Elastic Contact Problem of Straight Beams

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For a problem of contact between straight beams with a sufficiently narrow contact domain, an equation of condition of contact is derived using the beam theory and the analyses of two-dimensional problems of contact as a simplification of contact state in the cross section of a beam. From this equation it is found analytically that the above simplification is sufficiently effective for the practical use and that the length of contact domain and the distribution of contact pressures in the direction of beam axis have similar tendencies to those in the problem of contact between circular plates. Numerical calculations are carried out for the influence coefficient of bending displacement on the contact surface and for the contact state of a beam on an elastic foundation.

1. Introduction

The theoretical analyses have been made of the elastic contact between straight beams under various assumptions of the contact pressure and the contact region; especially the contact has been assumed to be over the entire length of the beam. On the other hand, a problem of a slightly curved beam on an elastic foundation was treated by assuming that the domain in contact is straight. However, in this study any method to determine the length of contact domain was not mentioned. So far as these assumptions are used, however, a satisfactory solution will not be obtained in some cases. For instance, in order to compare a problem of contact between beams with one of the contact state between circular plates, it is necessary to analyse this problem under the condition which, even though simplified, sufficiently describes the actual state of contact. In the present study, regarding the smooth contact between straight beams, the extent of contact domain and the distribution of contact pressures are investigated using the theory of elasticity in comparison with the contact problem of circular plates. In order to simplify the analysis, an approximate expression of contact state in the transverse direction is introduced and the beam theory is applied. The accuracy of the solution derived from these simplifications is discussed.

2. Equation of condition of contact in the axial direction

As illustrated in Fig.1(a), two beams with constant cross section are subjected to line loads q(x) and p(x) which are symmetric with regard to the z-axis in the coordinate plane z and are in smooth contact with each other without twisting. The contact domain (Fig.1(b)) is symmetric as for the z- and y-axes and sufficiently slender in the direction of x-axis. Its unknown length l, width, and contact pressure are denoted by w, 2α(x) and q(x,y) respectively and q(x) and p(x) are distributed in the range −l≤x≤0 and 0≤x≤l respectively. Resultant forces q(x) and p are defined in the forms

\[ q(x) = \int_{-l}^{0} p(x,y) dy \quad p = \int_{-l}^{0} q(x,y) dy \]

In the present study, regarding the boundary curve of a cross section of the beam, two cases will be treated. One is a case in which the boundary curve has a rounding of radius r or r in the contact portion as shown in Fig.1(c) and the other a case of a flat contact with constant width 2a as in Fig.1(d). In the former case with a concave rounding the corresponding radius r or r is taken negative. The problem for Fig.1(d) may be treated in section 2.5.

![Fig.1 Contact state of beams and coordinate system](image)

2.1 Beam on the elastic foundation

Let us consider a case when beam 2 is regarded as a so-called elastic foundation because the dimension of cross section of beam 2 is much larger than that of beam 1. In this case, concerning the point on the x-axis in the contact surface, displacement w(x) in the x-direction can be expressed approximately by the following equation which presents a normal displacement on the surface of a half-space under compressive load p(x,y):

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the displacement $w(x)$ in the $x$-direction will be expressed simply as a deflection due to the bending moment in the form:

$$w(x) = \frac{c}{2} \int_{-l}^{l} \int_{-\delta}^{\delta} [\cos(z) - \cos(z)] \, dz \, dx$$

where $I$ is the secondary cross-sectional moment and when beam 1 is subjected to a concentrated load, $w(x)$ in Eq. (5) may be taken zero.

Substitution of Eqs. (1) and (5), with $\delta = \ell$, assumed into the condition $w(x) = w_0(x)$ of continuity of the displacement on the contact surface yields:

$$\frac{1}{2} \int_{-l}^{l} \int_{-\delta}^{\delta} [\cos(z) - \cos(z)] \, dz \, dx$$

where $l = \ell^2$ and $\delta$ is an unknown constant depending on the zero point with respect to the displacement in the $x$-direction. $H(x, z)$ and $f(x)$ are given in the form:

$$H(x, z) = \frac{1}{2} \int_{-\delta}^{\delta} [\cos(z) - \cos(z)] \, dz \, dx$$

$$(x-z)^2$$

In Eqs. (6) and (7) the following dimensionless quantities (8) are introduced for the sake of convenience:

$$\frac{q(x)}{E_1} = q_0(x) / E_1$$

These quantities are finally denoted by $\phi(x)$, $\xi$, $\eta$, $\delta$, $\varphi(x)$, $\psi(x)$, $\zeta(x)$, and $\theta(x)$.

In the following these notations (8) shall be used in Eqs. (9) except Eqs. (15), (18), and (20). Next, the function $w(x)$ has to satisfy the relation of equilibrium between contact stress and given load

$$\int_{-l}^{l} \int_{-\delta}^{\delta} [\cos(z) - \cos(z)] \, dz \, dx = \frac{1}{4} \int_{-l}^{l} [1 - \cos(x)] \, dx$$

and the condition that at both ends $x=\ell$ of contact domain the contact pressure becomes zero yields

$$w(1) = 0$$

Thus the problem is reduced to one of finding the solutions $\ell$ and $w(x)$ from Eqs. (6) and (9) together with condition (10) and it is readily known that these solutions depend on $(1-w)/(E_1/E_2)$ and $(\ell^2)/\delta$.

Particularly, in the case of a beam loaded by a concentrated force, only a value of $c\ell_1(1-w)/(E_1/E_2)$ being given the solutions $(\ell^2)/(1-w)/(1-w)$ and $w(x)$ are determined. And the rate of change of each solution for the value of $c\ell_1(1-w)/(E_1/E_2)$ is as small as explained later. Hence writing the solutions in the following forms (11) and (12), $\alpha$ in Eq. (11) can be regarded as a constant in a wide range of $(1-w)/(1-w)$ and $\beta$ in Eq. (12) a constant function of $x$ in the same range:

$$\frac{1}{2} \int_{-\delta}^{\delta} [\cos(z) - \cos(z)] \, dz \, dx$$

Accordingly, from the definition of $w(x)$ in Eq. (9), it is readily seen that the pressure distribution $f(x)$ of contact and the displacement $w(x)$ of contact
surface vary approximately in proportion to \( P \) in a wide range of \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) Eqs. (11), (12), and the above proportional relations have similar tendency to those in the case of a thin circular plate on the elastic foundation under a concentrated load.  

Next, an approximate calculation will be tried in order to find out the range of variable \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) in which Eqs. (11) and (12) are formed as explained before. Let us consider, for instance, the case when the ratio of two values of \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) is 1:10 and for this large value of 10 the solutions are given by \( \delta, = 800 \) and \( \delta u(0)=800 \). Then the solutions \( i \) and \( \varepsilon \) for this result may be estimated approximately from Eqs. (6), (8), (9), (10), and (2) in the following. In this estimation, according to Eqs. (7), (9), and (10) and under such a tendency of \( u(\varepsilon) \) to simplify the calculations, the \( \delta \) of \( u(\varepsilon) \) can be assumed by a linear function \( \varepsilon \), because there is no effect of this assumption of \( u(\varepsilon) \) on the general tendency of the solution obtained as the result.  

(i) Writing by \( \delta = X_1(\varepsilon) \), the amount of the left-hand side of Eq. (6) which is calculated from the relations \( i=n, = m, = 0 \), \( \delta = \delta \varepsilon \), and \( \delta v(0)=800 \), \( X_1(0) - X_1(1) \) is evaluated approximately as \( \sim 0.15 \varepsilon \).  

(ii) Similarly to the amount \( \delta = X_1(\varepsilon) \) of the left-hand side of Eq. (6) which is calculated from the relations \( i=n, = m, = 0 \), \( \delta = \delta \varepsilon \), and \( \delta u(0)=800 \), \( X_1(0) - X_1(1) \) is estimated approximately as \( \sim 0.05 \varepsilon \).  

(iii) Then, setting with \( m = n \) and \( \delta = \delta = \delta \varepsilon \), the solutions \( \delta u(\varepsilon) \) and \( \delta \) satisfying Eqs. (6) and (9) under the assumption \( \varepsilon = 0.06, \varepsilon = n \), the first equation which should be satisfied by \( m(\varepsilon) \) and \( \delta = \delta \varepsilon \) in Eq. (9) is calculated from the right-hand side and is taken as \( 0 \), \( m(\varepsilon) \) and \( \delta = \delta \varepsilon \) respectively and the second equation is the expression \( \int_0^1 m(\varepsilon) d\varepsilon = 0 \) which is derived from Eq. (9).  

This linear approximation of \( G(\varepsilon) \) is based on the consideration that from Eq. (2) \( G(\varepsilon) \) is a simply increasing function having a small rate of change for \( \varepsilon \) and that the value of \( u(\varepsilon) \) changes gradually against \( P \) from the physical meaning in the definition of \( u(\varepsilon) \) in Eq. (8). Comparing the above-mentioned first equation with Eq. (6) used in section (ii) and considering that \( m(\varepsilon) \) satisfies the relation \( \int_0^1 m(\varepsilon) d\varepsilon = 0 \) and does not vary with \( \varepsilon \), it is known that the above two equations are approximately satisfied by taking \( \delta = 0.06 \varepsilon \) and \( \delta u(0)=m(0) \) \( = 0.02 \, m(0) \sim 0.02 \, m(0) \). From the above the solutions \( m(0)=0 \) and \( m(0)=0 \) are obtained and hence the condition (10) is not satisfied by the above \( u(\varepsilon) \). But \( m(\varepsilon) \) is such a small quantity compared with \( \varepsilon \) as described above and it is concluded that the above solutions \( \varepsilon = 0.06, \varepsilon = n, \) and \( \delta u(0)=m(0) \) are nearly equal to the exact solutions. Next, for the solutions \( \delta u(0)=m(0) \) approximate solutions \( \varepsilon = 0.06, \varepsilon = n, \) and \( \delta u(0)=m(0) \) are obtained respectively in the same manner as above.  

These tendencies with respect to the rate of change of \( \varepsilon \) or \( u(\varepsilon) \) for \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) or \( \delta u(0)=m(0) \) are known generally from Eqs. (6) and (2). When the value of \( \delta u(0)=m(0) \) is less than 200 a similar calculation to the above can not be made owing to the relation \( \delta = \delta \varepsilon \). But from Eq. (9) it is known that in this case the rate of change of the solution becomes larger than that of the above case. In the case of short contact region, however, the simplification by Eqs. (3) and (4) comes into effect.  

Next, in order to check the effect of the simplification by Eq. (3) on the accuracy of the solution, let us assume the expression \( \rho(x) = u(x) = \rho(0) \) instead of Eq. (3). Then the expression corresponding to Eq. (2) can be presented in the form  

\[
G(\varepsilon) = \int_0^1 (1 - \varepsilon^2) \sinh (1/\varepsilon) d\varepsilon
\]

\[
= 3 + 2 \ln (2\pi) + 1 + 2 \ln (1 + \varepsilon) - 2 \ln (1 - \varepsilon) - 2 \ln (1 - \varepsilon)
\]

For the case of \( \delta u(0)=m(0) \), by using the similar method to that described before, it is estimated that the difference between the solution \( \varepsilon \) or \( u(\varepsilon) \) derived from Eq. (10) and the one from Eq. (2) is less than 0.05 per cent regardless of the amount of \( \varepsilon \) for each case. Similarly, taking uniformly distributed pressure \( \rho(x) = \rho(x) \), the difference of solution is less than 0.5 per cent. Even if a difference of value of \( \varepsilon \) is taken into account in the above calculation, the effect of this difference on the solution is small owing to such a small rate of change of \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) as described in the proceeding. From the above discussion it is evident that Eq. (3) and the constant \( c \) given by Eq. (4) have a sufficient utility in practice for simplification of the condition of contact. This utility can be understood naturally from Eqs. (3) and (4) being related directly with the contact state in the transverse direction, provided that the contact domain is sufficiently slender. But in the case of short contact domain, for instance, a case of large radii \( r_1 \) and \( r_2 \) in contact portion, it is necessary to analyze the problem by taking the exact condition of contact into account.  

For the general case as for lording on the beam, by using a similar manner to that developed above, it is concluded that the rate of change of \( \varepsilon \) or \( u(\varepsilon) \) is small for variable \( c/(1-\varepsilon^2)E_1/E_2^{3/4}/P \) and that the simplification of contact stress by Eqs. (3) and (4) is sufficiently effective in practice.  

Accordingly, an error in a solution results mainly from the use of Eq. (5), i.e., the error due to the application of the beam theory and the neglect of local displacement on the contact surface of beam. It will be seen that this error has similar quality and quantity to those in the problem of a thin plate on the elastic foundation. When \( E_1/E_2 \) in Eq. (11) is near zero, the error produced by the neglect of local displacement on the contact surface becomes large. On the other hand, displacement
on the surface of a half-space can be used approximately instead of this local displacement, and then taking 
\(\psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)\) instead of \(\psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)\) the following equation can be obtained in the same manner as in the case of Eq.(11):

\[ I_{x} = \left(1 - \psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)\right) I_{x} \]  

From this, it is readily known that the similar results to those derived from Eq.(11) can be obtained from Eq.(14), that is, \(\psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)\) instead of \(\psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)\). Since the local displacement on the contact surface of a beam is less than that on the surface of a half-space, almost errors in \(I/I_{x}\) of Eq.(14) are ones due to the deflection derived from the beam theory. However, the error in \(I/I_{x}\) due to this deflection is far less than the error in this deflection itself. For example, in the case of a concentrated load if the true bending displacement is 20 per cent larger than that given by the beam theory, the true solution \(I/I_{x}\) is estimated approximately by using Eq.(6) as \((1 - 0.2)^{2} = 0.95\) times the solution derived from the beam theory, i.e., the error of \(I/I_{x}\) is about 5 percent. In the above calculation the value of \(\epsilon_{b}\) remains constant because the errors of \(\epsilon_{b}\) are produced owing to the beam theory, because the difference in the above-mentioned bending displacement is not such a constant as 20 per cent used above but varies with the point on the beam. In the case of a concentrated load, the length of the contact range becomes minimum and the error of solution caused by the beam theory is maximum.

2.2 Contact between beams

In the following, concerning the point on the \(z\)-axis in the contact surface of beam 1, the displacement \(w(x)\) in the \(z\)-direction is resolved into \(w_{1}(x)\) and \(w_{2}(x)\). Where \(w_{1}(x)\) is the displacement when beam 1 is subjected to the load \(p(x, y)\) on the contact surface and to the same load \(p(x, y)\) on the side, opposite to this contact surface. \(w_{2}(x)\) is given approximately by the following equation (15) which expresses the displacement on the surface of an infinite plate with a thickness \(bd\), which is equal to the height of cross section of beam 1:

\[ w_{2}(x) = \frac{p_{0}}{2} \int_{a}^{b} \frac{\sinh t}{\sinh (2t)} \left[ \sqrt{1 + \left(\frac{x}{2}\right)^{2}} - 1 \right] \frac{1}{d} \sinh t \, dt \, dx \]  

Next, \(w_{0}(x)\) is the displacement of beam 1 under the loads \(q(x)\) and \(p(x, y)\) acting on the side opposite to the contact surface and \(w_{0}(x)\) is simply expressed by Eq.(5).

Similarly, the displacement \(w(x)\) for beam 2 is known and substituting \(w_{1}(x)\) and \(w_{2}(x)\) into the condition \(w(x) = w_{0}(x)\) of displacement on the contact surface, the following equation (16) is obtained in a similar way to the case of Eq.(6):

\[ \sum_{i=1}^{3} \left\{ \sum_{j=1}^{3} G_{ij}(\psi_{i}(x) + \psi_{j}(y) + \psi_{3}(z)) \right\} \psi(x) + \int_{a}^{b} \left[ H(x, z) \psi(z) \right] \, dz = k(x) + \Delta \]  

in which

\[ H(x, z) = \frac{1}{2} \sum_{i=1}^{3} \left( \begin{array}{c} 1 - \psi_{i}(x) + \psi_{2}(y) + \psi_{3}(z) \\ \sinh t/\sinh (2t) + 2t/\sinh (2t) + 2t/\sinh (2t) + 2t/\sinh (2t) \end{array} \right) \int_{a}^{b} \left[ \sinh (2t) \, dt \right] \]

where \(k(x)\) is the secondary moment of cross section of beam 2. Denoting by \(d_{b}\) or \(2d_{b}\) the height of cross section of beam 1 or 2 respectively, the deflection quantities \(2d_{b}/I_{b}\) and \(2d_{b}/I_{b}\) are introduced and finally in Eqs.(16) and (17) these quantities are denoted by \(d_{b}/I_{b}\). Local displacement of a point on the contact surface depends mainly on the contact pressure acting in the vicinity of this point and hence if the terms of integral in Eq.(17) are neglected, the effect of this procedure on the accuracy of solution is small. Eqs.(9) and (10) should be satisfied for the present problem also. From Eqs.(16), (9), (10), and using the same method as that in section 2.1, the point load \(q(x)\) of beam 2 is solved and that the solutions \(I/I_{x}\) and \(w(x)\) depend on \(p/P\) and the rate of change of solution for \(p/P\) is small and hence similar results to those in section 2.1 are applicable to the present problem also. For a particular case of two beams under equal loading, the solution depends mainly on \((1 - \psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z)) / (1 + \psi_{1}(x) + \psi_{2}(y) + \psi_{3}(z))\) and \(p/P\) and there exist similar tendencies to the case of contact between circular plate as follows: Dependence of the solution on the value \(\psi_{3}(z)\) becomes smaller with an increasing ratio \(I/I_{b}\) and the solution is nearly independent of \(E_{b}/E_{0}\) when \(I/I_{b}\) is 1. On the other hand, when \(E_{b}/E_{0}\) is 0 the contact surface reduces to the area of a plane, and hence in the case of \(I/I_{b}=1\) the point load on the \(z\)-axis in the contact surface remains nearly on a straight line after deformation even when the value of \(E_{b}/E_{0}\) is arbitrary.

2.3 In the case of plane contact with constant width (Fig. 4 (d))

First, let us consider a simple type of contact pressure being given by the equation

\[ p(x,y) = \frac{q(x)}{(\pi^{2}r^{2})} \]  

Then in the same manner as in section 2.1, the expression corresponding to Eq.(2) becomes

\[ G(x) = \int_{a}^{b} \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^{2}}} \sinh t \, dt \, dx = 2 \log(4) + \frac{1}{4\pi} - \frac{9}{25} \frac{1}{80} + \frac{1}{25} \]  

Equation (18) gives the pressure distribution of contact between a rigid flat with flat bottom and a half-space, Using Eq.(1) and Eqs.(9) in Eqs.(5) of Eq.(16) instead of Eq.(2) and \(G(x) = f(2\pi/4Pw(x)/(m\pi))\), the solution can be obtained in the same manner as in section 2.1 or 2.2. Similarly the following equation (19) is obtained by assuming the expression (20):

\[ G(x) = \frac{q(x)}{\pi^{2}r^{2}} \]  

\[ G(x) = \frac{2}{\pi^{2}} \int_{a}^{b} \frac{1}{\sqrt{1 + \left(\frac{x}{2}\right)^{2}}} \sinh t \, dt \, dx = 2 \log(4) + \frac{1}{2} - \frac{3}{64} \frac{1}{80} + \frac{5}{256} \]  

\[ G(x) = \frac{q(x)}{\pi^{2}r^{2}} \]
It is known by the method, analogous to that used in section 2.1, that a difference between the solution derived from Eqs. (18) and (19) and the one from Eqs. (20) and (21) is less than 3.5 per cent in the case of \( h/w = 80 \). In many practical cases the contact pressure in the \( z \) direction distributes in a form intermediate between Eqs. (18) and (20). Selecting either Eq. (18) or (20) according to the condition of the problem, for instance a magnitude of \( K_{1}/K_{2} \), Eq. (18) or (20) can be a sufficiently effective solution for the distribution of contact pressures. Using Eqs. (9), (10), (12/13), and either Eq. (6), or (16) in a similar method to that in section 2.1 or 2.2, it can be concluded that the solutions \( U/I \) and \( w(z) \) are independent of the intensity of \( P \) and \( q(z) \), and that those of Eqs. (12), (14), and the same tendencies of solutions as in section 2.1 and 2.2 are developed independently of \( P \). These tendencies are as those in the case of contact between circular plates. Next, concerning the change of the solution with the value of width \( 2a \) of contact domain, for instance when \( 2a \) changes by three times in the case of \( 1/2a = 80 \), the corresponding changes of \( U/I \) and \( w(z) \) are less than 3.5 per cent and 2 per cent respectively in similar calculation to that in section 2.1.

3. Numerical calculation

3.1 Influence coefficient of bending displacement on the contact surface of beam with rectangular or circular cross section

In order to get an exact solution without use of the beam theory, an exact value of influence coefficient \( K \) in Eqs. (7, 10) becomes necessary instead of \( K \) in Eq. (7,-c). In the following, these coefficient \( K(x, z) \) and \( K(x, z) \) are calculated for an infinitely long beam with rectangular or circular cross section respectively. For a narrow rectangular cross section and \( K(x, z) \) are given from the theory of elasticity in the formulae:

\[
K(x, z) = \frac{3K}{4x} \left[ \sinh(z/l) - \sinh(z/l) \right] \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2x} \sum_{a=1}^{\infty} \frac{1}{a} \cos(ax) \left[ 1 - \cos \left( ax \right) \right] dt
\]

\[
K(x, z) = \frac{1}{4x} \left[ \sum_{a=1}^{\infty} \frac{1}{a} \right] \sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2x} \sum_{a=1}^{\infty} \frac{1}{a} \cos(ax) \left[ 1 - \cos \left( ax \right) \right] dt
\]

in which

\[
P(x) = \sqrt{2E(x) + \rho_{L}(x)} + \pi N(x)\]

\[
L(x) = (x^2 + 1) + \pi N(x) + \pi \left( N(x) - 1 \right) N(x)
\]

\[
M(x) = \sqrt{2E(x) + \rho_{L}(x)} + \pi N(x) \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{a=1}^{\infty} \frac{1}{a} \cos(ax) \left[ 1 - \cos \left( ax \right) \right] dt
\]

\[
N(x) = \frac{\pi}{2} \left( N(x) - 1 \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{a=1}^{\infty} \frac{1}{a} \cos(ax) \left[ 1 - \cos \left( ax \right) \right] dt
\]

where \( 2k \) is height of rectangular cross section and \( x \) are radius and Poisson’s ratio of circular cross section and \( \text{Li}(z) \) is a modified Bessel function.

Fig. 2 Tendencies of \( K_{1}(1.1) \) for \( b/h \) or \( l/l \)

In order to get the tendency of \( K \) to \( b/h \) in Fig. 2, \( K \) are dotted instead of \( K \) for \( b/h = 2.1 \) assuming that the bending displacement of contact surface of a beam with rectangular cross section is nearly equal to that of a beam with circular cross section whose both the cross-sectional area and the secondary cross-sectional moment are equal to the ones of this rectangular cross section respectively. Since a dotted point of \( K \) nearly lies on an extension of the curves presented for \( b/h = 0.0 \) and the same \( l/l \) as that for this \( K \), these curves together with each extension are regarded as the expressions for the tendencies of \( K \) to \( b/h \) or \( l/l \). Concerning \( K_{1}(1,e) \) and \( K_{1}(1,e) \) without \( z = e^{-1} \), another calculation to those in Fig. 2 can be obtained and the following explanation can be made. Making a similar calculation to that in chapter 2 by using \( K \) and \( K \) instead of \( K \) given by Eqs. (7,-c) for Fig. 1, it is known that the rate of change of \( l/l \) or \( w(z) \) for \( t/P \) is small for the general case of loading. Furthermore, in the case of contact between two beams whose lengths, dimensions of cross sections and loadings are all equal respectively, the same results as those in section 2.2 for the case of \( l/l = 1 \) are obtained independently of the shape of their cross sections from Eqs. (16), (19), (10), and the expression of displacement. If the beams are in contact as shown Fig. 1 in the above description, the solutions are independent of \( P \) and then all the conditions of \( t/P \) may be omitted.

In Fig. 2 an error of \( K \) originating from the beam theory increases with a decrease of \( b/h \) and this tendency is mainly due to the neglect of the deflection caused by the shearing force of beam. If the error of \( U/I \) given by Eq. (14) is estimated from Eqs. (12), (14), \( K_{1}(1,e) \) and \( K_{1}(1,e) \) in the same evaluation as in section 2.1 and is shown in similar way to that in Fig. 2, it may be seen that this error has the same tendency as in the graph in Fig. 2. The above conclusions hold also for a beam with a different
cross section whose shape is approximately a rectangle or circle, because the influence coefficient of the beam may be approximately given by such an equivalent rectangular cross section as shown in Fig. 2.

3.2 Method and results of calculation

For a case of Fig. 1(e) numerical calculations are made of a beam on the elastic foundation loaded by a concentrated force. Equations (6) and (9) are solved numerically by the method of successive approximation assuming $w(x, y) = G(x, y)$ in Eq. (6). In this manner calculations are carried out with good convergence since in the first term of the left-hand side of Eq. (6) the magnitude of non-linear part is small as compared with the linear part according to Eq. (2). For example, assuming by linear distribution $w(x, y) = 1 - x$ the first assumed function of $w$ in $G(x, y)$, sufficient convergence can be obtained by three repetitions of this assuming procedure. Then Eq. (10) can be satisfied some repeats of assumption of $w$ in Eq. (14). The representative coordinate points necessary for the simultaneous linear equation used in solving numerically Eqs. (6) and (9), are taken by dividing equally the range $x = 0$ to 0.05, 0.05 to 0.05, 0.05 to 0.05, 0.05 to 0.05, 0.05 to 0.05, 0.05 to 0.05, and 0.05 to 0.05 in sections respectively. And in each section, after approximation of $w(x, y)$ by a quadratic function of $x$, $y$ in $G(x, y)$, Eq. (6) is integrated with the aid of the formulas of infinite integral.

Integral in Eq. (9) is calculated by using First Simpson's Integral Formula. $K_0(x, y)$ used in Eq. (23) can be expressed in the vicinity of $x = y = 0$ in the form for $\alpha = 0.3$.

\[ F(x) = 1 + 0.12466266 + 0.3643195 \times 10 \]  

and hence dividing the infinite series in Eq. (23) into two parts, such as

\[ \int_{-\infty}^{\infty} F(at^3)(1 - \cos (at))(1 - \cos (bt)) \, dt \]

and the remainder, the latter is calculated by using Lagrange's six points interpolation method. Values of the function used in this interpolation are calculated by using the following expression of gradual development of $K_0(x, y)$ to accelerate the convergence of integral in Eq. (23):

\[ K_0(x) = (1 - \pi^2) \sum \frac{1}{(1 - \pi^2)^2} \sum \frac{1}{(1 - \pi^2)^2} \]

where $E$, $\nu$ is a constant depending on $x$, $y$. If necessary, a rapid convergence of the infinite series in Eq. (23) can be effected by employing a method of acceleration of the convergence. On the other hand, dividing $F(at^3)$ into two parts, $E_1$ expressed by Eq. (25) and the remainder, the integral

\[ \int_{-\infty}^{\infty} F(at^3)(1 - \cos (at))(1 - \cos (bt)) \, dt \]

is calculated for the range near $\pm 0$ and the remainder respectively. Equation (22) can be calculated in similar manner to the case of Eq. (23). In all calculations Poisson's ratio 0.3 is used.

In the following, the results of calculation will be explained by graphs. The solution for a beam on the elastic foundation loaded by a concentrated force is shown in Fig. 3~Fig. 6. In Fig. 3, the graph of beam theory gives the values of $\alpha$ in Eq. (14). It is fully confirmed that the rate of change in $\alpha$ for $\left(\frac{(1 - \pi^2)E_1}{(1 - \pi^2)E_0}\right)^{1/4}$ is such a small magnitude as evaluated in section 2.1 and that an error of Eq. (14) caused by the beam theory is a small quantity, as explained in section 2.1. The dotted line presents the slenderness of contact domain derived from the beam theory by the ordinate $A(0, 0)$.

Let us consider, as an example, a case when diameter of circular cross-sectional beam $2r = 1$ cm, radius of rounding portion on the foundation $r_0 = 0.2$, $\lambda = 100$, $E = 2.1 \times 10^{11}$ kg/cm² and concentrated load $P = 1$ kg.

In this case the dimensionless value $\left(\frac{H}{\lambda E_0} + \frac{1}{(1 - \pi^2)E_0}\right)^{1/4}$ becomes $1.89 \times 10^3$ and then $\alpha = 3.44$ and $A(0, 0) = 0.365 \times 10^3$ are obtained from Fig. 3. Next, for the case when in the above example only $P$ varies in the range 0.1~10 kg, i.e., value of $\left(\frac{(1 - \pi^2)E_1}{(1 - \pi^2)E_0}\right)^{1/4}$ changes in $1.89 \times 10^{-10}$, the corresponding half length $l_0$ cm of contact domain is known from Fig. 3 and is shown in Fig. 4 compared with a radius 1 cm of contact circle between half-space $\alpha$ and infinite plate of width $1$ cm subjected to a concentrated load $P = 1$ kg.

![Fig. 3 Variation of $H/(\lambda E_0) + (1-\pi^2)E_0/E_1^{1/4}$ or $\lambda A(0,0)$ with $\left(\frac{(1-\pi^2)E_1}{(1-\pi^2)E_0}\right)^{1/4}$](image)

![Fig. 4 Relation between $l_0$ and $P$, kg](image)
Graph of Eq.(14) in Fig. 4 is plotted using the following expression derived from Eq.(14) in section 2.1:

\[ t = 6.6\sqrt{2}/(1 - \nu^2) + (1 - \nu)(E_0/E_2) \]

The radius of contact circle of the plate does not vary with a changing intensity of \( P \) and is given for a thin plate by the equation (15):

\[ r = 6.6\sqrt{2}/(1 - \nu^2) + (1 - \nu)(E_0/E_2) \]

It is known that both values of \( E_0 \) for beam and plate are the same order as shown in Fig. 4. Next, Fig. 5 shows the error of \( L/I \) which is calculated using a value of \( L/I \) obtained from Eq.(14) and the one from Eqs.(6),(9),(10),(22), and (23). In Fig. 5, plotting the error of circular cross section instead of the error for \( b/h = 2.1 \) with the same consideration as in the case of Fig. 2, it is known that the error increases with a decrease of \( b/h \) as described in section 2.1. And the magnitude of this error and tendency for \( b/h \) are known also. In Fig. 6 an example of \( w(x) \) is shown and from this it may be seen that \( w(x) \) nearly distributes linearly and that the error of \( w(x) \) due to the beam theory is small as explained in section 2.1. Furthermore, in the other case there are the same tendencies of \( w(x) \) as in Fig. 6 and the variation of \( s(0) \) which comes from the beam theory in a wide range \( 0.1 < (1 - \nu)(E_0/E_2) \) is less than 1 per cent. This magnitude of the variation is less than that evaluated in section 2.1 and Eq.(12) is confirmed sufficiently. Next, Fig. 7 shows \( L/I \) of the contact between two beams which are loaded each by a concentrated force and whose cross sections are in size for the case of rectangular or circular cross section and are \( t = h \) for the graph of beam theory. It will be seen that both the magnitude of error due to beam theory and the rate of change of \( L/I \) for \( cP/P \) are small as described in section 2.2. These solutions presented in Fig. 7 are nearly equal to the ones obtained from Eq.(6) by taking \( E_0/E_2 = 1 \).

In the above, concerning the calculation of Eqs.(17,18), the effect of neglecting both terms of integral in these equations on the accuracy of \( L/I \) or \( s(0) \) is less than 1 per cent. This tendency is such as described in section 2.2 and effect of neglect decreases with an increase of length of contact domain.

![Graph showing the relation between error of L/I and b/h](image)

![Graph showing the contact pressure distribution in the axial direction](image)

**4. Conclusions**

(1) For a problem of contact between straight beams with constant cross section, analyses of two two-dimensional problems of contact one of which is the contact between parallel circular cylinders and the other that between a flat-ended stamp and a half-space, can be employed as an effective simplification of contact state in the transversal direction, provided that the contact region is sufficiently slender. The former analysis is used for a problem of a rounding contact portion and the latter for a problem of flat contact.
(2) From an analysis of the contact state in the axial direction using the above-mentioned simplifications and the beam theory, it is found that the length of contact domain, the distribution of contact pressures and the displacement on the contact surface have similar tendencies to those in the case of contact between circular plates.

(3) Finally, numerical calculations are carried out about the influence coefficient of bending displacement on the contact surface of beam with a rectangular or circular cross section and about the contact state of a beam on the elastic foundation loaded by a concentrated force, the results being shown graphically.

References

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