532.595.7:624.042.7

Paper No. 204-16

A Seismic Response Analysis of a Cylindrical
Liquid Storage Tank Including the Effect of Sloshing*
(2nd Report, Analysis Based on Energy Method)

By Katsuhisa FUJITA**

This paper presents a seismic response analysis method for a cylindrical liquid storage tank subjected to a horizontal earthquake, which is based on an energy method with the coupling effect between sloshing and bulging taken into consideration. The kinetic energy and the strain energy of an empty tank shell are estimated by the axisymmetric shell finite element method. On the other hand, the kinetic energy and the potential energy of the liquid in the tank are estimated analytically by superposition of two type of velocity potentials, that is, one obtained by assuming the tank wall as rigid and the other obtained by considering the liquid to be coupled with the tank shell neglecting the oscillation of liquid free surface.

The numerical results obtained by this method are compared with those obtained by the method described in the former report, which treats Bernoulli's equation as the boundary condition of liquid free surface, and it is shown that there are little differences between them.

1. Introduction

Increasing attention has been paid to the aseismic design of important industrial facilities. The study on the seismic response analysis of the liquid storage tank has been making rapid progress. The papers published in the past introduce simple dynamic design methods based on Housner's theory⁽¹⁾⁽²⁾, studies based on the velocity potential theory⁽³⁾⁽⁴⁾, dynamic analysis methods considering the coupled vibration between tank shell and liquid by using the analytical solution⁽⁵⁾⁽⁷⁾, and ones by the finite element method⁽⁸⁾⁽⁹⁾. The author has also already presented some papers⁽¹⁰⁾⁽¹²⁾ which make the aseismic design method more reasonable.

In the present paper, the kinetic energy and the strain energy of an empty tank are calculated for each degree of natural mode as in the former paper⁽¹²⁾. And, the liquid in a tank is assumed to follow the velocity potential theory, in which the velocity potential is expressed analytically by superposing that obtained by considering the coupled vibration between the tank shell and the liquid with the effect of sloshing ignored, on that expressing a free surface oscillation obtained by assuming the tank wall to be rigid as in the former paper. The tank shell and the liquid are both dealt with by an energy method instead of treating Bernoulli's equation as the boundary condition of a liquid free surface as in the former paper. These kinetic energy and the potential energy are substituted into Lagrange's equation of motion, and a seismic response analysis method including the effect of sloshing (oscillation of liquid free sur-

face) and bulging (oscillation of tank shell) is derived. Furthermore, numerical calculations are done and the results are compared with those in the former paper to investigate their differences and the effects of sloshing and bulging on seismic response.

2. Method

2.1 Basic equations

2.1.1 Tank shell

It is supposed that a horizontal earthquake acts on the rigid bottom of a cylindrical liquid storage tank which has tank shell of thickness varied stepwise as shown in Fig. 1. In this figure, (r, θ, x) denotes a cylindrical coordinate system, R the tank radius, l the height of tank, h the wall thickness of tank, H the liquid depth and $\ddot{z}$ a horizontal seismic acceleration. The displacement components $u(\theta, x, t)$, $v(\theta, x, t)$ and $w(\theta, x, t)$ of a tank shell are expressed by the following series in terms of normal modes u_{ks} , v_{ks} and w_{ks} of an empty tank shell as in the former paper.⁽¹²⁾

$$\left. \begin{aligned} u(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^n u_{ks}(x) q_{ks}(t) \cos(k\theta) \\ v(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^n v_{ks}(x) q_{ks}(t) \sin(k\theta) \\ w(\theta, x, t) &= \sum_{k=1}^{\bar{n}} \sum_{s=1}^n w_{ks}(x) q_{ks}(t) \cos(k\theta) \end{aligned} \right\} \dots\dots (1)$$

where, q_{ks} is a normal coordinate of the k -th circumferential mode and s -th axial mode, n , $\bar{n}$ are the numbers of axial modes and circumferential ones respectively, and t denotes time. The suffixes k and s or the like follow the usage specified above, hereafter.

The strain energy U_E and kinetic energy K_E stored in a cylindrical tank are also expressed by the following equation as in the former paper:

$$U_E = \frac{1}{2} \sum_{k=1}^{\bar{n}} \sum_{s=1}^n K_{ks} q_{ks}^2 \dots\dots\dots (2)$$

* Received 26 May, 1980

** Senior Engineer, Takasago Technical Institute, Technical Headquarters, Mitsubishi Heavy Industries, LTD., Takasago, Hyogo Pref., 676, Japan

$$K_E = \frac{1}{2} \sum_{k=1}^{\bar{n}} \sum_{s=1}^n M_{ks} \dot{q}_{ks}^2 \quad \dots\dots\dots (3)$$

where, K_{ks} , M_{ks} are obtained from Eqs. (5), (7) of Reference (11) respectively, and $\dot{q}_{ks}$ is the derivative of q_{ks} with respect to time t .

Fig.1 Cylindrical liquid storage tank

Besides, the exciting energy F_E due to a horizontal earthquake $\ddot{z}$ which acts on the tank bottom base is shown as follows.

$$F_E = \left\{ \pi R \rho \int_0^h \sum_{s=1}^n (w_{1s} - v_{1s}) q_{1s} dx \right\} \ddot{z} \quad \left. \begin{array}{l} \text{(for } k=1) \\ \text{(for } k \neq 1) \end{array} \right\} \dots\dots (4)$$

$$= 0$$

where, ρ is the density of a shell.

2.1.2 Liquid in the tank

As in the former paper⁽¹²⁾, the velocity potential Φ including the effect of free surface oscillation will be obtained below. The boundary conditions are given as follows:

$$(i) \quad (\partial \Phi / r \partial \theta)_{\theta=0, \pi} = 0 \quad \dots\dots\dots (5)$$

$$(ii) \quad (\partial \Phi / \partial x)_{x=0} = 0 \quad \dots\dots\dots (6)$$

$$(iii) \quad (\partial \Phi / \partial r)_{r=R} = -\dot{w} - \dot{z} \cos \theta$$

$$= - \sum_{k=1}^{\bar{n}} \sum_{s=1}^n w_{ks} \dot{q}_{ks} \cos(k\theta) - \dot{z} \cos \theta \quad \dots\dots (7)$$

The equation of continuity requires that Φ must satisfy the following Laplace's equation:

$$r^2 \Phi = \partial^2 \Phi / \partial r^2 + \partial \Phi / r \partial r$$

$$+ \partial^2 \Phi / r^2 \partial \theta^2 + \partial^2 \Phi / \partial x^2 = 0 \quad \dots\dots\dots (8)$$

The solution of Eq. (8) is expressed as follows:

$$\Phi = \sum_{k=1}^{\bar{n}} \left(\sum_{p=1}^m {}_1\Phi_{kp} \dot{Q}_{kp} + \sum_{s=1}^n {}_2\Phi_{ks} \dot{q}_{ks} \right) - r \dot{z} \cos \theta \quad \dots\dots (9)$$

where, Q_{kp} denotes a normal coordinate of sloshing in the k -th circumferential mode and the p -th radial mode. For Eq. (9) to satisfy the boundary conditions of Eqs. (5) through (7), the velocity potential ${}_1\Phi_{kp}$ which represents the free surface oscillation of rigid wall tank and the velocity potential ${}_2\Phi_{ks}$ which represents a coupled

vibration between the tank shell and the liquid having a free surface unmoved can be expressed by

$$\left. \begin{array}{l} {}_1\Phi_{kp} = {}_1\phi_p^{(k)} \cos(k\theta) = J_k(\kappa_p^{(k)} r) \cosh(\kappa_p^{(k)} x) \cos(k\theta) \\ {}_2\Phi_{ks} = {}_2\phi_s^{(k)} \cos(k\theta) = - \sum_{i=0}^{\infty} \frac{2I_k(\lambda_i r/H)}{\lambda_i I_k'(\lambda_i R/H)} \cos\left(\frac{\lambda_i x}{H}\right) \\ \times \int_0^H w_{ks} \cos\left(\frac{\lambda_i x}{H}\right) dx \cos(k\theta) \end{array} \right\} \dots\dots\dots (10)$$

where J_k denotes a Bessel function of the first kind of order k , I_k a modified Bessel function of the first kind of order k . And,

$$\left\{ \begin{array}{l} \left[\frac{d}{dr} \{ J_k(\kappa_p^{(k)} r) \} \right]_{r=R} = 0, \quad \lambda_i = \left(i + \frac{1}{2} \right) \pi \\ I_k'(\lambda_i R/H) = I_{k-1}(\lambda_i R/H) - \frac{kH}{\lambda_i R} I_k(\lambda_i R/H) \end{array} \right\} \dots\dots (11)$$

In the former paper, the liquid pressure was obtained analytically and used to obtain the virtual work done on the tank shell. Then, a coupled vibration analysis method considering coupling effects between sloshing and bulging is derived by using Bernoulli's equation as the boundary condition of a liquid free surface. On the other hand, in the present paper, a coupled vibration analysis method is proposed in which the tank shell and the liquid are both treated by an energy method.

The potential energy U_L of the liquid is expressed as follows:

$$U_L = \frac{1}{2} \int_0^R \int_0^{2\pi} \rho_L g \eta^2 r d\theta dr \quad \dots\dots\dots (12)$$

where, ρ_L denotes the density of the liquid, η the wave height of liquid free surface, and the following relation holds.

$$\frac{\partial \eta}{\partial t} = - \left(\frac{\partial \Phi}{\partial x} \right)_{x=H} \quad \dots\dots\dots (13)$$

Integrating Eq. (13) with respect to time t for once and substituting it into Eq. (12) with help of Eq. (9), the following equation is obtained:

$$U_L = \frac{\rho_L g}{2} \int_0^R \int_0^{2\pi} \left[- \sum_{k=1}^{\bar{n}} \left\{ \sum_{p=1}^m \left(\frac{\partial {}_1\Phi_{kp}}{\partial x} \right)_{x=H} Q_{kp} \right. \right. \\ \left. \left. + \sum_{s=1}^n \left(\frac{\partial {}_2\Phi_{ks}}{\partial x} \right)_{x=H} q_{ks} \right\} \right]^2 r d\theta dr \quad \dots\dots\dots (14)$$

where, the constant of the integration can be eliminated by taking $\eta=0$ when $Q_{kp}=q_{ks}=0$.

The following relation holds between the velocity potential Φ and the liquid velocity components u_ℓ , v_ℓ , w_ℓ :

$$(\dot{u}_\ell, \dot{v}_\ell, \dot{w}_\ell) = - \text{grad } \Phi \quad \dots\dots\dots (15)$$

The kinetic energy K_L of the liquid in a tank is expressed as follows:

$$K_L = \frac{\rho_L}{2} \int \int \int (\dot{u}_\ell^2 + \dot{v}_\ell^2 + \dot{w}_\ell^2) r dr d\theta dx \quad \dots\dots (16)$$

When Eq. (15) is substituted into Eq. (16) and Green's theorem is applied, the following equation is obtained:

$$K_L = \frac{\rho_L}{2} \int_0^H \int_0^{2\pi} \left(\Phi \frac{\partial \Phi}{\partial r} \right)_{r=R} R d\theta dx$$

$$+\frac{\rho_L}{2}\int_0^R\int_0^{2\pi}\left(\Phi\frac{\partial\Phi}{\partial x}\right)_{x=H}rd\theta dr \dots\dots\dots (17)$$

Substituting Eq. (9) into Eq. (17) and using the boundary condition of Eqs. (6), (7) yields,

$$\begin{aligned} K_L = & \frac{\rho_L}{2}\int_0^R\int_0^{2\pi}\left[\sum_{k=1}^n\left\{\sum_{p=1}^m(1\Phi_{kp})_{r=R}\dot{Q}_{kp}\right.\right. \\ & + \sum_{s=1}^n\left\{2\Phi_{ks}\right\}_{r=R}\dot{q}_{ks}-R\dot{z}\cos\theta\left.\right\} \\ & \times\left\{-\sum_{k'=1}^n\sum_{s'=1}^m w_{k's'}\dot{q}_{k's'}\cos(k'\theta)-\dot{z}\cos\theta\right\}Rd\theta dx \\ & + \frac{\rho_L}{2}\int_0^R\int_0^{2\pi}\left[\sum_{k=1}^n\left\{\sum_{p=1}^m(1\Phi_{kp})_{x=H}\dot{Q}_{kp}-r\dot{z}\cos\theta\right\}\right] \\ & \times\left[\sum_{k'=1}^n\left\{\sum_{p'=1}^m\left(\frac{\partial_1\Phi_{k'p'}}{\partial x}\right)_{x=H}\dot{Q}_{k'p'}\right.\right. \\ & + \sum_{s=1}^n\left\{\left(\frac{\partial_2\Phi_{ks}}{\partial x}\right)_{x=H}\dot{q}_{ks}\right\}\left.\right]rd\theta dr \\ & \dots\dots\dots (18) \end{aligned}$$

2.2 Eigenvalue analysis

Suppose the tank is in a free vibration. In this case, the term $\dot{z}$ in Eq. (7) must be eliminated because the bottom of tank keeps still. Therefore, terms involving $\dot{z}$ in Eqs. (9), (18) can be discarded. Substituting the strain energy U_E , the potential energy U_L and the sum of the kinetic energy (K_E+K_L) into Lagrange's equation of motion, the following equation is obtained:

$$\frac{d}{dt}\{\partial(K_E+K_L)/\partial\dot{X}\}+\partial(U_E+U_L)/\partial X=0 \dots\dots (19)$$

where, X denotes Q_{kp} or q_{ks} .

Substituting Eqs. (2), (3), (14) and Eq. (18) into Eq. (19) with terms involving $\dot{z}$ discarded, the following equation is obtained:

$$([M_1]+[M_2])\left\{\ddot{Q}_k\right\}+([K_1]+[K_2])\left\{Q_k\right\}=0 \dots\dots (20)$$

where,

$$\left. \begin{aligned} [M_1] &= \begin{bmatrix} 0 & 0 \\ M_{kp}^L & 0 \\ 0 & 0 \\ 0 & M_{ks} \end{bmatrix} \\ [M_2] &= \begin{bmatrix} 0 & a_{ps}^{(k)}+b_{ps}^{(k)} \\ a_{sp}^{(k)}+b_{sp}^{(k)} & e_{ss}^{(k)}+f_{ss}^{(k)} \end{bmatrix} \\ [K_1] &= \begin{bmatrix} 0 & 0 \\ K_{kp}^L & 0 \\ 0 & 0 \\ 0 & K_{ks} \end{bmatrix} \\ \times [K_2] &= \begin{bmatrix} 0 & c_{ps}^{(k)} \\ c_{sp}^{(k)} & g_{ss}^{(k)} \end{bmatrix} \end{aligned} \right\} \dots\dots (21)$$

$$\begin{aligned} M_{kp}^L &= \pi\rho_L\kappa_p^{(k)}\sinh(\kappa_p^{(k)}H)\int_0^R(1\phi_p^{(k)})_{x=H}J_k(\kappa_p^{(k)}r)rd r \\ a_{ps}^{(k)} &= \frac{\pi R\rho_L}{2}\int_0^H(1\phi_p^{(k)})_{r=R}(-w_{ks})dx \end{aligned}$$

$$\begin{aligned} b_{ps}^{(k)} &= \frac{\pi\rho_L}{2}\int_0^R(1\phi_p^{(k)})_{x=H} \\ & \times\left\{\left(\frac{\partial_2\phi_s^{(k)}}{\partial x}\right)_{x=H}\right\}rd r, \quad K_{ks}=\omega_{ks}^2M_{ks} \\ K_{kp}^L &= \pi g\rho_L\kappa_p^{(k)}\sinh(\kappa_p^{(k)}H) \\ & \times\int_0^R\left\{\left(\frac{\partial_1\phi_p^{(k)}}{\partial x}\right)_{x=H}\right\}J_k(\kappa_p^{(k)}r)rd r = \omega_{kp}^2M_{kp}^L \\ c_{ps}^{(k)} &= \pi g\rho_L\kappa_p^{(k)}\sinh(\kappa_p^{(k)}H) \\ & \times\int_0^R\left\{\left(\frac{\partial_2\phi_s^{(k)}}{\partial x}\right)_{x=H}\right\}J_k(\kappa_p^{(k)}r)rd r \\ e_{ss}^{(k)} &= \frac{\pi R\rho_L}{2}\int_0^H(z\phi_s^{(k)})_{r=R}(-w_{ks})dx \\ f_{ss}^{(k)} &= \frac{\pi R\rho_L}{2}\int_0^H(z\phi_s^{(k)})_{r=R}(-w_{ks'})dx = e_{ss}^{(k)} \\ g_{ss}^{(k)} &= \pi g\rho_L\int_0^R\left\{\left(\frac{\partial_2\phi_s^{(k)}}{\partial x}\right)_{x=H}\left(\frac{\partial_2\phi_s^{(k)}}{\partial x}\right)_{x=H}\right\}rd r \end{aligned} \dots\dots (22)$$

and where ω_{ks} denotes the natural circular frequency of an empty tank shell, and ω_{kp} the natural circular frequency of the liquid free surface of rigid wall tank. Assuming $[M_2]=[K_2]=0$ in Eq. (20) yields two separate dynamic equations, one for the free surface oscillation of liquid in a rigid wall tank, and the other for an empty tank. By substituting $\left\{\begin{smallmatrix} Q_k \\ q_k \end{smallmatrix}\right\} = \{\phi_k\}_c e^{i\omega_k t}$ into Eq. (20), an eigenfrequency equation is obtained as follows:

$$(-\omega_k^2([M_1]+[M_2])+[K_1]+[K_2])\{\phi_k\}_c=0 \dots\dots (23)$$

Solving the above equation, $(m+n)$ pairs of eigenvalues ω_{kc} and eigenvectors $\{\phi_k\}_c$ are obtained. The c -th normal mode of displacement components of the coupled system, $(u_k)_c$, $(v_k)_c$, $(w_k)_c$, and the c -th mode of a liquid pressure and a wave height, $(p_k)_c$ and $(\eta_k)_c$, are obtained as follows:

$$\left. \begin{aligned} (u_k)_c &= \sum_{s=1}^n u_{ks}(\phi_{k,m+s})_c, \\ (v_k)_c &= \sum_{s=1}^n v_{ks}(\phi_{k,m+s})_c \\ (w_k)_c &= \sum_{s=1}^n w_{ks}(\phi_{k,m+s})_c \\ (p_k)_c &= -\rho_L\omega_k^2\left\{\sum_{p=1}^m 1\phi_p^{(k)}(\phi_{kp})_c \right. \\ & \quad \left. + \sum_{s=1}^n 2\phi_s^{(k)}(\phi_{k,m+s})_c\right\} \\ (\eta_k)_c &= -\left\{\sum_{p=1}^m\left(\frac{\partial_1\phi_p^{(k)}}{\partial x}\right)_{x=H}(\phi_{kp})_c \right. \\ & \quad \left. + \sum_{s=1}^n\left(\frac{\partial_2\phi_s^{(k)}}{\partial x}\right)_{x=H}(\phi_{k,m+s})_c\right\} \end{aligned} \right\} (c=1\sim m+n) \dots\dots\dots (24)$$

2.3 Seismic response analysis

Substituting the strain energy U_E of Eq. (2), the potential energy U_L of Eq. (14), the sum of the kinetic energy (K_E+K_L) of Eq. (3) and (18), and the exciting energy F_E into Lagrange's equation of motion, the following equation is obtained:

$$([M_1]+[M_2])\left\{\ddot{Q}_k\right\}+([K_1]+[K_2])\left\{Q_k\right\}=-\{F\}\ddot{z} \dots\dots\dots (25)$$

where,

$$\left. \begin{aligned} [F] &= [d_p, h_i]^T \\ d_p &= -\frac{\pi R \rho_L}{\kappa_p^{(1)}} J_1(\kappa_p^{(1)} R) \sinh(\kappa_p^{(1)} H) \quad (k=1) \\ &= 0 \quad (k \neq 1) \\ h_i &= \pi R \rho \int_0^H h(w_{1i} - v_{1i}) dx + \frac{\pi R^2 \rho_L}{2} \int_0^H w_{1i} dx \\ &\quad - \frac{\pi R \rho_L}{2} \int_0^H (z \phi_i^{(1)})_{r=R} dx \\ &\quad - \frac{\pi \rho_L}{2} \int_0^R \left\{ \left(\frac{\partial z \phi_i^{(1)}}{\partial x} \right)_{x=H} \right\} r^2 dr \quad (k=1) \\ &= 0 \quad (k \neq 1) \end{aligned} \right\} \dots (26)$$

In order to apply a modal analysis method to Eq. (25), the solution of Eq. (25) is assumed to be expressed by:

$$\left\{ \begin{matrix} Q \\ q \end{matrix} \right\} = \sum_{c=1}^{m+n} \{\phi\}_c T_c(t) = [\phi] \{T(t)\} \dots (27)$$

The subscript k is omitted hereafter because the force term in the right hand side of Eq. (25) vanishes when $k \neq 1$ as is known from Eq. (26).

Multiplying $[\phi]^T$ from the left hand side on both sides of Eq. (25) into which Eq. (27) is substituted, $(m+n)$ pairs of dynamic equations which are independent of each other are obtained on account of the orthogonality of the normal functions for symmetric matrices. Adding a damping force term to each equation ignoring the coupling effect due to damping force, the following equation is obtained:

$$\ddot{T}_c(t) + 2h_c \omega_c \dot{T}_c(t) + \omega_c^2 T_c(t) = -\beta_c \ddot{z} \dots (28)$$

where,

$$\left. \begin{aligned} \omega_c^2 &= K_c / M_c, \quad \beta_c = \{\phi\}_c^T [F] / M_c \\ M_c &= \{\phi\}_c^T ([M_1] + [M_2]) \{\phi\}_c \\ K_c &= \{\phi\}_c^T ([K_1] + [K_2]) \{\phi\}_c \end{aligned} \right\} \dots (29)$$

and where $T_c(t)$ denotes the normal coordinate of the c -th coupled vibration mode, h_c the damping coefficient ratio of the c -th mode, and β_c the participation factor of the c -th mode.

When an acceleration response S_{A_c} for the c -th coupled vibration mode can be read from a response spectrum, the response displacement and the response acceleration in the radial direction, the response pressure and the response wave height are given in the following expressions by taking the square root sum of squares method.

$$\left. \begin{aligned} w(\theta, x) &= \left[\sum_{c=1}^{m+n} \left\{ \beta_c \frac{S_{A_c}}{\omega_c^2} \sum_{s=1}^n w_s(\phi_{m+s})_c \cos \theta \right\}^2 \right]^{1/2} \\ \ddot{w}(\theta, x) + \ddot{z} \cos \theta &= \left[\sum_{c=1}^{m+n} \left\{ \beta_c S_{A_c} \sum_{s=1}^n w_s(\phi_{m+s})_c \cos \theta \right\}^2 \right]^{1/2} \\ p(r, \theta, x) &= \rho_L \left[\sum_{c=1}^{m+n} \left\{ \beta_c S_{A_c} \left\{ \sum_{p=1}^m \phi_p(\phi_p)_c \right. \right. \right. \\ &\quad \left. \left. + \sum_{s=1}^n z \phi_s(\phi_{m+s})_c \right\} \cos \theta \right\}^2 \right]^{1/2} \\ \eta(r, \theta) &= \left[\sum_{c=1}^{m+n} \left\{ \beta_c \frac{S_{A_c}}{\omega_c^2} \left\{ \sum_{p=1}^m \left(\frac{\partial \phi_p}{\partial x} \right)_{x=H} (\phi_p)_c \right. \right. \right. \\ &\quad \left. \left. + \sum_{s=1}^n \left(\frac{\partial z \phi_s}{\partial x} \right)_{x=H} (\phi_{m+s})_c \right\} \cos \theta \right\}^2 \right]^{1/2} \end{aligned} \right\} \dots (30)$$

The other values are similar. Otherwise, alternate expressions obtained by a modal analysis method for time-history response can be replaced instead of these responses.

3. Numerical examples

In the former paper, the virtual work exerted by the liquid on the tank shell was treated analytically, and the equations of motion of a tank shell were obtained by an energy method. Coupling this equations of motion with Bernoulli's equation considered as the boundary condition of a liquid free surface, a dynamic response analysis method considering both the sloshing and bulging was shown. In the present paper, on the other hand, the liquid is also treated by an energy method instead of treating as the boundary condition of a liquid free surface. Regarding the difference between two approaches, it is found that if, for example, the partial integration is carried out with regard to time t after taking the first variation of Q_{kp} and q_k in Eq. (18), the equation corresponding to the virtual work done on the tank shell in the former paper is included as a part of the equation of the present paper. As is known from a comparison between Eq. (21) and Eq. (34) shown below, the former paper does not contain all the terms related to the coupling terms unlike the present paper but omits some coupling terms. However, numerical examples to be shown later indicate only slight differences; then it is understood that only terms of secondary significance are ignored.

As there are differences in dealing with the liquid in a tank as mentioned above, the theory of the liquid of small wave height will be discussed in close examination. According to the Reference (13), the equation of motion for a liquid free surface is given by

$$\left(\frac{p}{\rho_L} \right)_{x=H+\eta} = \left(\frac{\partial \Phi}{\partial t} \right)_{x=H+\eta} - \frac{1}{2} (\dot{u}_i^2 + \dot{v}_i^2 + \dot{w}_i^2)_{x=H+\eta} - g\eta \dots (31)$$

where η denotes the wave height of liquid free surface at time t measured from the point (r, θ, H) on the stationary liquid surface with respect to the coordinate system as shown in Fig. 2. Considering the pressure of a liquid free surface to be equal to the atmospheric pressure and ignoring the second order terms, the following equation is obtained.

$$\eta = \frac{1}{g} \left(\frac{\partial \Phi}{\partial t} \right)_{x=H} \dots (32)$$

Similarly, the liquid velocity in the direction parallel to the x -axis can be considered to be equal to the velocity in the normal direction of the liquid free surface itself, so Eq. (13) which was mentioned earlier holds as follows:

$$\frac{\partial \eta}{\partial t} = - \left(\frac{\partial \Phi}{\partial x} \right)_{x=H} \dots (13)$$

The boundary condition for a liquid free surface which was discussed in the former paper is obtained from Eqs. (13) and (32) as follows:

Fig.2 Wave height of liquid free surface

$$\left(\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial x} \right)_{z=H} = 0 \quad \dots\dots\dots (33)$$

While Eq. (13) is used in the present paper under the assumption of small wave height, Eq. (33), which represents the boundary condition for a liquid free surface, is not used in the present paper. Because of the above-mentioned difference, the coefficient matrices in the former paper corresponding to Eq. (21) are shown as follows:

$$\left. \begin{aligned} [M_1] &= \begin{bmatrix} \tilde{M}_{kp} & 0 & 0 \\ 0 & \tilde{M}_{kp} & 0 \\ 0 & 0 & M_{ks} \end{bmatrix} \\ [M_2] &= \begin{bmatrix} 0 & 0 \\ 2a_{ip}^{(k)} & e_{is}^{(k)} + f_{is}^{(k)} \end{bmatrix} \\ [K_1] &= \begin{bmatrix} \tilde{K}_{kp} & 0 & 0 \\ 0 & \tilde{K}_{kp} & 0 \\ 0 & 0 & K_{ks} \end{bmatrix} \\ \times [K_2] &= \begin{bmatrix} 0 & \tilde{c}_{pi}^{(k)} \\ 0 & 0 \end{bmatrix} \end{aligned} \right\} \dots\dots (34)$$

$$\begin{aligned} \tilde{M}_{kp} &= M_{kp} / \pi \rho_L \kappa_p^{(k)} \sinh(\kappa_p^{(k)} H), \\ \tilde{K}_{kp} &= K_{kp} / \pi \rho_L \kappa_p^{(k)} \sinh(\kappa_p^{(k)} H) \\ \tilde{c}_{pi}^{(k)} &= c_{pi}^{(k)} / \pi \rho_L \kappa_p^{(k)} \sinh(\kappa_p^{(k)} H) \end{aligned}$$

As is evident from a comparison between Eq. (34) and Eq. (21), some of elements of $[M_2]$ and $[K_2]$ of Eq. (34) are dropped or replaced and reduced to asymmetric matrices.

3.1 Relation between the tank sizes and its vibration characteristics

Table 1 shows a comparison between the present solution and the former one with respect to natural frequencies of lower three sloshing modes and lower five bulging modes by taking $H/R=2, 1, 0.5$ as parameters. We restrict hereafter ourselves to the case $k=1$ unless otherwise specified. The remaining items used in this numerical example are $R=1000$ cm, $h=1$ cm, $E=206$ Gpa ($=2.1 \times 10^6$ kgf/cm²), $\nu=0.3$, $\rho=7.875 \times 10^3$ kg/m³ ($=8.03 \times 10^{-6}$ kgf.s²/cm⁴), $\rho_L=1.00 \times 10^3$ kg/m³ ($=1.020 \times 10^{-6}$ kg.s²/cm⁴) and $H=l$ (full tank). The result shows that the present solution and the former one⁽¹²⁾ coincide completely up to five significant digits with respect to sloshing mode, whereas the difference between the present solution and the former one with respect to bulging mode is observed to commence with the 3rd or 4th digit but is very small for practical purposes.

Figure 3(a) shows a comparison between the present solution and the former one with regard to the 1st through 3rd sloshing modes of the radial displacement of tank wall, the liquid pressure exerted on the tank side shell and the bottom plate of tank and the wave height of liquid free surface. They are in good agreement with each other, and only a small difference in the displacement mode shape near the free end of the tank side shell is observed. In this figure, the displacement of tank shell and the wave height are drawn in cm, and the liquid pressure in kgf/cm². As the wave height η and the liquid pressure p are connected by the relation $p = \rho_L g \eta \approx 10^{-3} \eta$ according to Eq. (31), (32) in the unit specified above, the liquid pressure and the displacement are expressed on an enlarged scale of 10^3 times that of wave height.

Figure 3(b) shows a comparison with respect to the 1st through 3rd bulging modes of both solutions. They are in close agreement with each other. Only a small difference of the wave height is observed. The mode shape of wave height of the present solution is more smooth than those in the former solution except near tank shell. The

Table 1 Relation between the tank sizes and the natural frequencies, 1st circumferential mode (rigid-free)

H/R	No. of mode	Sloshing mode (Hz)			Bulging mode (Hz)				
		1st	2nd	3rd	1st	2nd	3rd	4th	5th
2	Former sol.	0.213 42	0.363 76	0.460 29	3.692 1	11.439	14.339	17.928	19.630
	Present sol.	0.213 42	0.363 76	0.460 29	3.689 4	11.438	14.341	17.931	19.631
1	Former sol.	0.208 38	0.363 78	0.460 30	7.291 2	13.885	19.420	23.493	27.118
	Present sol.	0.208 38	0.363 78	0.460 30	7.290 0	13.885	19.421	23.494	27.122
0.5	Former sol.	0.182 12	0.362 04	0.460 22	11.837	21.307	28.269	33.810	39.298
	Present sol.	0.182 12	0.362 04	0.460 22	11.836	21.308	28.273	33.828	39.380

slight difference in natural frequencies is considered due to that in the mode shapes of the wave height. And, it should be noted that the comparison between both solutions is made under the common numerical scheme because they are not perfect analytical solutions.

In order to investigate the effect of tank sizes on mode shapes, a comparison between the present solution and the former one is made in Fig.4 with respect to the 1st mode of sloshing and bulging mode for the case of $H/R=2$ and $H/R=0.5$. As for the difference between both solutions the tendency is similar to that for $H/R=1$ in Fig.3. The remaining items of numerical data except for value of H/R are the same as those in Table 1.

3.2 Seismic response

For the sake of simplicity, numerical calculations are carried out with the same acceleration response spectrum value for each degree of freedom assumed as $S_{Ac}=100$

cm/s^2 (gal). The numerical data are the same as those used in Table 1 but restricted to $H/R=1$.

Table 2(a), (b) show the seismic response values calculated by the square root sum of square method with the same acceleration response spectrum $S_{Ac}=100 \text{ cm/s}^2$ (gal) for each degree of freedom of sloshing and bulging mode. Table 2(a) shows little difference between the present solution and the former one with respect to the response liquid pressure of tank side shell, but the response radial acceleration and the response radial displacement near the free end of tank obtained by the present solution are slightly smaller than those by the former solution. The seismic response analysis is carried out within $c=14$ degrees of freedom, i.e. 5 degrees for sloshing mode and 9 for bulging. When the number of the degrees of freedom increases, the response values near the free end of tank decrease and tend to converge both in the present solution and the former one. So, it is considered that the response values by the present solution are more accurate. As for the liquid pressure on bottom plate and the wave height of liquid free surface, Table 2(a) shows little difference between the present solution and the former one, so the difference in the wave height of bulging mode gives no meaningful effect to the seismic response.

Moreover, in order to investigate the effects of sloshing and bulging on the seismic response, Table 3(a), (b) shows the response with an acceleration response spectrum $S_{Ac}=100 \text{ cm/s}^2$ (gal) for the sloshing mode but $S_{Ac}=0$ for the bulging mode. From this table, it is known that the present solution and the former one give nearly the same response. However, comparing the response in Table 3 with one in Table 2, there is considerable difference with respect to the radial displacement of side shell near the bottom plate but is less difference at the free end. The same tendency is observed in the liquid pressure of side shell, so it can be said that the effects of sloshing are great near the free end of tank. Also, the wave height of liquid free surface is almost the same and it is understood that the sloshing mode has predominant effects on the wave height. This reaffirms that the difference in the wave height of bulging mode between the present solution and the former one has little meaningful effect on the responses. On the other hand, the liquid pressure on bottom of tank in Table 3 is fairly small as compared with that in Table 2. The response acceleration in the radial direction of side shell in Table 3 is very small, which indicates that the effects of bulging mode are great.

In order to investigate the above-mentioned results further in detail, Table 4(a), (b) show results of response analysis with an accele-

Fig.3(a) Comparison between the present solution and the former one with respect to 1st through 3rd sloshing modes, ($H/R=1.0$, 1st circumferential mode)

Fig.3(b) Comparison between the present solution and the former one with respect to 1st through 3rd Bulging modes, ($H/R=1.0$, 1st circumferential mode)

Table 2 Response analysis by the response spectrum method (both sloshing and bulging included)

(a)						
x	Present solution			Former solution		
	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²
	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$
0.1Z	4.339 1	84.512	7.281 8	4.353 2	84.914	7.307 1
0.2Z	5.612 7	106.09	7.590 8	5.617 7	106.45	7.617 0
0.3Z	5.746 9	104.90	7.688 6	5.760 5	105.30	7.716 3
0.4Z	6.658 8	119.38	7.738 5	6.680 3	119.93	7.768 0
0.5Z	6.948 3	120.07	7.638 1	6.955 4	121.02	7.668 9
0.6Z	7.314 7	118.48	7.454 6	7.352 0	118.95	7.485 5
0.7Z	7.649 5	115.98	7.279 0	7.651 4	117.08	7.309 3
0.8Z	7.807 4	108.39	7.232 2	7.873 4	110.08	7.260 1
0.9Z	8.098 2	96.762	7.565 1	8.038 4	100.97	7.584 3
Z	7.926 0	133.68	8.587 4	8.758 5	185.85	8.586 2

(b)				
r	Present solution		Former solution	
	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm
	$\times 10^{-2}$		$\times 10^{-2}$	
0.2R	1.439 7	29.252	1.445 3	29.248
0.4R	2.885 5	52.014	2.896 7	52.016
0.6R	4.346 2	69.700	4.362 8	69.698
0.8R	5.803 1	81.487	5.824 5	81.486
R	6.896 5	85.800	6.920 8	85.897

ration response spectrum value $S_{Ac}=100$ cm/s² (gal) only for the bulging mode. These tables also show the results obtained by Reference (11) in which the oscillation of a liquid free surface is ignored. Table 4(a) shows that the radial displacement of tank side shell near the fixed point is not so different from that of Table 2 and it is understood that the bulging mode has great effects on it. The same tendency is observed in the liquid pressure of side shell. Also, the response acceleration of side shell is almost the same as that in Table 2, and it is found that it depends predominantly on bulging mode. It is also known from Table 4 that the present solution is in close agreement with the former one as well as the numerical solution by Reference (11) in which bulging alone is taken into account with the oscillation of liquid free surface ignored. Furthermore, the wave height of liquid free surface in Table 4(b) is so small as compared with that in Table 2(b) and it is found that the effects of bulging mode on the wave height can be ignored.

4. Conclusions

The following conclusions may be obtained from the foregoing considerations.

- (1) There are little differences between the present solution and the former one with respect to vibration characteristics, i.e. natural frequency and mode. Regarding the natural frequency, only a slight difference is observed in that of the bulging mode. Also as for the natural mode, only a small difference is found in the wave height of bulging mode. All that can be said is that the present solution is somewhat more accurate.
- (2) According to the seismic response analysis with numerical data of the tank dimensions close to the actual ones, the deformation and the liquid pressure near the free end of tank side shell are apt to be influenced greatly by sloshing. Meanwhile, the deformation of side shell near the bottom plate and the liquid pressure on bottom plate are liable to depend on bulging. Also, it can be said that the response wave height of liquid free surface is predominantly influenced by sloshing, and hence the difference between the present solution and the former one with respect to the wave height of bulging mode shows no meaningful effect on the response.

The above-mentioned results may be probably modified to some extent depending on the tank dimensions and on the

Fig.4 Comparison between the present solution and the former one with respect to 1st sloshing mode and 1st bulging mode (1st circumferential mode)

response spectrum as a seismic input, but it seems that these tendencies are maintained qualitatively.

(3) The difference between the present solution and the former one is observed only in the wave height and has little influence on the seismic response. Therefore, both the present solution and the former one can

be applied to the actual design. However, while the coefficient matrices of equations of motion in the present solution which are more complicated than those in the former one, are symmetric ones, and hence more accurate mode of wave height can be obtained from the present solution.

(4) According to the numerical results by the present solution and the former one, it is known that the coupling effect between sloshing and bulging is very weak. So, it can be said that even if the sloshing and the bulging would be dealt with separately for the seismic response analysis, identical results might be obtained. However, as both sloshing and bulging can be dealt with simultaneously in the present solution, it can be considered that the present solution is more reasonable as to the seismic response analysis.

References

- (1) Housner, G.W., Bull. of the Seismolog'1 Soc. Am., vol.47, No.1 (1957-1), 15.
- (2) U.S. AEC, TID Report-7024 (1963)
- (3) Senda, K. and Nakagawa, K., Proc. 4-th J. Nat. Congr. Appl. Mech. (1954)
- (4) Sogabe, K., et al., Jour. Japan Soc. Mech. Engrs. (in Japanese), 79-689 (1976-4), 305.
- (5) Veletsos, A.S. and Yang, J.Y., U.S.-Jpn. Semi. Earthquake Eng. Res. Emph. on Lifeline Sys. Jpn., (1976-4)
- (6) Mouzakis, T., Ph. D. Dissertation, Univ. Massachusetts, Dep. of Civil Eng., (1976)
- (7) Bauer, H.P. and Siekmann, J., Inge-Arch., 40 (1971), 226.
- (8) Shaaban, S.H. and Nash, W.A., Univ. Massachusetts Rep. of Nat. Sci. Found. (1976-7)
- (9) Sakai, F. and Sakoda, H., Proc. 4-th Jpn. Earthquake Eng. Symp., (1975), 623.
- (10) Fujita, K. and Shiraki, K., Proc. 4-th Int. Conf. on Stru. Mech. in Reactor Tech. (1977-8), K5/4.
- (11) Fujita, K., Bulletin of the JSME, Vol. 24, No.192 (1981-6), 1029.
- (12) Fujita, K., Bulletin of the JSME, Vol. 24, No.195 (1981-9), 1634.

Table 3 Response analysis by the response spectrum method (sloshing only)

(a)

x	Present solution			Former solution		
	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$
0.1Z	1.951 6	3.349 4	2.693 6	1.972 7	3.386 3	2.693 7
0.2Z	2.604 8	4.472 0	2.830 2	2.603 8	4.470 2	2.830 2
0.3Z	2.946 5	5.063 1	3.062 8	2.963 1	5.092 0	3.062 9
0.4Z	3.589 8	6.169 5	3.399 4	3.619 2	6.221 0	3.399 5
0.5Z	4.127 8	7.100 6	3.851 5	4.132 0	7.107 7	3.851 6
0.6Z	4.708 5	8.104 7	4.434 6	4.760 1	8.195 4	4.434 7
0.7Z	5.404 2	9.317 2	5.168 9	5.403 4	9.315 9	5.169 0
0.8Z	6.016 3	10.396	6.080 6	6.099 4	10.543	6.080 7
0.9Z	6.854 0	11.913	7.203 8	6.783 8	11.789	7.203 9
Z	7.110 4	12.486	8.586 1	8.023 5	14.127	8.586 2

(b)

r	Present solution		Former solution	
	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm
	$\times 10^{-2}$		$\times 10^{-2}$	
0.2R	0.823 50	29.252	0.823 51	29.248
0.4R	1.564 2	52.014	1.564 3	52.016
0.6R	2.148 8	69.700	2.148 9	69.698
0.8R	2.521 2	81.487	2.521 2	81.486
R	2.648 5	85.799	2.648 5	85.896

Table 4 Response analysis by the response spectrum method (bulging only)

(a)

x	Present solution			Former solution ¹⁾			Reference (11) ²⁾		
	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²	Radial disp. of side shell cm	Radial accel. of side shell gal	Liquid press. on side shell kg/cm ²
	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$	$\times 10^{-2}$		$\times 10^{-2}$
0.1Z	3.875 5	84.512	6.765 3	3.880 6	84.914	6.792 5	3.878 2	84.495	6.787 5
0.2Z	4.971 7	106.09	7.043 4	4.977 8	106.45	7.071 7	4.975 0	106.08	7.066 6
0.3Z	4.934 0	104.90	7.052 2	4.940 0	105.30	7.082 4	4.937 6	104.90	7.077 4
0.4Z	5.608 3	119.38	6.951 8	5.615 0	119.93	6.984 6	5.612 5	119.37	6.979 7
0.5Z	5.589 3	120.07	6.595 9	5.595 1	121.02	6.631 5	5.593 3	119.98	6.627 3
0.6Z	5.597 7	118.48	5.992 1	5.603 0	118.95	6.030 5	5.602 4	118.43	6.028 9
0.7Z	5.413 8	115.98	5.125 0	5.417 3	117.08	5.168 0	5.417 9	115.76	5.168 3
0.8Z	4.975 9	108.39	3.915 6	4.978 7	110.08	3.966 7	4.980 6	107.90	3.969 4
0.9Z	4.313 1	96.762	2.310 0	4.312 3	100.97	2.371 8	4.315 7	94.721	2.375 3
Z	3.502 0	133.68	0.149 04	3.512 0	185.85	0.0107 34	3.514 2	91.702	6.475 5

(b)

r	Present solution		Former solution ¹⁾		Reference (11) ²⁾	
	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm	Liquid press. on bottom plate kg/cm ²	Wave height of liquid free surface cm
	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	$\times 10^{-2}$	
0.2R	1.181 0	1.030 3	1.187 8	1.210 4	1.187 2	-
0.4R	2.424 7	2.108 3	2.438 0	1.999 1	2.436 7	-
0.6R	3.777 8	3.631 1	3.796 9	3.684 3	3.794 7	-
0.8R	5.226 9	5.981 6	5.250 5	6.086 9	5.247 2	-
R	6.367 7	16.027	6.394 0	10.738	6.389 4	-

1): Solution obtained by using Bernoulli's equation as the boundary condition of liquid free surface.

2): Solution in which the oscillation of liquid free surface is ignored.

(13) Lamb, H., Hydrodynamics, 6-th Ed. (1932), 363, Cambridge Univ. Press.