Vibrations of Cylindrical Shells with Varying Thickness*

(Continued)

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In this paper, the free vibrations of a cylindrical shell which is symmetric about the cross-section at the center and whose thickness varies quadratically are analyzed by both the classical theory and an improved one. The equations of vibration, like in the previous paper, are solved exactly by a series solution and three boundary conditions are considered, that is, both ends are clamped, simply-supported and free. The effects of the variations of thickness, etc. upon the natural frequencies and the mode shapes are clarified. And at the same time, the improved shell theories are compared with the classical ones.

1. Introduction

Many researches on the vibrations of cylindrical shells have already been done. On the analyses by classical theory (thin cylindrical shell theory), there are an excellent work by Forsberg⁽¹⁾, the one by Warburton⁽²⁾ and papers by authors^{(3),(4)}. On the other hand, there are works by Herrmann and Mirsky⁽⁵⁾⁻⁽⁸⁾ and the one by Gazis⁽⁹⁾ on the analyses by improved theory (thick cylindrical shell theory) considering the effects of the shear deformation and the rotatory inertia besides the components of displacements on a middle surface. Recently, not only the investigations on the free vibration problems and the wave motion problems but also the analyses on the impulses⁽¹⁰⁾ and the forced vibrations⁽¹¹⁾⁻⁽¹³⁾ of thick cylindrical shells have been done. However, all these researches restrict themselves to cylindrical shells with uniform thickness and it seems that the vibrations of a cylindrical shell with varying thickness have not been clarified yet. From this point of view, we have analyzed, in the previous paper⁽¹⁴⁾, the free vibrations of a cylindrical shell of which one end is clamped and the thickness varies exponentially by using both the classical theory and an improved one. And we have clarified the influences of the variation of thickness, the boundary condition of another end and other parameters upon the natural frequencies.

In this paper, we analyze the free vibrations of a cylindrical shell which is symmetric about the cross section at the center and whose thickness varies quadratically by using both the classical and the improved theories in order to clarify more clearly the characteristics of vibration of the cylindrical shell. Just as in the previous paper, we solve the equations of vibration exactly by a series solution, and consider three kinds of

boundary conditions, that is, both ends are clamped, simply supported and free. And we clarify the influences of the variation of thickness and other parameters upon the natural frequencies and the mode shapes. Moreover, comparing the results by the classical theory with those by the improved theory, we clarify the effects of the shear deformation and the rotatory inertia upon the natural frequencies and the mode shapes.

2. Improved theory

2.1 Equations of vibration and boundary conditions

The middle surface of a cylindrical shell is shown in Fig.1. The base circle is taken at the middle cross section of the shell and its center is denoted by 0. We take the coordinate axis x' along the generator on the middle surface, the coordinate axis z along radius toward the center and the angular coordinate θ in the circumferential direction in order that the coordinates may constitute the right hand system. The radius R of the base circle is taken to be the representative length, and $x=x'/R$, μ and $2\ell(=2\mu R)$ denote respectively the nondimensional coordinate,

Fig.1 Coordinates

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the nondimensional length and the length of the shell. Let h_0 denote the thickness of the shell at $x'=0$ and $h(x, \theta)$ at any point where $h/h_0=H$. Let p and t denote the circular frequency and the time, respectively. We assume that the displacements in the x' , θ and z directions are expressed as $(\bar{u} + z\bar{u}_0/R) \times \sin pt$, $(\bar{v} + z\bar{v}_0/R) \sin pt$ and $\bar{w} \sin pt$, in which $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{u}_0$ and $\bar{v}_0$ are functions of x and θ .

Now, let us consider the vibration with $2n$ nodal generators ($n \neq 1$). We put as follows.

$$\left. \begin{aligned} \bar{u}, \bar{w}, \bar{u}_0 &= \sum_{n=1}^{\infty} \{u_n(x), w_n(x), u_{0n}(x)\} \cos n\theta \\ \bar{v}, \bar{v}_0 &= \sum_{n=1}^{\infty} \{v_n(x), v_{0n}(x)\} \sin n\theta \end{aligned} \right\} \dots\dots\dots (1)$$

Then, the equations of vibration are expressed from Eq.(6) in the previous paper⁽¹⁴⁾ as

$$\left. \begin{aligned} \alpha^4 H \left(u_n - \frac{H^2}{\beta} u_{0n} \right) + \frac{d}{dx} \left[H \left\{ \frac{du_n}{dx} + (1-2\zeta)(nv_n - w_n) - \frac{H^2}{\beta} \frac{du_{0n}}{dx} \right\} \right] \\ + n\zeta H \left\{ \frac{dv_n}{dx} - n \left(1 + \frac{H^2}{\beta} \right) u_n - \frac{nH^2}{\beta} u_{0n} \right\} &= 0 \\ \alpha^4 H \left(v_n - \frac{H^2}{\beta} v_{0n} \right) + \frac{d}{dx} \left[\zeta H \left\{ \frac{dv_n}{dx} - nu_n - \frac{H^2}{\beta} \frac{dv_{0n}}{dx} \right\} \right] - nH(1-2\zeta) \frac{du_n}{dx} \\ - nH \left(1 + \frac{H^2}{\beta} \right) (nv_n - w_n) - \frac{H^3 n^2}{\beta} v_{0n} - KH \left(1 + \frac{H^2}{\beta} \right) (v_n + v_{0n} - nw_n) &= 0 \\ \alpha^4 H w_n + \frac{d}{dx} \left[KH \left(u_{0n} + \frac{dw_n}{dx} \right) \right] + nKH \left(1 + \frac{H^2}{\beta} \right) (v_n + v_{0n} - nw_n) \\ + (1-2\zeta)H \frac{du_n}{dx} + H \left(1 + \frac{H^2}{\beta} \right) (nv_n - w_n) + \frac{nH^3}{\beta} v_{0n} &= 0 \\ \alpha^4 H^3 (u_{0n} - u_n) + \frac{d}{dx} \left[H^3 \left\{ -\frac{du_n}{dx} + \frac{du_{0n}}{dx} + n(1-2\zeta)v_{0n} \right\} \right] \\ + n\zeta H^3 \left(\frac{dv_{0n}}{dx} - nu_{0n} - nu_n \right) - \beta KH \left(u_{0n} + \frac{dw_n}{dx} \right) &= 0 \\ \alpha^4 H^3 (v_{0n} - v_n) + \frac{d}{dx} \left[\zeta H^3 \left\{ \frac{dv_{0n}}{dx} - nu_{0n} - \frac{dv_n}{dx} \right\} \right] + nH^3 \left\{ w_n \right. \\ \left. - n(v_n + v_{0n}) - (1-2\zeta) \frac{du_{0n}}{dx} \right\} - KH(\beta + H^2)(v_n + v_{0n} - nw_n) &= 0 \end{aligned} \right\} \dots\dots\dots (2)$$

where

$$D_0 = \frac{Eh_0^3}{12(1-\nu^2)}, \quad \alpha_0^4 = \frac{\gamma h_0 p^2 R^4}{g D_0}, \quad \beta = \frac{12R^2}{h_0^2}, \quad \zeta = \frac{1-\nu}{2}, \quad K = \zeta k', \quad \alpha^4 = \alpha_0^4 / \beta \quad \dots\dots (3)$$

And E , γ/g , ν and k' are the modulus of longitudinal elasticity, the density, Poisson's ratio and the shear coefficient. The boundary conditions at both ends ($x=\pm\mu$) of the shell are expressed from Eq.(7) in the previous paper as

$$T_1 u_n = T_2 v_n = T_3 w_n = M_1 u_{0n} = M_2 v_{0n} = 0 \quad \dots\dots\dots (4)$$

where

$$\left. \begin{aligned} T_1 &= \beta H \left\{ \frac{du_n}{dx} + (1-2\zeta)(nv_n - w_n) \right\} - H^3 \frac{du_{0n}}{dx}, \quad T_2 = \beta \zeta H \left(\frac{dv_n}{dx} - nu_n \right) - \zeta H^3 \frac{dv_{0n}}{dx} \\ T_3 &= \beta \zeta k' H \left(u_{0n} + \frac{dw_n}{dx} \right), \quad M_1 = H^3 \left\{ -\frac{du_n}{dx} + \frac{du_{0n}}{dx} + (1-2\zeta)nv_{0n} \right\} \\ M_2 &= \zeta H^3 \left(\frac{dv_{0n}}{dx} - nu_{0n} - \frac{dv_n}{dx} \right) \end{aligned} \right\} \dots\dots (5)$$

2.2 Solutions of equations of vibration

Now let us consider such a case that the thickness varies quadratically as $H=1+kx^2$. In general, the cylindrical shell is geometrically symmetric about the cross section at $x=0$ when H is an even function of x and two kinds of solutions are obtained as the solutions of Eq.(2), that is, a solution in which u_n is an odd function of x and another in which u_n is an even function of x . When the boundary conditions of both ends of the shell are the same, the vibrations are divided into a symmetric vibration and an antisymmetric one about the cross section at $x=0$. Then, the former solution is the solution for symmetric vibration and the latter is the one for antisymmetric vibration. Let us consider them separately.

(i) In the case that u_n is an odd function of x , we put as follows;

$$u_n = \sum_{m=0}^{\infty} A_m x^{2m+1}, \quad v_n = \sum_{m=0}^{\infty} B_m x^{2m}, \quad w_n = \sum_{m=0}^{\infty} C_m x^{2m}, \quad u_{0n} = \sum_{m=0}^{\infty} G_m x^{2m+1}, \quad v_{0n} = \sum_{m=0}^{\infty} H_m x^{2m} \quad \dots\dots\dots (6)$$

where A_m , B_m , C_m , G_m and H_m are undetermined coefficients which are determined as follows. Namely, substituting Eq.(6) into Eq.(2), we have

$$\left. \begin{aligned} \sum_{n=0}^{\infty} [K(2m+1)(2m+2)C_{m+1} + f_{mc}]x^{2m} &= 0 \\ \sum_{n=0}^{\infty} [-(2m+2)(2m+3)A_{m+1} - K\beta(2m+2)C_{m+1} + (2m+2)(2m+3)G_{m+1} \\ &+ n(1-\zeta)(2m+2)H_{m+1} + f_{md}]x^{2m+1} = 0 \\ \sum_{n=0}^{\infty} [-\zeta(2m+1)(2m+2)B_{m+1} + \zeta(2m+1)(2m+2)H_{m+1} + f_{me}]x^{2m} &= 0 \end{aligned} \right\} \dots\dots\dots (7)$$

where

$$\begin{aligned} f_{ma} &= \left\{ \alpha^4 + k(2m+1)(2m+2) - n^2\zeta \left(1 + \frac{1}{\beta} \right) \right\} A_m + k \left\{ \alpha^4 - n^2\zeta \left(1 + \frac{3}{\beta} \right) \right\} A_{m-1} \\ &\quad - \frac{3\zeta n^2 k^2}{\beta} A_{m-2} - \frac{\zeta n^2 k^3}{\beta} A_{m-3} + 2kn[1-2\zeta + m(1-\zeta)]B_m - k(1-2\zeta)(2m+2)C_m \\ &\quad - \frac{1}{\beta} [\{ \alpha^4 + 3k(2m+1)(2m+2) + n^2\zeta \} G_m + 3k \{ \alpha^4 + k(2m-1)(2m+2) + n^2\zeta \} G_{m-1} \\ &\quad + \{ 3k^2(\alpha^4 + n^2\zeta) + k^3(2m-3)(2m+2) \} G_{m-2} + k^3(\alpha^4 + n^2\zeta) G_{m-3}] \\ f_{mb} &= -n(1-\zeta)(2m+1)A_m - kn[2\zeta + (1-\zeta)(2m-1)]A_{m-1} + \left\{ \alpha^4 + k\zeta(2m)(2m+1) \right. \\ &\quad \left. - (n^2 + K) \left(1 + \frac{1}{\beta} \right) \right\} B_m + k \left\{ \alpha^4 - (n^2 + K) \left(1 + \frac{3}{\beta} \right) \right\} B_{m-1} - \frac{3k^2}{\beta} (n^2 + K) B_{m-2} \\ &\quad - \frac{k^3}{\beta} (n^2 + K) B_{m-3} + n(1+K) \left(1 + \frac{1}{\beta} \right) C_m + kn(1+K) \left(1 + \frac{3}{\beta} \right) C_{m-1} \\ &\quad + \frac{3k^2 n}{\beta} (1+K) C_{m-2} + \frac{k^3 n}{\beta} (1+K) C_{m-3} - \frac{1}{\beta} \left[\left\{ \alpha^4 + 3k\zeta(2m)(2m+1) + n^2 + K\beta \left(1 + \frac{1}{\beta} \right) \right\} H_m \right. \\ &\quad \left. + \left\{ 3k\alpha^4 + 3k^2\zeta(2m-2)(2m+1) + 3kn^2 + kK\beta \left(1 + \frac{3}{\beta} \right) \right\} H_{m-1} \right. \\ &\quad \left. + \{ 3k^2\alpha^4 + k^3\zeta(2m-4)(2m+1) + 3k^2(n^2 + K) \} H_{m-2} + k^3(\alpha^4 + n^2 + K) H_{m-3} \right] \\ f_{mc} &= (1-2\zeta)(2m+1)A_m + k(1-2\zeta)(2m-1)A_{m-1} + n(1+K) \left(1 + \frac{1}{\beta} \right) B_m \\ &\quad + kn(1+K) \left(1 + \frac{3}{\beta} \right) B_{m-1} + \frac{3k^2 n}{\beta} (1+K) B_{m-2} + \frac{k^3 n}{\beta} (1+K) B_{m-3} \\ &\quad + \left\{ \alpha^4 + 2kKm(2m+1) - \left(1 + \frac{1}{\beta} \right) (1+n^2K) \right\} C_m + k \left\{ \alpha^4 - \left(1 + \frac{3}{\beta} \right) (1+n^2K) \right\} C_{m-1} \\ &\quad - \frac{3k^2}{\beta} (1+n^2K) C_{m-2} - \frac{k^3}{\beta} (1+n^2K) C_{m-3} + K(2m+1)G_m + kK(2m+1)G_{m-1} \\ &\quad + n \left\{ K + \frac{1}{\beta} (1+K) \right\} H_m + kn \left\{ K + \frac{3}{\beta} (1+K) \right\} H_{m-1} + \frac{3k^2 n}{\beta} (1+K) H_{m-2} + \frac{k^3 n}{\beta} (1+K) H_{m-3} \\ f_{md} &= - \left\{ \alpha^4 + 3k(2m+1)(2m+2) + n^2\zeta \right\} A_m - 3k \{ \alpha^4 + k(2m-1)(2m+2) + n^2\zeta \} A_{m-1} \\ &\quad - \{ 3k^2(\alpha^4 + n^2\zeta) + k^3(2m-3)(2m+2) \} A_{m-2} - k^3(\alpha^4 + n^2\zeta) A_{m-3} - 2kK\beta m C_m \\ &\quad + \{ \alpha^4 + 3k(2m+1)(2m+2) - n^2\zeta - K\beta \} G_m + \{ 3k(\alpha^4 - n^2\zeta) + 3k^2(2m-1)(2m+2) \\ &\quad - k\beta K \} G_{m-1} + \{ 3k^2(\alpha^4 - n^2\zeta) + k^3(2m-3)(2m+2) \} G_{m-2} + k^3(\alpha^4 - n^2\zeta) G_{m-3} \\ &\quad + 6kn[1-2\zeta + m(1-\zeta)]H_m + 6k^2n[1-3\zeta + m(1-\zeta)]H_{m-1} + 2k^3n[1-4\zeta + m(1-\zeta)]H_{m-2} \\ f_{me} &= - \left\{ \alpha^4 + 3k\zeta(2m)(2m+1) + n^2 + K\beta \left(1 + \frac{1}{\beta} \right) \right\} B_m - \left[3k \{ \alpha^4 + \zeta k(2m-2)(2m+1) + n^2 \} \right. \\ &\quad \left. + kK\beta \left(1 + \frac{3}{\beta} \right) \right] B_{m-1} - \{ 3k^2(\alpha^4 + n^2 + K) + \zeta k^3(2m-4)(2m+1) \} B_{m-2} \\ &\quad - k^3(\alpha^4 + n^2 + K) B_{m-3} + n \left\{ 1 + K\beta \left(1 + \frac{1}{\beta} \right) \right\} C_m + kn \left\{ 3 + K\beta \left(1 + \frac{3}{\beta} \right) \right\} C_{m-1} + 3k^2n(1+K)C_{m-2} \\ &\quad + k^3n(1+K)C_{m-3} - n(1-\zeta)(2m+1)G_m - 3kn[2\zeta + (1-\zeta)(2m-1)]G_{m-1} \\ &\quad - 3k^2n[4\zeta + (1-\zeta)(2m-3)]G_{m-2} - k^3n[6\zeta + (1-\zeta)(2m-5)]G_{m-3} \\ &\quad + \left\{ \alpha^4 + 3k\zeta(2m)(2m+1) - n^2 - K\beta \left(1 + \frac{1}{\beta} \right) \right\} H_m + \left\{ 3k(\alpha^4 - n^2) - kK\beta \left(1 + \frac{3}{\beta} \right) \right. \\ &\quad \left. + 3k^2\zeta(2m-2)(2m+1) \right\} H_{m-1} + \{ 3k^2(\alpha^4 - n^2 - K) \} H_{m-2} \\ &\quad + k^3\zeta(2m-4)(2m+1)H_{m-3} + k^3(\alpha^4 - n^2 - K)H_{m-3} \end{aligned} \dots\dots\dots (8)$$

From Eq.(7), the coefficients C_{m+1} , H_{m+1} , B_{m+1} , G_{m+1} and A_{m+1} ($m \geq 0$) are obtained in turn as follows with A_0 , B_0 , C_0 , G_0 and H_0 left undetermined provided that we take $A_q = B_q = C_q = G_q = H_q = 0$ ($q < 0$), i.e.;

$$\left. \begin{aligned} C_{m+1} &= -f_{mc}/\{K(2m+1)(2m+2)\}, & H_{m+1} &= -(f_{mb}+f_{me})/\left\{\zeta\left(1-\frac{1}{\beta}\right)(2m+1)(2m+2)\right\} \\ B_{m+1} &= -(f_{mb}+\frac{1}{\beta}f_{me})/\left\{\zeta\left(1-\frac{1}{\beta}\right)(2m+1)(2m+2)\right\} \\ G_{m+1} &= -[n(1-\zeta)(2m+2)(B_{m+1}+H_{m+1})-(2m+2)\{(1-2\zeta)+K\beta\}C_{m+1} \\ &\quad +f_{ma}+f_{md}]/\left\{\left(1-\frac{1}{\beta}\right)(2m+2)(2m+3)\right\} \\ A_{m+1} &= -\left[n(1-\zeta)(2m+2)B_{m+1}-(1-2\zeta)(2m+2)C_{m+1}\right. \\ &\quad \left.-\frac{1}{\beta}(2m+2)(2m+3)G_{m+1}+f_{ma}\right]/\{(2m+2)(2m+3)\} \end{aligned} \right\} \dots\dots\dots(9)$$

Hence, we have five pairs of independent solutions.

(ii) In the case that u_n is an even function of x , we put as follows;

$$u_n = \sum_{m=0}^{\infty} A_m x^{2m}, \quad v_n = \sum_{m=0}^{\infty} B_m x^{2m+1}, \quad w_n = \sum_{m=0}^{\infty} C_m x^{2m+1}, \quad u_{0n} = \sum_{m=0}^{\infty} G_m x^{2m}, \quad v_{0n} = \sum_{m=0}^{\infty} H_m x^{2m+1} \dots\dots\dots(10)$$

where A_m , B_m , C_m , G_m and H_m are undetermined coefficients. Substituting Eq.(10) into Eq.(2), we obtain the coefficients G_{m+1} , A_{m+1} , C_{m+1} , H_{m+1} and B_{m+1} ($m \geq 0$) in turn as follows with A_0 , B_0 , C_0 , G_0 and H_0 left undetermined, i.e.;

$$\left. \begin{aligned} G_{m+1} &= -(f_{ma}+f_{md})/\left\{\left(1-\frac{1}{\beta}\right)(2m+1)(2m+2)\right\} \\ A_{m+1} &= -(f_{ma}+\frac{1}{\beta}f_{md})/\left\{\left(1-\frac{1}{\beta}\right)(2m+1)(2m+2)\right\} \\ C_{m+1} &= -\{(1-2\zeta)(2m+2)A_{m+1}+K(2m+2)G_{m+1}+f_{mc}\}/\{K(2m+2)(2m+3)\} \\ H_{m+1} &= \{n(1-\zeta)(2m+2)(A_{m+1}+G_{m+1})-(f_{mb}+f_{me})\}/\left\{\zeta\left(1-\frac{1}{\beta}\right)(2m+2)(2m+3)\right\} \\ B_{m+1} &= \left\{n(1-\zeta)(2m+2)\left(A_{m+1}+\frac{1}{\beta}G_{m+1}\right)-(f_{mb}+\frac{1}{\beta}f_{me})\right\}/\left\{\zeta\left(1-\frac{1}{\beta}\right)(2m+2)(2m+3)\right\} \end{aligned} \right\} \dots\dots\dots(11)$$

where

$$\left. \begin{aligned} f_{ma} &= \left\{\alpha^4+k(2m)(2m+1)-n^2\zeta\left(1+\frac{1}{\beta}\right)\right\}A_m+k\left\{\alpha^4-n^2\zeta\left(1+\frac{3}{\beta}\right)\right\}A_{m-1} \\ &\quad -\frac{3\zeta n^2 k^2}{\beta}A_{m-2}-\frac{\zeta n^2 k^3}{\beta}A_{m-3}+n(1-\zeta)(2m+1)B_m+kn\{(2m+1)-\zeta(2m+3)\}B_{m-1} \\ &\quad -(1-2\zeta)(2m+1)(C_m+kC_{m-1})-\frac{1}{\beta}[\{\alpha^4+3k(2m)(2m+1)+n^2\zeta\}G_m+3k\{\alpha^4+k(2m-2)(2m+1) \\ &\quad +n^2\zeta\}G_{m-1}+\{3k^2(\alpha^4+n^2\zeta)+k^3(2m-4)(2m+1)\}G_{m-2}+k^3(\alpha^4+n^2\zeta)G_{m-3}] \\ f_{mb} &= -2kn\{m+\zeta(1-m)\}A_m+\left\{\alpha^4+k\zeta(2m+1)(2m+2)-(n^2+K)\left(1+\frac{1}{\beta}\right)\right\}B_m \\ &\quad +k\left\{\alpha^4-(n^2+K)\left(1+\frac{3}{\beta}\right)\right\}B_{m-1}-\frac{3k^2}{\beta}(n^2+K)B_{m-2}-\frac{k^3}{\beta}(n^2+K)B_{m-3}+n(1+K)\left(1+\frac{1}{\beta}\right)C_m \\ &\quad +kn(1+K)\left(1+\frac{3}{\beta}\right)C_{m-1}+\frac{3k^2 n}{\beta}(1+K)C_{m-2}+\frac{k^3 n}{\beta}(1+K)C_{m-3} \\ &\quad -\frac{1}{\beta}\left[\left\{\alpha^4+3k\zeta(2m+1)(2m+2)+n^2+K\beta\left(1+\frac{1}{\beta}\right)\right\}H_m\right. \\ &\quad \left.+\left\{3k\alpha^4+3k^2\zeta(2m-1)(2m+2)+3kn^2+kK\beta\left(1+\frac{3}{\beta}\right)\right\}H_{m-1}\right. \\ &\quad \left.+\{3k^2\alpha^4+k^3\zeta(2m-3)(2m+2)+3k^2(n^2+K)\}H_{m-2}+k^3(\alpha^4+n^2+K)H_{m-3}\right] \end{aligned} \right\}$$

$$\begin{aligned}
f_{me} &= k(1-2\zeta)(2m)A_m + n(1+K)\left(1+\frac{1}{\beta}\right)B_m + kn(1+K)\left(1+\frac{3}{\beta}\right)B_{m-1} \\
&\quad + \frac{3k^2n}{\beta}(1+K)B_{m-2} + \frac{k^3n}{\beta}(1+K)B_{m-3} + \left\{\alpha^4 + kK(2m+1)(2m+2)\right. \\
&\quad \left.- \left(1+\frac{1}{\beta}\right)(1+n^2K)\right\}C_m + k\left\{\alpha^4 - \left(1+\frac{3}{\beta}\right)(1+n^2K)\right\}C_{m-1} - \frac{3k^2}{\beta}(1+n^2K)C_{m-2} \\
&\quad - \frac{k^3}{\beta}(1+n^2K)C_{m-3} + kK(2m+2)G_m + n\left\{K + \frac{1}{\beta}(1+K)\right\}H_m \\
&\quad + kn\left\{K + \frac{3}{\beta}(1+K)\right\}H_{m-1} + \frac{3k^2n}{\beta}(1+K)H_{m-2} + \frac{k^3n}{\beta}(1+K)H_{m-3} \\
f_{md} &= -\left\{\alpha^4 + 3k(2m)(2m+1) + n^2\zeta\right\}A_m - 3k\left\{\alpha^4 + k(2m-2)(2m+1)\right. \\
&\quad \left.+ n^2\zeta\right\}A_{m-1} - \{3k^2(\alpha^4 + n^2\zeta) + k^3(2m-4)(2m+1)\}A_{m-2} - k^3(\alpha^4 + n^2\zeta)A_{m-3} \\
&\quad - K\beta\{(2m+1)C_m + k(2m-1)C_{m-1}\} + \{\alpha^4 + 3k(2m)(2m+1) - n^2\zeta - K\beta\}G_m + \{3k(\alpha^4 - n^2\zeta) \\
&\quad + 3k^2(2m-2)(2m+1) - k\beta K\}G_{m-1} + \{3k^2(\alpha^4 - n^2\zeta) + k^3(2m-4)(2m+1)\}G_{m-2} \\
&\quad + k^3(\alpha^4 - n^2\zeta)G_{m-3} + n(1-\zeta)(2m+1)H_m + 3kn\{(1-2\zeta)(2m+1) + \zeta(2m-1)\}H_{m-1} \\
&\quad + 3k^2n\{(1-2\zeta)(2m+1) + \zeta(2m-3)\}H_{m-2} + k^3n\{(1-2\zeta)(2m+1) + \zeta(2m-5)\}H_{m-3} \\
f_{me} &= -\left\{\alpha^4 + 3k\zeta(2m+1)(2m+2) + n^2 + K\beta\left(1+\frac{1}{\beta}\right)\right\}B_m - \left[3k(\alpha^4 + \zeta k(2m-1)(2m+2) + n^2)\right. \\
&\quad \left.+ kK\beta\left(1+\frac{3}{\beta}\right)\right]B_{m-1} - \{3k^2(\alpha^4 + n^2 + K) + \zeta k^3(2m-3)(2m+2)\}B_{m-2} - k^3(\alpha^4 + n^2 + K)B_{m-3} \\
&\quad + n\left\{1 + K\beta\left(1+\frac{1}{\beta}\right)\right\}C_m + kn\left\{3 + K\beta\left(1+\frac{3}{\beta}\right)\right\}C_{m-1} + 3k^2n(1+K)C_{m-2} \\
&\quad + k^3n(1+K)C_{m-3} - 6kn\{\zeta + (1-\zeta)m\}G_m - 6k^2n\{3\zeta - 1 + (1-\zeta)m\}G_{m-1} \\
&\quad - 2k^3n\{5\zeta - 2 + (1-\zeta)m\}G_{m-2} + \left\{\alpha^4 + 3k\zeta(2m+1)(2m+2) - n^2 - K\beta\left(1+\frac{1}{\beta}\right)\right\}H_m \\
&\quad + \left\{3k(\alpha^4 - n^2) - kK\beta\left(1+\frac{3}{\beta}\right) + 3k^2\zeta(2m-1)(2m+2)\right\}H_{m-1} + \{3k^2(\alpha^4 - n^2 - K) \\
&\quad + k^3\zeta(2m-3)(2m+2)\}H_{m-2} + k^3(\alpha^4 - n^2 - K)H_{m-3}
\end{aligned}
\tag{12}$$

Hence, we have five pairs of independent solutions. In this way, we obtain ten pairs of independent solutions. The general solutions of Eq.(2) are expressed by combining linearly these ten pairs of independent solutions. However, when the boundary conditions of both ends are the same, the vibrations are divided into a symmetric vibration and an antisymmetric one about the cross section at $x=0$. Just then, the expressions obtained by combining linearly the five pairs of independent solutions from Eq.(6) are the general solutions for symmetric vibration and those from Eq.(10) are for antisymmetric vibration, respectively.

3. Classical theory

3.1 Equations of vibration and boundary conditions

As in the analysis by the improved theory, we denote the radius of middle surface, the thickness and the length of a cylindrical shell by R , h and $2L(=2\mu R)$, respectively. And also we take the origin at the center O of the shell and x' , θ and z coordinates on the middle surface as shown in Fig.1. We employ a nondimensional coordinate $x=x'/R$ and denote the thickness at $x'=0$ and the one at any point by h_0 and h , where $h/h_0=H$. Let p and t denote the circular frequency and the time. Assuming that the displacements in the x' , θ and z directions are expressed as $(U, V, W)\sin pt$, we obtain the following equations of vibration and the boundary conditions at $x=\pm 1$ (14), i.e.;

$$\left. \begin{aligned}
&\alpha_0^4 HU + \frac{\partial}{\partial x} \left[\beta H \left\{ \frac{\partial U}{\partial x} + (1-2\zeta) \left(\frac{\partial V}{\partial \theta} - W \right) \right\} \right] + \zeta \frac{\partial}{\partial \theta} \left\{ \beta H \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial \theta} \right) \right\} = 0 \\
&\alpha_0^4 HV + \zeta \frac{\partial}{\partial x} \left\{ \beta H \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial \theta} \right) + 4H^3 \left(\frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial V}{\partial x} \right) \right\} + \frac{\partial}{\partial \theta} \left[\beta H \left\{ (1-2\zeta) \frac{\partial U}{\partial x} \right. \right. \\
&\quad \left. \left. + \frac{\partial V}{\partial \theta} - W \right\} + H^3 \left\{ (1-2\zeta) \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial \theta^2} + \frac{\partial V}{\partial \theta} \right\} \right] = 0 \\
&\alpha_0^4 HW + \beta H \left\{ (1-2\zeta) \frac{\partial U}{\partial x} + \frac{\partial V}{\partial \theta} - W \right\} - \frac{\partial^2}{\partial x^2} \left[H^3 \left\{ \frac{\partial^2 W}{\partial x^2} + (1-2\zeta) \left(\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial V}{\partial \theta} \right) \right\} \right] \\
&\quad - \frac{\partial^2}{\partial \theta^2} \left[H^3 \left\{ (1-2\zeta) \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial \theta^2} + \frac{\partial V}{\partial \theta} \right\} \right] - 4\zeta \frac{\partial^2}{\partial x \partial \theta} \left\{ H^3 \left(\frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial V}{\partial x} \right) \right\} = 0
\end{aligned} \right\} \tag{13}$$

$$\bar{T}_1 U = \bar{T}_2 V = \bar{T}_3 W = \bar{M}_1 \frac{\partial W}{\partial x} = 0 \quad (\text{boundary condition}) \tag{14}$$

where the definitions of α_0^4 , ζ and β are the same as in Eq.(3) and where

$$\left. \begin{aligned} \bar{T}_1 &= \beta H \left\{ \frac{\partial U}{\partial x} + (1-2\zeta) \left(\frac{\partial V}{\partial \theta} - W \right) \right\}, \quad \bar{T}_2 = \zeta \beta H \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial \theta} \right) + 4\zeta H^3 \left(\frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial V}{\partial x} \right) \\ \bar{T}_3 &= -\frac{\partial}{\partial x} \left[H^3 \left\{ \frac{\partial^2 W}{\partial x^2} + (1-2\zeta) \left(\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial V}{\partial \theta} \right) \right\} \right] - 4\zeta \frac{\partial}{\partial \theta} \left\{ H^3 \left(\frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial V}{\partial x} \right) \right\} \\ \bar{M}_1 &= H^3 \left\{ \frac{\partial^2 W}{\partial x^2} + (1-2\zeta) \left(\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial V}{\partial \theta} \right) \right\} \end{aligned} \right\} \dots\dots(15)$$

3.2 Solutions of equations of vibration

Now let us consider the vibration with $2n$ nodal generators ($n \geq 1$). We put as follows.

$$U, W = \sum_{n=1}^{\infty} [U_n(x), W_n(x)] \cos n\theta, \quad V = \sum_{n=1}^{\infty} V_n(x) \sin n\theta \dots\dots\dots(16)$$

Then, the equations of vibration are expressed from Eq.(13) as

$$\left. \begin{aligned} \alpha^4 H U_n + \frac{d}{dx} \left[H \left\{ \frac{dU_n}{dx} + (1-2\zeta)(nV_n - W_n) \right\} \right] + \zeta \left\{ H \left(n \frac{dV_n}{dx} - n^2 U_n \right) \right\} &= 0 \\ \alpha^4 H V_n - H \left\{ n(1-2\zeta) \frac{dU_n}{dx} + n^2 V_n - n W_n \right\} - \frac{H^3}{\beta} \left\{ n(1-2\zeta) \frac{d^2 W_n}{dx^2} - n^3 W_n + n^2 V_n \right\} \\ + \zeta \frac{d}{dx} \left[H \left(\frac{dV_n}{dx} - n U_n \right) + \frac{4}{\beta} H^3 \left(-n \frac{dW_n}{dx} + \frac{dV_n}{dx} \right) \right] &= 0 \\ \alpha^4 H W_n + H \left\{ (1-2\zeta) \frac{dU_n}{dx} + n V_n - W_n \right\} - \frac{d^2}{dx^2} \left[\frac{H^3}{\beta} \left\{ \frac{d^2 W_n}{dx^2} + (1-2\zeta)(-n^2 W_n + n V_n) \right\} \right] \\ + \frac{H^3}{\beta} \left\{ n^2(1-2\zeta) \frac{d^2 W_n}{dx^2} - n^4 W_n + n^3 V_n \right\} - 4\zeta \frac{d}{dx} \left\{ \frac{H^3}{\beta} \left(-n^2 \frac{dW_n}{dx} + n \frac{dV_n}{dx} \right) \right\} &= 0 \end{aligned} \right\} \dots\dots(17)$$

in which the definition of α^4 is the same as in Eq.(3). As in the improved theory, we consider such a case that $H=1+kx^2$. Then we have the following two kinds of solutions as the solutions of Eq.(17). We consider them separately.

(i) In the case that U_n is an odd function of x , we put as follows;

$$U_n = \sum_{m=0}^{\infty} \bar{A}_m x^{2m+1}, \quad V_n = \sum_{m=0}^{\infty} \bar{B}_m x^{2m}, \quad W_n = \sum_{m=0}^{\infty} \bar{C}_m x^{2m} \dots\dots\dots(18)$$

where $\bar{A}_m$, $\bar{B}_m$ and $\bar{C}_m$ are undetermined coefficients which are determined as follows. Substituting Eq.(18) into Eq.(17), we have

$$\left. \begin{aligned} \sum_{m=0}^{\infty} [(2m+2)(2m+3)\bar{A}_{m+1} + g_{ma}] x^{2m+1} &= 0, \quad \sum_{m=0}^{\infty} \left[\zeta \left(1 + \frac{4}{\beta} \right) (2m+1)(2m+2)\bar{B}_{m+1} + g_{mb} \right] x^{2m} = 0 \\ \sum_{m=0}^{\infty} \left[-\frac{1}{\beta} (2m+1)(2m+2)(2m+3)(2m+4)\bar{C}_{m+2} + g_{mc} \right] x^{2m} &= 0 \end{aligned} \right\} \dots\dots\dots(19)$$

where

$$\left. \begin{aligned} g_{ma} &= \{ \alpha^4 - \zeta n^2 + k(2m+1)(2m+2) \} \bar{A}_m + k \{ \alpha^4 - \zeta n^2 \} \bar{A}_{m-1} + n(1-\zeta)(2m+2)\bar{B}_{m+1} \\ &\quad + nk \{ (1-2\zeta)(2m+2) + 2\zeta m \} \bar{B}_m - (1-2\zeta)(2m+2)(\bar{C}_{m+1} + k\bar{C}_m) \\ g_{mb} &= -n(1-\zeta)(2m+1)\bar{A}_m - nk \{ (1-2\zeta)(2m-1) + \zeta(2m+1) \} \bar{A}_{m-1} + \left\{ \alpha^4 - n^2 \left(1 + \frac{1}{\beta} \right) \right. \\ &\quad + 2\zeta km(2m+1) \left(1 + \frac{12}{\beta} \right) \} \bar{B}_m + \left\{ k\alpha^4 - kn^2 \left(1 + \frac{3}{\beta} \right) + \frac{12}{\beta} \zeta k^2(2m-2)(2m+1) \right\} \bar{B}_{m-1} \\ &\quad - \frac{1}{\beta} \{ 3k^2 n^2 - 4\zeta k^3(2m-4)(2m+1) \} \bar{B}_{m-2} - \frac{k^3 n^2}{\beta} \bar{B}_{m-3} - \frac{n}{\beta} (1+2\zeta)(2m+1)(2m+2)\bar{C}_{m+1} \\ &\quad + \frac{n}{\beta} [n^2 + \beta - 6km \{ (1-2\zeta)(2m-1) + 4\zeta(2m+1) \}] \bar{C}_m + \frac{1}{\beta} [3kn^3 \\ &\quad + \beta kn - 3k^2 n(2m-2) \{ (1-2\zeta)(2m-3) + 4\zeta(2m+1) \}] \bar{C}_{m-1} \\ &\quad + \frac{1}{\beta} [3k^2 n^3 - k^3 n(2m-4) \{ (1-2\zeta)(2m-5) + 4\zeta(2m+1) \}] \bar{C}_{m-2} + \frac{k^3 n^3}{\beta} \bar{C}_{m-3} \\ g_{mc} &= (1-2\zeta) \{ (2m+1)\bar{A}_m + k(2m-1)\bar{A}_{m-1} \} - \frac{n}{\beta} (1+2\zeta)(2m+1)(2m+2)\bar{B}_{m+1} \\ &\quad + \frac{1}{\beta} [\beta n + n^3 - 6kn(2m+1) \{ (1-2\zeta)(m+1) + 4\zeta m \}] \bar{B}_m + \frac{1}{\beta} [\beta kn + 3kn^3 \\ &\quad - 3k^2 n(2m+1) \{ (1-2\zeta)(2m+2) + 4\zeta(2m-2) \}] \bar{B}_{m-1} + \frac{1}{\beta} [3k^2 n^3 - k^3 n(2m+1) \{ (1-2\zeta)(2m+2) \\ &\quad + 4\zeta(2m-4) \}] \bar{B}_{m-2} + \frac{1}{\beta} k^3 n^3 \bar{B}_{m-3} - \frac{1}{\beta} (2m+1)(2m+2) \{ 3k(2m+1)(2m+2) - 2n^2 \} \bar{C}_{m+1} \end{aligned} \right\}$$

$$\begin{aligned}
& + \frac{1}{\beta} [\beta(\alpha^4 - 1) - n^4 + 3kn^2(1 - 2\zeta)] \{ (2m+1)(2m+2) + 2m(2m-1) \} + 6km(2m+1) \{ 4\zeta n^2 \\
& - k(2m-1)(2m+2) \} \bar{C}_m + \frac{1}{\beta} [\beta k(\alpha^4 - 1) - 3kn^4 + 3k^2n^2(1 - 2\zeta)] \{ (2m+1)(2m+2) \\
& + (2m-2)(2m-3) \} + k^2(2m-2)(2m+1) \{ 12n^2\zeta - k(2m-3)(2m+2) \} \bar{C}_{m-1} \\
& + \frac{1}{\beta} [k^3n^2(1 - 2\zeta)] \{ (2m+1)(2m+2) + (2m-4)(2m-5) \} \\
& - 3k^2n^4 + 4\zeta k^3n^2(2m-4)(2m+1) \} \bar{C}_{m-2} - \frac{1}{\beta} k^3n^4 \bar{C}_{m-3}
\end{aligned}
\tag{20}$$

From Eq. (19), the coefficients $\bar{A}_{m+1}$, $\bar{B}_{m+1}$ and $\bar{C}_{m+2}$ ($m \geq 0$) are obtained in turn as follows with $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$ and $\bar{C}_1$ left undetermined provided that we take $\bar{A}_2 = \bar{B}_2 = \bar{C}_2 = 0$ ($q < 0$), i.e.;

$$\left. \begin{aligned}
\bar{B}_{m+1} &= -g_{mb} / \left\{ \zeta \left(1 + \frac{4}{\beta} \right) (2m+1)(2m+2) \right\}, \quad \bar{A}_{m+1} = -g_{ma} / \{ (2m+2)(2m+3) \} \\
\bar{C}_{m+2} &= g_{mc} / \left\{ \frac{1}{\beta} (2m+1)(2m+2)(2m+3)(2m+4) \right\}
\end{aligned} \right\} \dots\dots\dots (21)$$

Hence, we obtain four pairs of independent solutions.

(ii) In the case that U_n is an even function of x , we put as follows;

$$U_n = \sum_{m=0}^{\infty} \bar{A}_m x^{2m}, \quad V_n = \sum_{m=0}^{\infty} \bar{B}_m x^{2m+1}, \quad W_n = \sum_{m=0}^{\infty} \bar{C}_m x^{2m+1} \dots\dots\dots (22)$$

where $\bar{A}_m$, $\bar{B}_m$ and $\bar{C}_m$ are undetermined coefficients. Substituting Eq. (22) into Eq. (17), we obtain the coefficients $\bar{A}_{m+1}$, $\bar{B}_{m+1}$ and $\bar{C}_{m+2}$ ($m \geq 0$) in turn as follows with $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$ and $\bar{C}_1$ left undetermined provided that we take $\bar{A}_2 = \bar{B}_2 = \bar{C}_2 = 0$ ($q < 0$), i.e.;

$$\left. \begin{aligned}
\bar{A}_{m+1} &= -g_{ma} / \{ (2m+1)(2m+2) \}, \quad \bar{B}_{m+1} = -g_{mb} / \left\{ \zeta \left(1 + \frac{4}{\beta} \right) (2m+2)(2m+3) \right\} \\
\bar{C}_{m+2} &= g_{mc} / \left\{ \frac{1}{\beta} (2m+2)(2m+3)(2m+4)(2m+5) \right\}
\end{aligned} \right\} \dots\dots\dots (23)$$

where

$$\begin{aligned}
g_{ma} &= (\alpha^4 - \zeta n^2 + 2km(2m+1)) \bar{A}_m + (k\alpha^4 - \zeta kn^2) \bar{A}_{m-1} + n(1 - \zeta)(2m+1) \bar{B}_m \\
& + kn \{ (1 - 2\zeta)(2m+1) + \zeta(2m-1) \} \bar{B}_{m-1} - (1 - 2\zeta)(2m+1) (\bar{C}_m + k\bar{C}_{m-1}) \\
g_{mb} &= -n(1 - \zeta)(2m+2) \bar{A}_{m+1} - n(2km(1 - 2\zeta) + 2\zeta k(m+1)) \bar{A}_m + \left\{ \alpha^4 - n^2 \left(1 + \frac{1}{\beta} \right) \right. \\
& + \zeta k \left(1 + \frac{12}{\beta} \right) (2m+1)(2m+2) \} \bar{B}_m + \left\{ k\alpha^4 - kn^2 \left(1 + \frac{3}{\beta} \right) \right. \\
& + \frac{12\zeta k^2}{\beta} (2m-1)(2m+2) \} \bar{B}_{m-1} - \frac{1}{\beta} \{ 3k^2n^2 - 4\zeta k^3(2m-3)(2m+2) \} \bar{B}_{m-2} \\
& - \frac{k^3n^2}{\beta} \bar{B}_{m-3} - \frac{n}{\beta} (1 + 2\zeta)(2m+2)(2m+3) \bar{C}_{m+1} + \frac{n}{\beta} [n^2 + \beta - 6k(2m+1) \{ (1 - 2\zeta)m \\
& + 2\zeta(2m+2) \} \bar{C}_m + \frac{1}{\beta} [3kn^3 + \beta kn - 6k^2n(2m-1) \{ (1 - 2\zeta)(m-1) + 2\zeta(2m+2) \} \bar{C}_{m-1} \\
& + \frac{1}{\beta} [3k^2n^3 - 2k^3n(2m-3) \{ (1 - 2\zeta)(m-2) + 2\zeta(2m+2) \} \bar{C}_{m-2} + \frac{k^3n^3}{\beta} \bar{C}_{m-3} \\
g_{mc} &= (1 - 2\zeta)(2m+2) \bar{A}_{m+1} + 2km(1 - 2\zeta) \bar{A}_m - \frac{n}{\beta} (1 + 2\zeta)(2m+2)(2m+3) \bar{B}_{m+1} \\
& + \frac{1}{\beta} [\beta n + n^3 - 3kn(2m+2) \{ (1 - 2\zeta)(2m+3) + 4\zeta(2m+1) \} \bar{B}_m \\
& + \frac{1}{\beta} [\beta kn + 3kn^3 - 3k^2n(2m+2) \{ (1 - 2\zeta)(2m+3) + 4\zeta(2m-1) \} \bar{B}_{m-1} \\
& + \frac{1}{\beta} [3k^2n^3 - k^3n(2m+2) \{ (1 - 2\zeta)(2m+3) + 4\zeta(2m-3) \} \bar{B}_{m-2} \\
& + \frac{k^3n^3}{\beta} \bar{B}_{m-3} + \frac{1}{\beta} (2m+2)(2m+3) \{ 2n^2 - 3k(2m+2)(2m+3) \} \bar{C}_{m+1} \\
& + \frac{1}{\beta} [\beta(\alpha^4 - 1) - n^4 - 6k(2m+1)(2m+2) \{ km(2m+3) - 2\zeta n^2 \} + 3kn^2(1 - 2\zeta) \{ (2m+2)(2m+3) \\
& + 2m(2m+1) \} \bar{C}_m - \frac{1}{\beta} [\beta k(1 - \alpha^4) + 3kn^4 + k^2(2m-1)(2m+2) \{ k(2m-2)(2m+3) - 12\zeta n^2 \} \\
& - 3k^2n^2(1 - 2\zeta) \{ (2m+2)(2m+3) + (2m-1)(2m-2) \} \bar{C}_{m-1} + \frac{k^2n^2}{\beta} [k(1 - 2\zeta) \{ (2m+2)(2m+3) \\
& + (2m-3)(2m-4) \} + 4k\zeta(2m-3)(2m+2) - 3n^2] \bar{C}_{m-2} - \frac{k^3n^4}{\beta} \bar{C}_{m-3}
\end{aligned}
\tag{24}$$

Hence, we have four pairs of independent solutions. In this way, we obtain eight pairs of independent solutions. The general solutions of Eq.(17) are expressed by combining linearly these eight pairs of independent solutions. However, when the boundary conditions of both ends are the same, the expressions obtained by combining linearly the four pairs of independent solutions from Eq.(18) are the general solutions for symmetric vibration and those from Eq.(22) are for antisymmetric vibration, respectively.

4. Numerical calculations

We have made numerical calculations for cylindrical shells with both ends clamped, simply supported and free and have obtained the natural frequencies and the mode shapes. The boundary conditions of both ends considered for the improved theory are $u_n = v_n = w_n = u_{0n} = v_{0n} = 0$ (clamped), $u_n = v_n = w_n = M_1 = M_2 = 0$ (simply supported) and $T_1 = T_2 = T_3 = M_1 = M_2 = 0$ (free) and for the classical theory, $U = V = W = \partial W / \partial x = 0$ (clamped), $U = V = W = M_1 = 0$ (simply supported) and $T_1 = T_2 = T_3 = M_1 = 0$ (free). We have carried out the calculations by the following procedure

for both the theories. Namely, we have given, at first, n , α , β , k , ζ and k' (only for the improved theory) any value and searched for the values of nondimensional length μ which satisfy the frequency equation. We have assumed $\zeta = 0.35$ (Poisson's ratio 0.3) and $k' = \pi^2/12$. And we have carried out the calculations for the range where $\beta = 500, 1000$, $k = -0.2, 0, 0.2$, $n = 1 \sim 10$, $\alpha_0^4/\beta = 0.05 \sim 5.0$. In the improved theory, we have calculated u_n , v_n and w_n of Eqs.(6) and (10) by retaining 70 terms of the coefficients A_m , B_m , C_m and in the classical theory, we have done U_n , V_n and W_n of Eqs.(18) and (22) by retaining 50 terms of the coefficients $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_n$, respectively. We have determined the five pairs of independent solutions in the improved theory by putting one of A_0 , B_0 , C_0 , G_0 and H_0 equal to unity and others equal to zero. And in the classical theory, we have determined the four pairs of independent solutions by putting one of $\bar{A}_0$, $\bar{B}_0$, $\bar{C}_0$ and $\bar{C}_1$ equal to unity and others equal to zero. The frequency equation for the improved theory, for example, is obtained as follows. Combining linearly the five pairs of independent solutions $(u_{nj}, v_{nj}, w_{nj}, u_{0nj}, v_{0nj})_{j=1 \sim 5}$

Fig.2 Frequency curve (both ends clamped, symmetric vibration, $k=0.2$, $\beta=1000$)

Fig.4 Frequency curve (both ends simply supported, symmetric vibration, $k=0.2$, $\beta=1000$)

Fig.3 Frequency curve (both ends clamped, antisymmetric vibration, $k=0.2$, $\beta=1000$)

Fig.5 Frequency curve (both ends free, symmetric vibration, $k=0.2$, $\beta=1000$)

obtained from Eq.(6) or Eq.(10), we have

$$\left. \begin{aligned} u_n &= \lambda_1 u_{n1} + \lambda_2 u_{n2} + \lambda_3 u_{n3} + \lambda_4 u_{n4} + \lambda_5 u_{n5} \\ v_n &= \lambda_1 v_{n1} + \lambda_2 v_{n2} + \lambda_3 v_{n3} + \lambda_4 v_{n4} + \lambda_5 v_{n5} \\ &\vdots \end{aligned} \right\} \dots (25)$$

where $\lambda_1 \sim \lambda_5$ are arbitrary constants. Considering the boundary conditions mentioned above, we obtain a frequency equation in the form of a fifth order determinant. We can obtain similarly a frequency equation for the classical theory in the form of a fourth order determinant. On the convergency of series solutions, as in the previous paper, the values of β and k have large effect, but the values of n and α have little effects. In both the classical theory and the improved one, the convergency becomes better as k becomes smaller, that is, the cylindrical shell approaches a shell with uniform thickness and also as β becomes smaller. The range where the values of μ can be obtained is about $\alpha=1.8$. The accuracies of the functions of displacements, moment, etc. are a few digits if $\alpha > 1.8$ and therefore, it is impossible to obtain the natural frequency, because some digits drop out in solving the frequency equation.

Fig.6 Frequency curve (both ends clamped, symmetric vibration, $k=0.2$, $\beta=500$)

Fig.7 Frequency curve (improved theory, both ends clamped, symmetric vibration, $\beta=1000$)

The relation between the nondimensional frequency and the nondimensional length μ of a cylindrical shell for the first and the second modes of vibration are shown in Fig.2 to 5, in which we compare the results of the classical theory with those of the improved theory in the cases where $k=0.2$, $\beta=1000 (R/h_0 \approx 9.1)$ and $n=2, 4, 6$. The frequency curves by the classical theory are known to be higher than the ones by the improved theory for all boundary conditions of both ends clamped, simply supported and free. The difference of results between both theories becomes larger as μ becomes smaller (the shell becomes short), as n becomes larger and also as the vibration becomes higher mode. But the difference becomes smaller as μ becomes larger, as n becomes smaller and also as the vibration becomes lower mode. Moreover, it is seen from the figures that the frequency curves rise in the order of $n=2, 4, 6$ and also in the order of the boundary conditions of both ends simply supported, clamped and free.

Figure 6 shows the frequency curves for symmetric vibration in the case that both ends are clamped and $\beta=500 (R/h_0 \approx 6.5)$, at which the thickness is very large in

Fig.8 Frequency curve (improved theory, both ends clamped, antisymmetric vibration, $\beta=1000$)

Fig.9 Frequency curve (improved theory, both ends simply supported, symmetric vibration, $\beta=1000$)

comparison with the radius of the middle surface. The tendency of the frequency curves is almost similar in Fig.2, but as a whole, the curves come above the ones for $\beta=1000$ and the difference of results between both theories is large. In this case, it is obvious that we can not neglect the effects of the shear deformation and the rotatory inertia any more.

The influence of k denoting a parameter of thickness variation upon the nondimensional frequencies (by improved theory) is shown in Figs.7 to 12. In each figure, are shown the frequency curves for the first and the second modes of vibration in the cases where $\beta=1000$ and $n=2, 4, 6$. As seen from the figures, with $k=0$ (uniform thickness) as a turning point, the frequencies for $k=0.2$ become higher compared with those for $k=0$ and on the contrary, the frequencies for $k=-0.2$ become lower. The influence of k upon the frequencies becomes larger as n becomes larger or as the vibration becomes higher mode and the frequency becomes higher.

Fig.10 Frequency curve (improved theory, both ends simply supported, antisymmetric vibration, $\beta=1000$)

Fig.11 Frequency curve (improved theory, both ends free, symmetric vibration, $\beta=1000$)

Figure 13 denotes the influence of n (half number of nodal generators) upon the natural frequencies, in which we show the first frequency curves for symmetric vibration in the case that both ends are clamped and $k=-0.2, \beta=1000$. The frequency curves for $n=1, 2, 4$ are comparatively close to each other and the frequency curves rise rapidly as n becomes larger than 4. The frequency curves for $n=3$ come just below the ones for $n=2$, which are omitted in the figure on account of undistinguishableness.

Figures 14 and 15 show the distributions of mode shapes, forces (T_1, T_2, T_3) and moments (M_1, M_2) for cylindrical shells with clamped ends along the x axis and Figs.16 and 17 do the ones for cylindrical shells with both ends free. They are the results for the first and the second modes of vibration obtained by the improved theory in the cases where $k=0.2, \beta=1000$ and $n=4$. In order to save the space, we omit the mode shapes, forces, etc. by the classical theory, which are very similar to those by the improved theory. In these calculations, we have taken λ_1 of Eq.(25) to be unity.

Fig.12 Frequency curve (improved theory, both ends free, antisymmetric vibration, $\beta=1000$)

Fig.13 Frequency curve (improved theory, both ends clamped, symmetric vibration, $k=-0.2, \beta=1000$)

5. Conclusions

In this paper, we have analyzed the free vibrations of a cylindrical shell which is geometrically symmetric about the cross section at the center and whose thickness varies by using both the classical theory and the improved one. As numerical examples, we have obtained the natural frequencies and the mode shapes of the cylindrical shells of which the thicknesses vary quadratically in the axial direction and both ends are clamped, simply supported and free. The numerical calculations were carried out by using ACOS Series 77 NEAC System 900 Computer of Tohoku University. Compared with the previous paper, in which we used the Computer of Yamagata University, we have obtained, in this paper, the relations between the frequency and the length of the cylindrical shell over a wide range. And it became

possible to use them as a design data in practical applications. The results can be summarized as follows:

- (1) Comparing the results by the classical theory with those by the improved one, we find that the difference of frequencies between both the theories is remarkable when the cylindrical shell is short, when its thickness is large and when its vibration is higher mode. Consequently, it is advisable to use the improved theory in a design of cylindrical shell with varying thickness.
- (2) The natural frequencies become high as the thickness becomes large, but they become lower as the thickness becomes small. This tendency becomes prominent as n becomes large and as the vibration becomes higher mode and the natural frequencies become high.
- (3) The frequency curves for $n=1$ to 4 (first mode) are close to each other and the

Fig.14 Displacements, force and moment of a cylindrical shell with both ends clamped (improved theory, symmetric vibration, $k=0.2$, $\beta=1000$, $n=4$, $\alpha_0^4/\beta=5.0$)

Fig.15 Displacements, force and moment of a cylindrical shell with both ends clamped (improved theory, antisymmetric vibration, $k=0.2$, $\beta=1000$, $n=4$, $\alpha_0^4/\beta=5.0$)

Fig.16 Displacements, force and moment of a cylindrical shell with both ends free (improved theory, symmetric vibration, $k=0.2$, $\beta=1000$, $n=4$, $\alpha_0^4/\beta=3.0$)

Fig.17 Displacements, force and moment of a cylindrical shell with both ends free (improved theory, antisymmetric vibration, $k=0.2$, $\beta=1000$, $n=4$, $\alpha_0^4/\beta=3.0$)

frequency curves rise rapidly as n becomes larger than 4.

(4) The mode shapes by both the classical theory and the improved one are little different. Therefore, apart from the frequencies, it is sufficient to use the classical theory when we examine the mode shapes.

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