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Effects of Friction in Tensile and Compressive Stress Problems for a Rigid Circular Disk in an Infinite Plate*

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The plane stress problem of an infinite plate with a circular hole containing a rigid disk under uniaxial tension or compression is numerically solved. Effect of friction on the contact surface between the hole and the disk is considered. A general form of a stress function expressed in the polar coordinates is used and unknown coefficients of the stress function are adjusted successively so that the stress function satisfies boundary conditions. Distributions of stresses and displacements around the hole are shown in figures. Effects of friction on the distributions are discussed.

Key Words: Elasticity, Stress Analysis, Infinite Plate, Circular Hole, Rigid Disk, Effect of Friction

1. Introduction

The plane stress problem of an infinite plate which has a circular hole containing a disk and is subjected to uniform tensile or compressive stress at infinity, was solved by M. Stippes et al.⁽¹⁾, J. Margetson & L. W. Morland⁽²⁾, T. Yamamoto & T. Tsumura^{(3)~(6)} and also by the authors⁽⁷⁾⁽⁸⁾. These solutions were obtained under the assumption that there is neither friction nor adhesion between the hole and the disk. As far as the authors know, there has been no paper which treated effects of friction.

In this paper, introducing an empirically confirmed rule for friction in the hole-disk interface, stresses and displacements around the hole are analyzed numerically and the influence of friction is discussed. It is assumed that the disk is rigid and that the diameters of the hole and the disk are identical in the unstressed state. The numerical method is the same as that in the previous papers⁽⁷⁾⁽⁸⁾, i.e. a method using a general form of a stress function expressed in the polar coordinates, in which coefficients of the stress function are adjusted successively so that the stress function satisfies boundary conditions around the hole.

Since states of contact are quite different between tensile loading and compressive loading, these two cases must be treated separately, although the same method of solution can be applied.

2. Method of solution

An infinite plate with a hole containing a rigid disk is subjected to uniaxial tension or compression as shown in Fig.1. (σ_r , σ_θ , $\tau_{r\theta}$), (u_r , u_θ), E and ν denote stress components in the polar coordinates, displacement components, the modulus of elasticity and Poisson's ratio, respectively. μ denotes the coefficient of friction. α denotes contact angle in deformed state.

Nondimensional stress and displacement components around the hole are defined as follows;

$$\begin{aligned} \bar{\sigma}_r &= \left(\frac{\sigma_r}{\sigma_0} \right)_{r=a}, \quad \bar{\sigma}_\theta = \left(\frac{\sigma_\theta}{\sigma_0} \right)_{r=a}, \quad \bar{\tau}_{r\theta} = \left(\frac{\tau_{r\theta}}{\sigma_0} \right)_{r=a} \\ \bar{u}_r &= \left(\frac{u_r}{a} \right)_{r=a} / \left(\frac{\sigma_0}{E} \right), \quad \bar{u}_\theta = \left(\frac{u_\theta}{a} \right)_{r=a} / \left(\frac{\sigma_0}{E} \right) \end{aligned} \quad \dots\dots\dots (1)$$

(a) Case of tensile loading

(b) Case of compressive loading

Fig.1 Infinite plate with a circular hole containing a rigid disk

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where σ_0 is uniform stress at infinity and a is radius of the hole.

Instead of Coulomb's law, a law of friction as shown in Fig.2 is used, because Coulomb's law gives non-zero value to shearing stress $\tau_{r\theta}$ at the central point of the contact surface. In this law, the coefficient of friction μ increases in proportion to the relative displacement $\bar{u}_\theta$ if $\bar{u}_\theta < \bar{u}_\theta^*$, and takes a constant value μ^* if $\bar{u}_\theta > \bar{u}_\theta^*$. The validity of this law has been confirmed experimentally (11)(12).

The method of stress analysis is essentially the same as used in the previous papers (7)(8). A general form of a stress function expressed in the polar coordinates is adjusted successively so as to satisfy boundary conditions. First, stress and displacement components around the hole are calculated from the stress function of the uniform tensile or compressive stress state in an infinite plate without a hole. Then, the stress state is corrected so that the boundary conditions ($\bar{u}_r = 0$, $\tau_{r\theta} = \mu \bar{\sigma}_r$) on the contact surface and those ($\bar{\sigma}_r = \tau_{r\theta} = 0$) on the non-contact surface are satisfied alternately. These corrections are continued until all of the boundary conditions are numerically satisfied to a desired accuracy. The angle α shown in Fig.1 should be always corrected when the stress state is modified. The distributions of stresses and displacements around the hole which should be corrected in each stage are expressed in the form of trigonometric series obtained by harmonic analysis, and thus the stress function for the distributions is determined.

The part of the stress function is denoted by Φ , which gives excess stresses above the uniform stress state of an infinite plate without a hole. Since stresses given by Φ should vanish at infinity and displacement and rotation should be uniquely determined around the hole, the general form (9)(10) of Φ is expressed by the following equation:

$$\Phi = a_0 \log r + a_0' \theta + a_1' r^{-1} \cos \theta + c_1' r^{-1} \sin \theta + \sum_{n=2}^{\infty} (a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta + \sum_{n=2}^{\infty} (c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta \quad (2)$$

where a_0 , a_0' , a_n' and d_n' are undetermined coefficients. The distributions of $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and $\bar{u}_r$ which are obtained from Eq.(2) are symmetric with respect to the x- and y-axes, and $\tau_{r\theta}$ and $\bar{u}_\theta$ are anti-symmetric. The general forms of stress and displacement components around the hole are expressed in nondimensional forms as follows;

$$\left. \begin{aligned} \bar{\sigma}_r &= \bar{a}_0 - \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + (n-1)(n+2)\bar{b}_n'\} \cos n\theta \\ \bar{\sigma}_\theta &= -\bar{a}_0 + \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + (n-1)(n-2)\bar{b}_n'\} \cos n\theta \\ \tau_{r\theta} &= - \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + n(n-1)\bar{b}_n'\} \sin n\theta \\ \bar{u}_r &= (1+\nu) \left[-\bar{a}_0 + \sum_{n=2}^{\infty} \left\{ n\bar{a}_n' + \left(n-2 + \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \cos n\theta \right] \\ \bar{u}_\theta &= (1+\nu) \sum_{n=2}^{\infty} \left\{ n\bar{a}_n' + \left(n - \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \sin n\theta \end{aligned} \right\} \quad (3)$$

where $\bar{a}_0$, $\bar{a}_n'$, $\bar{b}_n'$ ($n = 2, 4, 6, \dots$) are undetermined coefficients and are related to those in Eq.(2) by the following equations:

$$\bar{a}_0 = a_0 a^{-2} \sigma_0^{-1}, \quad \bar{a}_n' = a_n' a^{-n-2} \sigma_0^{-1}, \quad \bar{b}_n' = b_n' a^{-n} \sigma_0^{-1} \quad (4)$$

Fig.2 Relation between coefficient of friction and relative displacement

Now, the distributions of stress and displacement components around the hole are determined numerically by using Eq.(3) through the following stages of calculation.

(1) The stress and displacement components around the hole in the uniform tensile or compressive stress state of an infinite plate without a hole are,

$$\left. \begin{aligned} \bar{\sigma}_r^{00} &= \pm \frac{1}{2} \pm \frac{1}{2} \cos 2\theta, & \bar{\sigma}_\theta^{00} &= \pm \frac{1}{2} \mp \frac{1}{2} \cos 2\theta, & \tau_{r\theta}^{00} &= \mp \frac{1}{2} \sin 2\theta \\ \bar{u}_r^{00} &= \pm \frac{1-\nu}{2} \pm \frac{1+\nu}{2} \cos 2\theta, & \bar{u}_\theta^{00} &= \mp \frac{1+\nu}{2} \sin 2\theta \end{aligned} \right\} \quad (5)$$

where the upper and lower signs denote the cases of tensile loading and compressive loading, respectively. As an example, the distributions of $\bar{\sigma}_r^{00}$, $\tau_{r\theta}^{00}$, $\bar{u}_r^{00}$ for the case of tensile loading and $\nu = 0.3$ are indicated by full lines in Fig.3.

Fig.3 Distributions of $\bar{\sigma}_r^{00}$, $\tau_{r\theta}^{00}$ and $\bar{u}_r^{00}$ on $r = a$ (tensile loading)

Fig.4 Distributions of $\bar{\sigma}_r^{10}$ and $\bar{u}_r^{10}$ on $r = a$ (tensile loading)

Fig.5 Distributions of $\bar{\sigma}_r^{''}$ and $\bar{u}_r^{''}$ on $r = a$ (tensile loading)

(2) $\bar{\tau}_{r\theta}$ should be zero on the non-contact surface and $\bar{u}_r$ should be zero on the contact surface. But Eq.(5) does not always satisfy these conditions. Therefore $\bar{\tau}_{r\theta}^{00}$ and $\bar{u}_r^{00}$ must be modified. The distributions of $\bar{\tau}_{r\theta}$ and $\bar{u}_r$ shown by broken lines in Fig.3 should be added to those of $\bar{\tau}_{r\theta}^{00}$ and $\bar{u}_r^{00}$, respectively. The distributions of $\bar{\tau}_{r\theta}$ and $\bar{u}_r$ are expressed in the form of trigonometric series by harmonic analysis, and the stress function for these distributions is determined. The resulting stress components are denoted by $\bar{\sigma}_r^{10}$, $\bar{\sigma}_\theta^{10}$, $\bar{\tau}_{r\theta}^{10}$ and the corresponding displacement components by $\bar{u}_r^{10}$, $\bar{u}_\theta^{10}$. The distributions of $\bar{\sigma}_r^{10}$ and $\bar{u}_r^{10}$ are indicated by full lines in Fig.4.

(3) It is unreal that a positive radial stress exists along the hole-disk interface. The part where $\bar{\sigma}_r^{10}$ is positive might be the non-contact one, and then the stress state must again be modified so that $\bar{\sigma}_r^{10}$ vanishes there. The distribution of $\bar{\sigma}_r$ indicated by the broken line in Fig.4(a) should be added to that of $\bar{\sigma}_r^{10}$, whilst $\bar{\tau}_{r\theta}^{10}$ is kept equal to zero. The procedures of this modification are the same as those in the stage (2). The corrected stress and displacement states are expressed by $\bar{\sigma}_r^{11}$, $\bar{\sigma}_\theta^{11}$, $\bar{\tau}_{r\theta}^{11}$, $\bar{u}_r^{11}$, $\bar{u}_\theta^{11}$. The distributions of $\bar{\sigma}_r^{11}$ and $\bar{u}_r^{11}$ are shown in Fig.5.

(4) It may occur in the resulting state that the radial displacement is negative along the interface, but it is unreasonable. Therefore, $\bar{u}_r^{11}$ must be modified so that the radial displacement vanishes on the part with negative $\bar{u}_r^{11}$. $\bar{\tau}_{r\theta}^{11}$ must be corrected simultaneously so that $\bar{\tau}_{r\theta}^{11}$ is equal to $\mu\bar{\sigma}_r^{11}$ on the same part. The corrected state is expressed by $\bar{\sigma}_r^{21}$, $\bar{\sigma}_\theta^{21}$, $\bar{\tau}_{r\theta}^{21}$, $\bar{u}_r^{21}$, $\bar{u}_\theta^{21}$.

(5) $\bar{\sigma}_r^{21}$ and $\bar{\tau}_{r\theta}^{21}$ are not equal to zero on the non-contact part after the correction in the stage (4). The state is, therefore, to be remodified so that the radial stress and the shearing stress vanish on the part.

(6) The modifications of the state in the stages (4) and (5) are repeated until all of the boundary conditions around the hole are satisfied to a desired accuracy.

The procedures of modification in the case of compressive loading are quite similar to those in tensile loading described above.

3. Numerical results

Numerical calculations are carried out under the condition that Poisson's ratio is 0.3 and μ^* shown in Fig.2 is 0.1 or 0.3. The stress and displacement distributions around the hole are calculated for the cases of tensile loading and compressive loading, giving various values to $\bar{u}_\theta^*$ shown in Fig.2. The results in the case of $\mu^* = 0.3$, $\bar{u}_\theta^* = 0.04$ under the conditions of tensile loading and compressive loading are shown in Figs.6 and 7, respectively. The solutions for the case of $\mu^* = 0$ given in the previous paper⁽⁷⁾ are also indicated by broken lines in these figures. The curves of stress and displacement for the case of $\mu^* = 0.1$ are closer to those for $\mu^* = 0$ than those for $\mu^* = 0.3$. No remarkable change is noticed in the stress and displacement distributions around the hole except in the neighbourhood of the central point of the contact surface, even if various values are given to $\bar{u}_\theta^*$ in the range between 0.01 and 0.4 for the case of tensile loading and between 0.01 and 0.15 for compressive loading. Table 1 gives the maximum stress, the maximum displacement and the contact angle for the case when the value of $\bar{u}_\theta^*$ is changed between 0.01 and 0.15 under the condition of tensile loading. Table 2 gives those for the case when it is changed between 0.01 and 0.06 under compressive loading. If $\bar{u}_\theta^*$ is larger than the above-mentioned value, the maximum stress and displacement are somewhat different from those shown in these tables. In the case of tensile loading $|\bar{\sigma}_r|_{\max}$ increases by about 2% and $|\bar{\tau}_{r\theta}|_{\max}$ decreases by about 7% if $\bar{u}_\theta^*$ increases from 0.15 to 0.4. In compressive loading $|\bar{\sigma}_r|_{\max}$ increases by about 4% and $(\bar{\tau}_{r\theta})_{\max}$ decreases by about 7% if $\bar{u}_\theta^*$ increases from 0.06 to 0.15. And the change of the other values in these tables is below 2% in both cases.

4. Considerations

The effects of friction on the distributions of stresses and displacements around the hole are discussed.

(a) Stress

(b) Displacement

Fig.6 Stress and displacement distributions around hole (tensile loading, $\mu^*=0.3$, $\bar{u}_\theta^*=0.04$)

Table 1 Maximum stress, maximum displacement and contact angle α (tensile loading, $\bar{u}_\theta^*=0.01 \sim 0.15$)

	$\mu^*=0$	$\mu^*=0.1$	$\mu^*=0.3$
$(\bar{\sigma}_\theta)_{\max}$	2.91	2.99	3.18
$ \bar{\sigma}_r _{\max}$	1.07	1.01	0.92
$ \bar{\tau}_{r\theta} _{\max}$	0.00	0.10	0.27
$(\bar{u}_r)_{\max}$	2.82	2.82	2.81
$ \bar{u}_\theta _{\max}$	1.66	1.66	1.65
α	70.4°	70.3°	69.0°

(1) Case of tensile loading

(a) The distribution of stresses in the range of $\theta > 60^\circ$ is affected by friction. In comparison with the frictionless contact ($\mu^*=0$), $(\bar{\sigma}_\theta)_{\max}$ increases by 3% for the case of $\mu^*=0.1$ and 9% for $\mu^*=0.3$.

(b) The effect of friction on the distribution of $\bar{u}_r$ is hardly discernible in Fig.6(b), though $\bar{u}_\theta$ is slightly affected.

(2) Case of compressive loading

(a) The effects of friction on the distributions of stresses and displacements are found in the whole range and are greater in compressive loading than in tensile loading.

(b) $|\bar{\sigma}_r|_{\max}$ is smaller by 9% in the case of $\mu^*=0.1$ and 20% in $\mu^*=0.3$ than in the friction-less contact.

(a) Stress

(b) Displacement

Fig.7 Stress and displacement distributions around hole (compressive loading, $\mu^*=0.3$, $\bar{u}_\theta^*=0.04$)

Table 2 Maximum stress, maximum displacement and contact angle α (compressive loading, $\bar{u}_\theta^*=0.01 \sim 0.06$)

	$\mu^*=0$	$\mu^*=0.1$	$\mu^*=0.3$
$ \bar{\sigma}_\theta _{\max}$	1.29	1.25	1.13
$ \bar{\sigma}_r _{\max}$	2.12	1.94	1.69
$(\bar{\tau}_{r\theta})_{\max}$	0.00	0.20	0.51
$(\bar{u}_r)_{\max}$	0.46	0.41	0.32
$(\bar{u}_\theta)_{\max}$	0.68	0.62	0.49
α	53.1°	54.9°	58.9°

(c) $(\bar{u}_r)_{\max}$ and $(\bar{u}_\theta)_{\max}$ also decrease by 10% in the case of $\mu^*=0.1$ and by 30% in $\mu^*=0.3$.

The maximum stress is found in $\bar{\sigma}_\theta$ for the case of tensile loading and in $|\bar{\sigma}_r|$ for compressive loading as shown in Tables 1 and 2. In the case of tensile loading the maximum stress occurs at the point of $\theta = \alpha$, i.e. at the boundary point between the contact part and the non-contact part and in compressive loading it occurs at the point of $\theta = 0$, i.e. at the central point of the contact part. The points where the maximum stress occurs, agree with those obtained in the case of $\mu^*=0$.

In the numerical analysis which yielded the above results, 200 terms are used in the

Fourier series. The validity of this treatment has been examined in the previous paper (7). When the corrections of stress and displacement on the contact surface and those on the non-contact surface are repeated four times in the case of tensile loading and five times in compressive loading, all of the boundary conditions around the hole are satisfied to a desired accuracy. From the same reasoning about the convergence of solution as in the previous paper (7), it is estimated that errors of the maximum stress and displacement in Tables 1 and 2 are within 0.5%. Since the curves of distribution of $\bar{\sigma}_r$, $\bar{\tau}_{r\theta}$ and $\bar{u}_r$ have a sharp point in the neighbourhood of the end of the contact surface, errors of their distributions expressed in the Fourier series take maximum values there. The maximum errors of $\bar{\sigma}_r$, $\bar{\tau}_{r\theta}$ and $\bar{u}_r$ are estimated within 0.7%, 0.3% and 1.5% as compared with the maxima of $\bar{\sigma}_r$, $\bar{\tau}_{r\theta}$ and $\bar{u}_r$, respectively.

The law of friction shown in Fig.2 is used in order to remove some difficulty of Coulomb's law appearing at the central point of the contact surface. This law is partially supported by experiments in surface engineering⁽¹¹⁾⁽¹²⁾, in which the critical relative displacement u_{θ}^* is found to be approximately constant for a particular material. Thus, $\bar{u}_{\theta}^*$, the nondimensional quantity of u_{θ}^* is not always constant. However, it is confirmed in the present results that the effect of $\bar{u}_{\theta}^*$ on the maximum stress and the maximum gap $(\bar{u}_r)_{\max}$ between the hole and the disk is very small although $\bar{u}_{\theta}^*$ changes in a wide range. Thus the results in Tables 1 and 2 are considered to be valid generally irrespective of $\bar{u}_{\theta}^*$ if the law of friction shown in Fig.2 is applied.

The mechanical phenomena involving friction are not reversible and have loading path dependence and nonlinearity. But the problem treated in this paper is special because all equations expressed in nondimensional forms do not contain the load σ_0 , and then the point of the boundary between the contact surface and the non-contact surface around the hole is obtained irrespective of σ_0 if the physical constants μ^* and $\bar{u}_{\theta}^*$ are independent of σ_0 . Thus the treatment used here is tenable. However it is evident that the present solutions are valid only in the case when the load increases monotonically within small deformation and are not applicable if an unloading process is involved.

5. Conclusions

The plane stress problem of an infinite plate with a circular hole containing a rigid disk under uniaxial tension or compression is numerically solved for the case when friction exists in the hole-disk interface. It is found that the numerical method described in the previous papers (7)(8) for the case when there exists no friction is also applicable when friction exists. The distributions of stresses and displacements around the hole are discussed in relation to friction property.

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