532.542:532.517

Paper No. 208-6

Transition to Turbulence in a Pulsatile Pipe Flow

Part 2, Characteristics of Reversing Flow

Accompanied by Relaminarization*

By Manabu IGUCHI** and Munekazu OHMI***

The instantaneous velocity distributions and pressure gradients in a reversing pulsatile flow and an oscillatory flow in which turbulent bursts follow by relaminarization in the same cycle are investigated. They are predicted with sufficient accuracy by the theory for a transient pulsatile laminar flow in the laminar phase. Meanwhile, in the phase where turbulence with higher frequency appears, they are well approximated by the well-known $1/7$ power law and the turbulent quasi-steady friction law respectively.

Key Words : Unsteady Flow, Pulsatile Pipe Flow, Transition to Turbulence, Relaminarization, Oscillatory Flow, Reversed Flow.

1. Introduction

Experimental results of wave forms and distributions of axial velocity components typical of transition to turbulence in a pulsatile pipe flow having a dimensionless frequency ω' of about 17^2 which is realized in a human aorta are shown in the first paper⁽¹⁾ of this research series. Especially, the change in the amplitude and phase of axial velocity component with time-averaged Reynolds number was discussed in detail. In some findings obtained there a relaminarization of flow is of particular interest. When a relaminarization of flow occurs in a pulsatile flow, the time-averaged distribution of axial velocity components follows neither the parabolic nor the $1/7$ power law velocity distribution and it becomes more blunt than the latter distribution. The reason for this phenomenon was not clear at that time.

This paper concerns a detailed examination of instantaneous axial velocity distributions in an oscillatory pipe flow ($Re_{ta} = 0$) and a pulsatile pipe flow for the purpose of clarifying the mechanism and characteristic of relaminarization which is likely to occur in the case where a reversing flow appears⁽¹⁾⁽²⁾. It becomes evident that the velocity at every radial position becomes zero at almost the same time after the occurrence of turbulent bursts and the turbulence decays out when the flow alters its direction both in an

oscillatory flow and in a pulsatile flow with a reversing flow. This phenomenon suggests that the velocity distribution in the laminar phase of a cycle in these flows can be described by the solution for transient laminar pulsatile flow and it follows the $1/7$ power law distribution in the turbulent phase. As expected the agreement of calculated with experimental results is satisfactory.

Additionally, the pressure gradient measurements are made by means of a semiconductor type difference pressure transducer. The comparison of experimental results with predicted ones based on the above mentioned transient laminar theory shows a good agreement in the laminar phase. Meanwhile in the phase where turbulence appears Blasius' friction law is applicable with sufficient accuracy.

2. Nomenclature

Main symbols used in this paper are listed as follows:

A_1 : velocity amplitude ratio = $|u_{m,os,1}| / u_{m,ta}$
 D : pipe diameter
 l : length of test section
 p : pressure
 Δp : pressure drop
 $\Delta p/l$: pressure gradient
 R : radius of pipe
 Re : instantaneous Reynolds number
 $= u_m D / \nu$
 Re_{os} : Reynolds number based on the amplitude of oscillatory component of cross-sectional mean velocity
 $= |u_{m,os,1}| D / \nu$
 Re_{ta} : time-averaged Reynolds number
 $= u_{m,ta} D / \nu$
 r : radial distance
 $r' = r/R$
 s : Laplace operator
 t : time
 $t' = t\nu/R^2$

* Received 16th October, 1980.

** Assistant, Faculty of Engineering, Osaka University, Yamadaoka 2-1, Suita, Osaka.

*** Professor, Faculty of Engineering, Osaka University.

- u : axial velocity component
 u_m : cross-sectional mean velocity
 x : axial distance
 α_i : i th zero of the first kind Bessel function of order 2, $J_2(z)$
 $\lambda_{g,t}$: turbulent quasi-steady friction factor = $0.3164/|Re|^{1/4}$
 μ : viscosity
 ν : kinematic viscosity
 ρ_0 : density
 τ_w : wall shear stress
 ω : angular frequency
 ω' : dimensionless frequency $\equiv R^2\omega/\nu$
 Subscripts and others
 os : oscillatory component
 ta : time-averaged value
 1 : fundamental wave in the finite Fourier expansion
 $\wedge$: quantities in the Laplace domain

3. Solution for transient pulsatile laminar flow when the flow rate, i.e., cross-sectional mean velocity is given

3.1 Theoretical fundamentals of transient pulsatile laminar flow

According to the theory for a locally as well as temporarily fully developed oscillatory pipe flow of an incompressible fluid,⁽³⁾ the molecular viscosity of fluid comes to play an important role in a thin layer very adjacent to the wall and hence the fluid in a central core region comes to oscillate like a solid body as the dimensionless frequency ω' increases. This phenomenon is known as "Richardson's annular effect" in honor of the discoverer of this effect. When turbulent bursts occur near the wall at a certain phase of a cycle in such an oscillatory pipe flow, the turbulence does not remain in its generation region but propagates along the pipe axis. Velocity wave forms measurements⁽²⁾⁽⁴⁾ support the propagation of turbulence from the vicinity of wall to the core region. As a result of this propagation the eddy viscosity ϵ influences the flow over the whole cross-section of a pipe.

In addition, the fact that the overall magnitude of eddy viscosity is larger than ten times that of molecular viscosity in a steady flow⁽⁵⁾ suggests a vanishingly small phase difference between oscillatory fluids near the wall and in the core region. In other words, the fluid in every section of a pipe oscillates in almost the same phase during the appearance of turbulent bursts and consequently the instantaneous axial velocity component vanishes at almost the same time when the flow rate, i.e., the instantaneous cross-sectional mean velocity alters its sign.

Turbulence with higher frequency decays out when the flow alters its direction due mainly to viscous dissipation and then the flow turns laminar. Therefore, in the oscillatory flow accompanied by transition to a turbulent flow and the following re-transition to a laminar flow in a half cycle, it can be considered that the fluid forgets its history at flow reversal and subsequently is accelerated under the laminar state till the phase when turbulent

bursts first occur. The same phenomenon as above may be observed in a pulsatile pipe flow accompanied by a reversing flow as well as onset of relaminarization. This possibility is realized experimentally as seen in Figs. 4 and 5 shown later.

3.2 Solution for the transient laminar pulsatile pipe flow

The equation of motion of an incompressible fluid in a locally fully developed laminar unsteady pipe flow can be described as follows:⁽⁶⁾

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (1)$$

Substitution of the expression for $\partial p/\partial x$ obtained by integrating Eq. (1) over a cross-section of a pipe into Eq. (1) gives a basic differential equation of motion. By performing the Laplace transformation of this equation with initial condition of zero axial velocity and then applying the boundary conditions of the form

$$u = 0 \text{ at wall, } \partial u/\partial r = 0 \text{ at center line}$$

an axial velocity component u can be expressed in the Laplace domain as follows:⁽⁷⁾⁽⁸⁾

$$\hat{u} = 2\hat{u}_m(1-r^2/R^2) + \sum_{i=1}^{\infty} \hat{u}_m \frac{s}{s + \alpha_i^2 \nu/R^2} \frac{4}{\alpha_i^2} \left\{ \frac{J_0(\alpha_i r/R)}{J_0(\alpha_i)} - 1 \right\} \quad (2)$$

where α_i is the i th zero of the first kind Bessel function of order 2, $J_2(z)$.

In the present paper an analysis will be made of the case when the cross-sectional mean velocity u_m is expressed in the form

$$u_m = \begin{cases} 0 & (t < 0) \\ u_{m,ta}(A + B \sin \omega t + C \cos \omega t) & (t \geq 0) \end{cases} \quad (3)$$

Here, we have $A = C = 0$ and $u_{m,ta}B = |u_{m,os,1}|$ for an oscillatory pipe flow and $A = 1$ for a pulsatile pipe flow. Inverse transformation of the expression derived from the substitution of the Laplace transformation of Eq. (3) into Eq. (2) yields

$$\begin{aligned} u/u_{m,ta} = & 2(1-r^2/R^2)(A + B \sin \omega' t' + C \cos \omega' t') \\ & + \sum_{i=1}^{\infty} \frac{4}{\alpha_i^2} \left\{ \frac{J_0(\alpha_i r')}{J_0(\alpha_i)} - 1 \right\} \left[\frac{B\omega' \alpha_i^2 + C\omega'^2}{\alpha_i^4 + \omega'^2} \cos \omega' t' \right. \\ & \left. + \frac{B\omega'^2 - C\omega' \alpha_i^2}{\alpha_i^4 + \omega'^2} \sin \omega' t' + \left\{ A + \frac{-B\omega' \alpha_i^2 + C\alpha_i^4}{\alpha_i^4 + \omega'^2} \right\} e^{-\alpha_i^2 t'} \right] \end{aligned} \quad (4)$$

From Eq. (4) the wall shear stress, $\tau_w = -\mu(\partial u/\partial r)_{r=R}$, can be calculated to give

$$\begin{aligned} \tau_w/(2\mu u_{m,ta}/R) = & 2(A + B \sin \omega' t' + C \cos \omega' t') \\ & + \sum_{i=1}^{\infty} \left[\frac{B\omega' \alpha_i^2 + C\omega'^2}{\alpha_i^4 + \omega'^2} \cos \omega' t' + \frac{B\omega'^2 - C\omega' \alpha_i^2}{\alpha_i^4 + \omega'^2} \sin \omega' t' \right. \\ & \left. + \left\{ A + \frac{-B\omega' \alpha_i^2 + C\alpha_i^4}{\alpha_i^4 + \omega'^2} \right\} e^{-\alpha_i^2 t'} \right] \quad (5) \end{aligned}$$

In calculation the series terms up to $i = 100$ are chosen.

The values of $|u_{m,os,1}|$, B , and C are determined from the finite Fourier expansion of a cross-sectional mean velocity which is obtained by setting $t = 0$ at the phase indicated by arrow in every (b) of Figs.1 through 6.

4. Comparison of experimental with calculated results and discussion

4.1 Experimental apparatus and procedure

Data presented here on an oscillatory and a pulsatile flow are the same as those reported in Refs.(1), (2), and (9). For details of the experimental apparatus and data processing procedure, the reader may refer to these references. Pressure gradient can be determined by applying the least squares method to measured values of pressure drop obtained by means of two semiconductor type difference pressure transducers. One is fixed to a certain measuring station of the test section and the other is set at 50 cm intervals along the test pipe.

A list of experimental conditions is seen in Table 1. Here, $\omega' = R^2\omega/\nu$ is the dimensionless frequency, $Re_{ta} = u_{m,ta}D/\nu$ the time-averaged Reynolds number, $Re_{os} = |u_{m,os,1}|D/\nu$ the Reynolds number based on the amplitude of oscillatory velocity component, and $A_1 = |u_{m,os,1}|/u_{m,ta}$ the velocity amplitude ratio. Regions [I], [II], and [III] denote the flows classified in the previous paper⁽¹⁾ based on the observation of velocity wave forms. These are summarized as follows:

- Region [I] : <laminar flow over one cycle>
- Region [II] : <low frequency and small amplitude disturbances appear in the accelerating phase in the core region or in the phase of higher velocity>
- Region [III] : <turbulence with higher frequency appears>

4.2 Velocity distribution

4.2.1 Oscillatory pipe flow

Examples accompanied by the occurrence of turbulent bursts in the decelerating phase on the velocity wave forms are chosen for three different dimensionless frequencies. Velocity wave forms, the cross-sectional mean velocity, and the distribution of axial velocity component in every example are shown in (a), (b), and (c) of Figs.1 through 3, respectively.

Table 1 A list of experimental conditions

Run	$\sqrt{\omega'}$	Re_{ta}	Re_{os}	A_1	region
1(47)*	4.29	0	9 850		[III]
2(51)*	7.93	0	20 600		[III]
3(55)*	11.24	0	21 800		[III]
8**	16.48	8 330	11 900	1.43	[II]
13**	16.48	10 900	19 700	1.81	[III]
14**	16.48	17 600	18 600	1.06	[III]

* run number corresponds that in Ref.(2)

** run number corresponds that in Ref.(1)

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8$, and 0.95

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow, ---: 1/7 power law)

(d) Fully developed axial velocity distribution (—: solution for temporarily fully developed pulsatile laminar flow)

Fig.1 Comparison of measured with theoretical results in an oscillatory pipe flow (Run 1)

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8,$ and 0.95

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow, ---: 1/7 power law)

(d) Fully developed axial velocity distribution (—: solution for temporarily fully developed pulsatile laminar flow)

Fig.2 Comparison of measured with theoretical results in an oscillatory pipe flow (Run 2)

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8,$ and 0.95

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow, ---: 1/7 power law)

(d) Fully developed axial velocity distribution (—: solution for temporarily fully developed pulsatile laminar flow)

Fig.3 Comparison of measured with theoretical results in an oscillatory pipe flow (Run 3)

From every (a) of Figs.1 through 3, it can be said that the time at which the flow alters its direction at every radial measuring position is almost the same. Consequently it can be concluded that the flows shown in Figs.1 through 3 possesses the properties to which the theory for transient pulsatile laminar flow applies.

The solid line in every (b) of Figs.1 through 3 denotes the finite Fourier expansion of cross-sectional mean velocity having components up to fundamental wave and is expressed in the form

$$u_m = |u_{m,os,1}| \cos(\omega t + \angle u_{m,os,1}) \dots\dots\dots (6)$$

The numerical value of Eq.(6) becomes zero twice in a cycle. Therefore, Eq.(4) must be calculated starting from the phase indicated by arrow. Here, it should be noted that the origin of abscissa ($t = 0$) in every (b) of Figs.1 through 3 is different from the instant of $t = 0$ in calculation of Eq.(4). The phase at which Eq.(4) is valid is represented by "L.Tr." (Laminar Transient) and the remaining phase is represented by "Tu." (Turbulent) because the turbulent quasi-steady relation is valid there.

Comparison of experimental with calculated results is shown in every (c) of Figs.1 through 3. A broken line is the 1/7 power law calculated by

$$u = (60/49) u_m (1 - r')^{1/7} \dots\dots\dots (7)$$

In this case, measured values of cross-sectional mean velocity should be substituted for u_m in Eq.(7). Comparison of (a) with (c) in every run reveals that the velocity distribution in the phase where turbulence does not appear (e.g., $\omega t/(\pi/6) = 2.5 \sim 5$ and $8.5 \sim 11$ in Fig.1) can be well represented by the solution for a transient pulsatile laminar flow, while in the phase where turbulence appears (e.g., $\omega t/(\pi/6) = 0 \sim 2.5$, $5 \sim 8.5$, and $11 \sim 12$ in Fig.1) it is well approximated by the 1/7 power law. The latter finding coincides with that previously reported by the authors⁽²⁾.

In every (d) of Figs.1 through 3 a temporarily fully developed velocity distribution for a laminar oscillatory pipe flow is shown for the case when u_m changes in a manner as shown by the solid line in every (b). This distribution can be derived from Eq.(4) by neglecting the transient term, i.e., the last term on the right hand side. A difference between transient and fully developed distributions is recognized in every example.

4.2.2 Pulsatile flow

Run 13 and Run 14 are cases when a reversing flow as well as a turbulent bursts occurs. Although the wave forms in these runs are already shown in Ref.(1), they are again shown. Both (a) of Figs.4 and 5 show that the time when the flow at every radial position alters its direction after the occurrence of turbulent bursts is almost the same over the whole cross-section of a pipe. In addition, though a large departure of wave form of axial velocity component u

from sinusoidal wave form can be seen, measured values of u_m are well approximated by the finite Fourier expansion having fundamental wave as seen in both (b). These facts guarantee the applicability of the solution for a transient pulsatile laminar flow to Run 13 and Run 14. The phase indicated by arrow denotes the time of $t = 0$ in calculation of Eq.(4). In the same manner as an oscillatory pipe flow, the phase where Eq.(4) and the turbulent quasi-steady relation are approximately valid are represented by "L.Tr." and "Tu.", respectively.

Every (c) of Figs.4 and 5 illustrates good agreement between experiments and calculations by Eq.(4) in the laminar phase. That is, the velocity distribution in the laminar phase [$\omega t/(\pi/6) = 0 \sim 1$ and $3.5 \sim 12$ in Fig.4 and $\omega t/(\pi/6) = 4.5 \sim 12$ in Fig.5] follows the transient laminar pulsatile flow theory with sufficient accuracy. Meanwhile that in the turbulent phase [$\omega t/(\pi/6) = 1 \sim 3.5$ in Fig.4 and $\omega t/(\pi/6) = 0 \sim 4.5$ in Fig.5] is well approximated by the well-known 1/7 power law velocity distribution. These results can explain the reason for the published finding that the time-averaged velocity distributions in Run 13 and Run 14 are appreciably different from both the parabolic and the 1/7 power law velocity distributions. In Fig.4(d) a temporarily fully developed velocity distribution calculated from Eq.(4) by neglecting the transient term is shown. It is obviously different from transient velocity distribution.

Since the present transient pulsatile laminar flow theory is derived assuming that the flow necessarily alters its direction and that the phase at which the flow alters its direction is equal over the whole cross-section of a pipe, if these conditions are satisfied no matter what the reason is, the present transient theory is applicable. Fortunately, a flow having wave forms which meet the above mentioned requirements can be observed in Run 8 (see Table 1). Figure 6 shows the comparison of above theory with measured velocity distribution of Run 8. In Fig.6(c) the measured velocity distribution at every phase except $\omega t/(\pi/6) = 8$ and 9 agrees favourably with the solution for a transient pulsatile laminar flow. It must be added that the fully developed velocity distribution which is not shown here is obviously different from the measured distribution.

4.3 Pressure gradient

The momentum integral equation can be expressed for both the laminar and the turbulent flows in the form⁽¹⁰⁾

$$\frac{\Delta p}{l} = \rho_0 \frac{du_m}{dt} + \frac{4}{D} \tau_w \dots\dots\dots (8)$$

From the left to the right hand side of Eq.(8) in order the authors named the pressure gradient term, inertia term, and viscous term, respectively.⁽¹¹⁾ As the change in u_m is known beforehand, it is sufficient to measure $\Delta p/l$ for the evaluation of wall shear

stress τ_w .

In what follows, experimental results of Run 13 and Run 14 will be shown. From the consideration in section 4.2 it is reasonable to say that the following turbulent quasi-steady relations are applicable to the phase where turbulence appears ⁽¹⁰⁾

$$\tau_w = \lambda_{q,t} \rho_0 u_m |u_m| / 8 \quad (9)$$

$$\lambda_{q,t} = 0.3164 / |Re|^{1/4}, \quad Re = u_m D / \nu \quad (10)$$

and in the other phase Eq.(5) is valid. In Eqs.(9) and (10) $\lambda_{q,t}$ is the turbulent quasi-steady friction factor and the absolute value of u_m is utilized to include the reversing flow. Figures 4(e) and 5(d) show the comparison of calculated with measured values of pressure gradient. Referring to the wave forms in both (a) of Figs.4 and 5, it can be considered that the pressure gradient in the phase where the occurrence of turbulent bursts is not observed [e.g., $\omega t / (\pi/6) = 0 \sim 1$ and $3.5 \sim 12$ in Fig.4 and $\omega t / (\pi/6) = 4.5 \sim 12$ in Fig.5] can be approximated by the theory for a transient pulsatile laminar flow. Meanwhile in the phase where turbulent bursts occur (e.g., $\omega t / (\pi/6) = 1 \sim 3.5$ in Fig.4 and $\omega t / (\pi/6) = 0 \sim 4.5$ in Fig.5) the turbulent quasi-steady relation which is represented by the broken line is valid.

In Fig.5(d) a temporarily fully developed laminar distribution is illustrated by the dotted line. Although the difference between the solid and dotted lines is

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8, 0.95$, and 0.975

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow, ---: 1/7 power law)

(d) Fully developed axial velocity distribution (—: solution for temporarily fully developed pulsatile laminar flow)

(e) Change in pressure gradient over one cycle (o: measured, —: solution for transient pulsatile laminar flow, ---: turbulent quasi-steady)

Fig.4 Comparison of measured with theoretical results in a pulsatile pipe flow (Run 13)

slight in $\omega t/(\pi/6) = 4.5 \sim 12$ the solution for a transient pulsatile laminar flow is adequate in a judgement based on the consideration of velocity distribution. Figure 5(e) shows the change in viscous term over one cycle. Experimental values agree well with the solution for a transient pulsatile laminar flow in the laminar phase and the turbulent quasi-steady relation in the turbulent phase.

5. Conclusions

(1) In a pulsatile flow which not only alters its direction but also is accompanied by a relaminarization in one cycle, the fluid begins to be accelerated under the laminar state from rest at almost the same time over the whole cross-section of a pipe and suddenly bursts into chaotic motion. Such a series of events are repeated in

every cycle. This phenomenon can be observed appreciably in an oscillatory pipe flow with zero mean component whose Reynolds number Re_{os} is larger than about $800\sqrt{\omega}$. Therefore, the discussion was first made on an oscillatory pipe flow and next on a pulsatile pipe flow. The results obtained in this investigation are summarized as follows:

(i) In an oscillatory pipe flow accompanied by relaminarization, the velocity distribution in the phase where turbulence does not appear can be well represented by the solution for a transient pulsatile laminar flow, Eq.(4), while in the phase where turbulent bursts occur it follows approximately the well-known 1/7 power law.

(ii) In a pulsatile flow in which reversing as well as relaminarization of flow occurs in one cycle, the velocity distribution and pressure gradient in the laminar

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8, 0.95$, and 0.975

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow, ---: 1/7 power law)

(d) Change in pressure gradient over one cycle (o: measured, —: solution for transient pulsatile laminar flow, ---: turbulent quasi-steady,: solution for temporarily fully developed pulsatile laminar flow)

(e) Change in viscous term $4\tau_w/D$ over one cycle (o: measured, —: solution for transient pulsatile laminar flow, ---: turbulent quasi-steady,: solution for temporarily fully developed pulsatile laminar flow)

Fig.5 Comparison of measured with theoretical results in a pulsatile pipe flow (Run 14)

(a) Velocity wave forms from top to bottom measured at $r/R = 0, 0.3, 0.6, 0.8, 0.95$, and 0.975

(b) Cross-sectional mean velocity (o: measured, —: finite Fourier expansion having fundamental wave)

(c) Axial velocity distribution (—: solution for transient pulsatile laminar flow)

Fig.6 Comparison of measured with theoretical results in a pulsatile pipe flow (Run 8)

phase agree favourably with the results calculated from Eq.(4) and from the combination of Eqs.(5) and (8), respectively. In the phase where turbulence appears the velocity distribution follows the $1/7$ power law and the pressure gradient can be calculated readily from the momentum integral equation and the turbulent quasi-steady representation for wall shear stress, Eqs. (9) and (10).

(2) There is a case where the axial velocity component vanishes at almost the same time over the whole cross-section of a pipe even if a turbulence with higher frequency does not appear. The solution for a transient pulsatile laminar flow is applicable to most of phases after the time at which the flow alters its direction.

References

- (1) Ohmi, M., et al., *Prepr. of Jpn. Soc. Mech. Eng.* (in Japanese), No. 800-17 (1980), p. 244; Ohmi, M., et al., *Bull. JSME*, Vol.25, No.200 (1982), p. 182.
- (2) Ohmi, M., et al., *Prepr. of Jpn. Hydraulics & Pneumatics Soc.* (in Japanese), (1979-11), p. 1; Ohmi, M., et al., *Bull. JSME*, Vol.25, No.201(1982) p. 365.
- (3) Sexl, T., *Z. Phys.*, Vol. 61, No. 6/7 (1930), p. 349.
- (4) Hino, M., et al., *J. Fluid Mech.*, Vol. 75, Part 2 (1976), p. 193.
- (5) Ohmi, M. and Usui, T., *Trans. Jpn. Soc. Mech. Eng.* (in Japanese), Vol. 41, No. 347 (1975), p. 2030.
- (6) Uchida, S., *Z. Angew. Math. u. Phys.*, Vol. 7, No. 5 (1956), p. 403.
- (7) Yokota, S. and Nakano, K., *J. Jpn. Hydraulics & Pneumatics Soc.* (in Japanese), Vol. 11, No. 2 (1980), p. 41.
- (8) Muto, T. and Nakane, K., *Trans. Jpn. Soc. Mech. Eng.* (in Japanese), Vol. 46, No. 404, B (1980), p. 610.
- (9) Ohmi, M., et al., *Prepr. Jpn. Soc. Mech. Eng.* (in Japanese), No. 790-15 (1979), p. 106.
- (10) Ohmi, M., et al., *Prepr. Jpn. Hydraulics & Pneumatics Soc.* (in Japanese), (1979-11), p. 5; Ohmi, M., et al., *Bull. JSME*, Vol. 25, No. 202 (1982), p.536.
- (11) Ohmi, M., et al., *Trans. Jpn. Soc. Mech. Eng.* (in Japanese), Vol.46, No. 404, B (1980), p. 619; Ohmi, M., et al., *Bull. JSME*, Vol. 23, No. 186 (1980), p. 2013.