreasing rate of $Nu_{h,x}$ becomes small with a decreasing H/W . This behavior of $Nu_{h,x}$ in respect to H/W would be corresponding to the decrease of the over-all influence of natural convection (average Nusselt number Nu) with a decreasing H/W as shown in Fig.5.

Fig.7 shows the non-dimensional temperature distribution in the Y-direction for $X = 0.5$ and $Ra = 10^5$ in order to examine the behavior of heat transfer in the fluid layer. It might be understood that the non-dimensional temperature distribution comes close to a linear relationship between the non-dimensional temperature T and the non-dimensional distance Y . This linearity of the fluid temperature distribution means the conductive heat transfer predominating mainly in the fluid layer. In contrast, as H/W increases, the temperature gradient of the fluid layer in the neighbourhood of the vertical walls ($Y = 0$ and W/H) becomes steeper and the convective heat transfer predominates mainly in the fluid layer.

The effect of Pr on Nu is presented in Fig.8 for $H/W = 0.2$ and $Ra = 10^5$. It might be said that the value of Nu increases linearly with an increasing Pr on the logarithmic graph. Nu is proportional to the 0.024 power

Fig.8 Relationship between Nu and Pr for $H/W = 0.2$ and $Ra = 10^5$

Fig.9 Relationship between Nu and H/W for $Pr = 10^3$ and $Ra = 10^4, 10^5$

of Pr .

Fig.9 shows the relationship between Nu and H/W together with those for $H/W > 1$ proposed in the authors' works^{(4)~(5)}. In the laminar region, if H/W approaches zero, the natural convection in the fluid layer becomes quiescent and the heat is transferred only by thermal conduction across the fluid layer. On the other hand, if H/W approaches ∞ , the mechanism of the heat transfer is essentially similar to the natural convective heat transfer between infinite parallel plates and the mode of heat transfer becomes similar to the conductive heat transfer. Therefore, it might be considered that the maximum value of Nu exists between $0 < H/W < \infty$. From these results, it could be seen that the maximum value of Nu in the present study exists between $1 < H/W < 1.5$ and also the tendency of Nu related to H/W is different according to H/W above or below 0.2 .

The following functional relationship given by a product of powers is derived from the results of the present analysis of the heat transfer through a fluid layer:

$$Nu = cPr^{\ell}(H/W)^mRa^n \quad (12)$$

All of the constant (c) and exponents (ℓ , m and n) are determined by the numerical computations, in which the deviation is within $\pm 7\%$ in the present study. The correlations derived from the analyses are as follows:

In the laminar region;

$$Nu = 0.223 Pr^{0.024} (H/W)^{0.19} Ra^{1/4}, \quad 0.2 < H/W < 1, 1 < Pr < 10^3, 10^3 < Ra \quad (13)$$

$$Nu = 0.851 Pr^{0.024} (H/W)^{1.02} Ra^{1/4}, \quad 0.03 < H/W < 0.2, 1 < Pr < 10^3, 10^5 < Ra \quad (14)$$

In the asymptotic region;

$$Nu = 0.280 Pr^{0.024} (H/W)^{1.75} Ra^{1/2}, \quad 0.03 < H/W < 0.2, 1 < Pr < 10^3, 10^2 < Ra < 10^6 \quad (15)$$

5. Conclusion

Numerical analyses have been carried out on a rectangular cavity having a relatively low aspect-ratio ($0.03 \leq H/W \leq 1$), whose two opposing vertical walls are kept at (but different) temperatures, while the upper

and bottom walls are thermally insulated. It can be seen that the Up-Wind finite difference technique is useful to calculate this kind of the natural convective heat transfer in a wide range of Ra , H/W or Pr . It is also clear that the aspect-ratio H/W plays an important role in evaluation of heat transfer in the cavity having a low aspect-ratio H/W . Moreover, the useful and convenient correlations of heat transfer in the asymptotic flow and the laminar flow region are obtained from the present analyses.

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