On the Shaft End Torque and the Unstable Vibrations of an Asymmetrical Shaft Carrying an Asymmetrical Rotor

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In a rotating asymmetrical shaft carrying an asymmetrical rotor, there occur two types of unstable regions. These unstable regions change with the orientation angle \( \zeta \) between the inequality of shaft stiffness and that of rotor inertia. The conditions under which unstable vibrations occur just as input energy into the rotating shaft system tends to increase the whirling amplitudes of the shaft, can be clearly ascertained. These conditions, which depend on the angle \( \zeta \), tend to show a good coincidence with those obtained by an analog computer. Moreover, when asymmetrical shaft stiffness \( A_H \) and asymmetrical rotor inertia \( A_J \) are chosen suitably, two types of unstable regions are eliminated.

Key Words: Vibration of Rotating Body, Asymmetrical Shaft, Asymmetrical Rotor, Unstable Vibration, Torque Applied to Shaft End, Negative Damping Coefficient

1. Introduction

Two types of unstable vibrations occur in a rotating asymmetrical shaft having an asymmetrical rotor in Refs. 1-36). The authors have reported analytical and experimental results in Ref. 2 according to which the width of these unstable regions and the magnitude of the negative damping coefficients change with the orientation angle between the inequality of shaft stiffness and that of rotor inertia; that is, the width of the statically unstable region becomes narrower as the orientation angle \( \zeta \) increases from zero to \( \pi / 2 \), and on the other hand the width of the dynamically unstable one becomes greater as \( \zeta \) increases.

In the present study, the condition is obtained, under which input energy supplied at the shaft end increases the whirling amplitudes of the shaft, so that these two types of unstable vibrations occur. The necessary condition for occurrence of instability depends on the angle \( \zeta \), and the analytical results agree qualitatively with the experimental ones obtained in the previous study. Furthermore, when inertia asymmetry \( A_J \) and stiffness asymmetry \( A_H \) are combined suitably, the condition under which the unstable region vanishes is realized. It is ascertained that the solutions of the unstable vibration obtained with an analog computer satisfy the necessary condition for instability.

2. Equations of motion

The principal moments of inertia about three principal axes of inertia \( SX \), \( SY \), and \( SZ \) through the center \( S \) of an asymmetrical rotor in Fig. 1 are denoted by \( I_1 \), \( I_2 \) and \( I_3 \), respectively, and \( I_4 = (I_1 + I_2) / 2 \), and \( I_5 = (I_1 - I_2) / 2 \). The mass of the rotor is \( m \), but the mass of the shaft is assumed to be negligibly small. Let the bearing center line be \( Oz \). The rectangular coordinate system \( O-xyz \) is parallel to the rectangular coordinate system \( S-XYZ \); thus, \( yz \)-plane coincides with \( XY \)-plane. The rectangular coordinate systems \( S-X_1Y_1Z_1 \) and \( S-X_2Y_2Z_2 \) are fixed to the rotor, the angular positions of which are denoted by Eulerian angles \( \theta \), \( \varphi \) and \( \psi \). The rectangular coordinate system \( S-NKZ \) is obtained by rotating the rectangular system \( S-XYZ \) about the vertical axis \( SZ \) by \( \varphi \), and then the system \( S-LKZ \) is obtained by inclining the coordinate system \( S-NKZ \) about the axis \( SK \) by \( \theta \). Next, the coordinate systems \( S-X_1Y_1Z_1 \) and \( S-X_2Y_2Z_2 \) are obtained by rotating the system \( S-LKZ \) about the axis of inertia \( SZ \) by \( \psi \) and \( \psi + \zeta \), respectively. The orientation angle between the inequality of shaft stiffness and that

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of rotor inertia is defined as \( \zeta = \angle X_2 \hat{S} X_3 = \angle Y_2 \hat{S} Y_1 \). The stiffnesses of the asymmetrical shaft are \( \alpha \pm \Delta \alpha \), \( \tau \pm \Delta \tau \), and \( \delta \pm \Delta \delta \), in which the lower negative signs correspond to the displacement in the \( X_2 \) direction, and the upper positive ones to that in the \( Y_2 \) direction. Let two components of displacement vector \( \hat{O}S \) be \( x \) and \( y \), and let projections of inclination angle \( \theta = \angle ZS \hat{Z} \) to \( xOz \) and \( yOz \) planes be \( \theta_x \) and \( \theta_y \), respectively.

When the terms higher than the 3rd order of small quantities \( \theta_3 \) and \( \theta_4 \) are neglected, the kinetic energy of the asymmetrical rotor \( T \) both of translation and rotation is represented as

\[
T = m_0(x\dot{x} + y\dot{y}) + I_x(\theta_x'' + \dot{\theta}_x \dot{\theta}_y - \dot{\theta}_y \dot{\theta}_x) + I_y(\theta_y'' + \dot{\theta}_y \dot{\theta}_x - \dot{\theta}_x \dot{\theta}_y) - AI[(\theta_x^2 - \dot{\theta}_x^2) \cos 2\theta + 2\dot{\theta}_x \dot{\theta}_y \sin 2\theta] \tag{1}
\]

where

\[
\Theta = \dot{\theta} + \phi \tag{2}
\]

is the rotational angle of the shaft end. Let the projections of the displacement vector \( \hat{O}S \) to \( X_2S_2 \) and \( Y_2S_2 \) planes be \( \theta_x' \) and \( \theta_y' \), respectively, and let the projection angles of inclination angle \( \dot{\theta} \) to \( X_2S_2 \) and \( Y_2S_2 \) planes be \( \theta_n \) and \( \theta_n' \), respectively. The potential energy of the asymmetrical shaft \( V \) is represented as

\[
V = (\alpha - \Delta \alpha)x^2 + 2(\tau + \Delta \tau)y + (\delta - \Delta \delta)\theta_n^2 + (\alpha + \Delta \alpha)x^2 + 2(\tau + \Delta \tau)y + (\delta + \Delta \delta)\theta_n'^2 \tag{3}
\]

The following relations in equation (3) are rewritten as follows: \( \theta_x' = \frac{x}{\theta_x} \cos(\theta + \zeta) + \frac{y}{\theta_x} \sin(\theta + \zeta) \), \( \theta_n' = \frac{x}{\theta_n} \sin(\theta + \zeta) + \frac{y}{\theta_n} \cos(\theta + \zeta) \)

The equations of motion \( x, \ y, \ \theta_x \) and \( \theta_y \) are obtained by using equations (1), (5) and (6):

\[
m_0\ddot{x} + \alpha x + \gamma \theta_x = \Delta x(x \cos(2\theta + \zeta) + x \sin(2\theta + \zeta) + \tau \theta_2 \cos(2\theta + \zeta) + \phi \theta_2 \sin(2\theta + \zeta))
\]

\[
m_0\ddot{y} + \gamma y + \gamma \theta_y = \Delta y(y \cos(2\theta + \zeta) - y \sin(2\theta + \zeta) + \tau \theta_2 \cos(2\theta + \zeta) - \phi \theta_2 \sin(2\theta + \zeta))
\]

\[
I\ddot{\theta}_x + I\dot{\theta}_x \dot{\theta}_y + \tau \dot{\theta}_x = A I \left( \dot{\theta}_x \cos(2\theta + \zeta) + \dot{\theta}_y \sin(2\theta + \zeta) \right) + \Delta \delta(\dot{\theta}_x \cos(2\theta + \zeta) + 2\dot{\theta}_y \sin(2\theta + \zeta))
\]

\[
I\ddot{\theta}_y - I\dot{\theta}_x \dot{\theta}_y + \tau \dot{\theta}_y = A I \left( \dot{\theta}_x \sin(2\theta + \zeta) - \dot{\theta}_y \cos(2\theta + \zeta) \right) + \Delta \delta(\dot{\theta}_x \sin(2\theta + \zeta) - \dot{\theta}_y \cos(2\theta + \zeta))
\]

Complex variables

\[
x = x + iy, \quad \ddot{x} = \dot{x} - i\dot{y}, \quad \ddot{\theta}_x = \dot{\theta}_x + i\dot{\theta}_y, \quad \ddot{\theta}_y = \dot{\theta}_x - i\dot{\theta}_y
\]

are introduced into equation (7), and the equations of motion of \( x \) and \( \theta_x \) are expressed as:

\[
m_0\ddot{x} + \alpha x + \gamma \theta_x = (\Delta x e^{-i\zeta} + \Delta x e^{i\zeta})
\]

\[
I\ddot{\theta}_x = I \left( \ddot{\theta}_x e^{-i\zeta} + \ddot{\theta}_x e^{i\zeta} \right)
\]

3. Mechanism for the occurrence of unstable vibrations

3.1 Increase in rate of total energy \( T + V \)

By differentiating equations (1) and (5) with respect to time \( t \) and using equation (6), the increase in rate of total energy and potential energy are obtained:

\[
\dot{T} = m_0(2x\dot{x} + 2y\dot{y}) + I_x(2\dot{x}\dot{\theta}_x + 2\dot{x}\dot{\theta}_y - 2\dot{\theta}_y \dot{\theta}_x) + I_y(2\dot{x}\dot{\theta}_y + 2\dot{x}\dot{\theta}_y - 2\dot{\theta}_x \dot{\theta}_y) - AI[(\dot{\theta}_x^2 - \dot{\theta}_y^2) \cos 2\theta + 2\dot{\theta}_x \dot{\theta}_y \sin 2\theta]
\]

\[
\dot{V} = (2\alpha x^2 + 2\gamma y + 2\gamma \theta_x) + (\Delta x^2 + (2\alpha x^2 - 2\gamma y) \sin(2\theta + \zeta)) - 2\alpha (2\dot{x}\dot{\theta}_x + 2\dot{x}\dot{\theta}_y - 2\dot{\theta}_x \dot{\theta}_y) \sin 2\theta + 2\alpha (2\dot{x}\dot{\theta}_x - 2\dot{x}\dot{\theta}_y - 2\dot{\theta}_x \dot{\theta}_y) \sin 2\theta,
\]

\[
(\Delta x^2 - \alpha y^2 - \gamma y - \gamma \theta_x) \sin(2\theta + \zeta)
\]

\[
- (\Delta x^2 - \alpha y^2 - \gamma y - \gamma \theta_x) \cos(2\theta + \zeta) - (\Delta \delta (2\dot{\theta}_x \dot{\theta}_x - 2\dot{\theta}_y \dot{\theta}_y - 2\dot{\theta}_x \dot{\theta}_y) \sin(2\theta + \zeta))
\]

\[
- (\Delta \delta (2\dot{\theta}_x \dot{\theta}_x - 2\dot{\theta}_y \dot{\theta}_y - 2\dot{\theta}_x \dot{\theta}_y) \sin(2\theta + \zeta))
\]
The use of complex variables (8) in equation (10) gives the following equations:

\[
\begin{align*}
\dot{T} &= m_0 \text{Re}[\dot{\mathbf{z}}] + \gamma \text{Re}[\dot{\mathbf{z}} \cdot \dot{\mathbf{z}}'] + \delta \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] - \delta \text{Re}[\dot{\mathbf{z}}'] \text{Re}[\dot{\mathbf{z}}] + \text{Im}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}] + (\text{Re}[\dot{\mathbf{z}}'] - \text{Re}[\dot{\mathbf{z}}])
\end{align*}
\]

\[
\begin{align*}
\dot{V} &= \text{Re} \left[ \text{Im}[\dot{\mathbf{z}}] - \gamma \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] - \delta \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] + \text{Im}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}] + (\text{Re}[\dot{\mathbf{z}}'] - \text{Re}[\dot{\mathbf{z}}]) \right]
\end{align*}
\]

(11)

The symbols \(\text{Re}[\cdot]\) and \(\text{Im}[\cdot]\) denote the real part and the imaginary one of a complex number \([\cdot]\). When the first equation in equation (9) is multiplied by \(i\), the second one by \(\overline{\dot{\theta}}\), and these are added together, the following equation is given by the right part of the derived equation:

\[
m_0 \text{Re}[\dot{\mathbf{z}}] + \alpha \text{Re}[\dot{\mathbf{z}}] + \gamma \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] + \delta \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] + \delta \text{Re}[\dot{\mathbf{z}}] + \text{Im}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}] + (\text{Re}[\dot{\mathbf{z}}'] - \text{Re}[\dot{\mathbf{z}}])
\]

(12)

The displacement \(\mathbf{x}'\) and the inclination angles \(\theta', \gamma\) in the rotating coordinate system \(O-x'y'z'\) turning at an angular velocity \(\omega\) are expressed by the following relations:

\[
\mathbf{x}' = x' + iy' = e^{i\omega t} \mathbf{x}, \quad \theta' = \theta + i \gamma t = \theta, e^{i\omega t}
\]

(13)

By substituting equations (12) and (13) into equation (11), the increase in rate of total energy \(T + V\) is given as

\[
T + V = -\omega \text{Im} \left[ \text{Re}[\dot{\mathbf{A}}(e^{i\omega t})] + (\text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] + 2 \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}] + 2 \text{Re}[\dot{\mathbf{z}}] \cdot \dot{\mathbf{z}}') e^{i\omega t} + (\text{Re}[\dot{\mathbf{z}}] - \text{Re}[\dot{\mathbf{z}}']) e^{i\omega t} \right]
\]

(14)

Torque \(T_r\) supplied at the shaft end is a generalized force with respect to the shaft end rotation \(\theta\). When equations (1) and (5) are substituted into Lagrange's equation of motion, and equations (6) and (13) are used, torque \(T_r\) is obtained as follows:

\[
T_r = -\text{Im} \left[ \text{Re}[\dot{\mathbf{A}}(e^{i\omega t})] + (\text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}'] + 2 \text{Re}[\dot{\mathbf{z}}' \cdot \dot{\mathbf{z}}] + 2 \text{Re}[\dot{\mathbf{z}}] \cdot \dot{\mathbf{z}}') e^{i\omega t} + (\text{Re}[\dot{\mathbf{z}}] - \text{Re}[\dot{\mathbf{z}}']) e^{i\omega t} \right]
\]

(15)

From equations (14) and (15), the relation

\[
\omega T_r = T + V
\]

(16)

is obtained. Equation (16) means that the time rate of work applied to the shaft end \(\omega T_r\) agrees with the increase in rate of total energy \(T + V\).

3.2 Frequency equation

For simplicity, the following dimensionless quantities are introduced:

\[
x' = \sqrt{I/m_0} x, y' = \sqrt{I/m_0} y, I_p/I = i, \quad \omega = \sqrt{\alpha/m_0}, \quad \gamma = \gamma', \quad \Delta I/I = \Delta i
\]

(17)

The primes in the dimensionless quantities (17) are omitted hereafter. The dots over the dimensionless mean the differential coefficient with respect to \(t\). The equations of motion (9) regarding \(z\) and \(\theta\) are rewritten, using these dimensionless quantities (18):

\[
\begin{align*}
\ddot{x} + \gamma \theta &= (\Delta I x + \gamma x' \theta') e^{i\omega t+c}
\end{align*}
\]

\[
\begin{align*}
\dot{\theta} - i \omega \dot{x} + \gamma x' + \theta' &= \Delta i \dot{x} + (\gamma \Delta I x + \gamma x' \theta') e^{i\omega t+c}
\end{align*}
\]

(18)

The existence of the rotating inequalities \(\Delta i\) and \(\Delta I\) yields a pair of natural frequencies \(p\) and \(2\omega - p\) (\(p\) is a conjugate complex number of \(p\)), and the solutions for the free vibration of equation (18) is expressed as follows:

\[
\begin{align*}
z = A e^{i\phi} + A' e^{i(2\omega - p)}, \quad \theta = Be^{i\phi} + B' e^{i(2\omega - p)}
\end{align*}
\]

(19)

where amplitudes \(A, A', B, B'\) are complex numbers. When equation (19) is substituted into equation (18), the 4th order determinant which consists of coefficients of \(A, \; A' e^{i\phi}, \; B, \; B' e^{i\phi}\) gives the frequency equation (20),

\[
\begin{align*}
F &= H(p) - \Delta I \gamma - \gamma \Delta I \\
&= -\Delta I \gamma - \gamma \Delta I \end{align*}
\]

(20)

where

\[
\begin{align*}
H(p) &= -1 - p^2, \quad G(p) = 1 + i\omega p - p^2
\end{align*}
\]

(21)

Expanding equation (20), a frequency equation is derived as follows:
\[ F = f(p)/(2\omega_{p} - p) + [-A_{1}^{2} (2\omega_{p} - p)^{2} H(p)H(2\omega_{p} - p) - A_{1}^{2} G(p)G(2\omega_{p} - p) - 2A_{2} A_{1} G(p)G(2\omega_{p} - p) - 2A_{1}^{2} H(p)H(2\omega_{p} - p)] \]

\[ + 2A_{2} A_{1} G(p)G(2\omega_{p} - p) + \delta_{1}^{2} A_{2} A_{1} H(p)H(2\omega_{p} - p) + 2\delta_{1}^{2} A_{2} A_{1} H(p)H(2\omega_{p} - p) - 2(\delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p)) \cos 2\gamma \]

\[ + (\delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p)) \cos 2\gamma \] = 0 \hspace{1cm} (22)

where

\[ f(p) = H(p)G(p)/\gamma^{2} \] \hspace{1cm} (23)

is a frequency equation for the case where \( A_{1} = 0 \) and \( A_{2} = 0 \).

### 3.3 Necessary condition for the occurrence of unstable vibration

The increase in rate of the total energy (14) is rewritten using the dimensionless quantities (17):

\[ \omega_{T_{u}} = T + \dot{Y} = -\omega \text{ Im}[\delta_{1} + \omega_{p} \delta_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p) + \delta_{1} A_{2} A_{1} H(p)] \] \hspace{1cm} (24)

The condition under which the outer energy is supplied to the shaft system and the unstable vibration occurs, coincides with the condition where a time average of the increase in rate of total energy represented by equation (24) is positive.

#### 3.3.1 Necessary condition for the occurrence of statically unstable vibration

When the root \( p \) derived from equation (22) is not a real number but an imaginary one, an unstable vibration occurs. In the statically unstable region in which the real part of a complex root \( p \) coincides with an angular velocity of shaft \( \omega \), whirl of natural frequencies in equation (19) may be replaced by \( p = \omega \pm i \text{Im} \), and solutions of free vibration are given as follows:

\[ x = A \text{e}^{i(\omega \text{Im} + \gamma t)}, \quad \theta = B \text{e}^{i(\omega \text{Im} + \gamma t)} \] \hspace{1cm} (25)

If the imaginary part \( \text{Im} \) of a complex root \( p \) is positive, then the first term of the right-hand side of equation (25) decreases exponentially with time as \( e^{-\text{Im} t} \) regardless of the initial condition. This first term in equation (25) may be neglected unlike the second one which increases exponentially as \( e^{\text{Im} t} \). Thus, only the second term in equation (25) is conserved, and the subscript 2 is omitted:

\[ x = A \text{e}^{i(\omega \text{Im} + \gamma t)}, \quad \theta = B \text{e}^{i(\omega \text{Im} + \gamma t)} \] \hspace{1cm} (26)

In the statically unstable region, the increase in rate of total energy (24) is rewritten by using equations (13) and (26) as follows:

\[ \omega_{T_{u}} = T + \dot{Y} = -\omega \text{ Im}[J] \] \hspace{1cm} (27)

where

\[ J = A_{1}^{2} (2\omega_{p} - p)^{2} B^{2} + (A_{1} A_{2} + 2\gamma A_{1} A_{2} + \delta A_{1} A_{2} B^{2})e^{-\text{Im} t} - i(\omega \text{ Im} B) \] \hspace{1cm} (28)

The condition under which unstable vibrations occur is given by the relation in which equation (27) is positive; that is,

\[ \text{Im}[J] = -\omega \text{ Im}[J] < 0 \] \hspace{1cm} (29)

Let the arguments of the complex numbers \( A \) and \( B \) be \( \arg A \) and \( \arg B \), respectively, and the imaginary part of equation (28) is expressed as follows:

\[ \text{Im}[J]/|B|^{2} = A_{1}^{2} (\omega - \text{Im}^{2} B^{2}) \cos 2\text{arg} A + 2A_{1} A_{2} \text{Im} \sin 2\text{arg} A \]

\[ + 2\gamma A_{1} A_{2} \text{Im} \sin 2\text{arg} A + (\delta A_{1} A_{2} B^{2}) \text{Im} \cos 2\text{arg} A \] \hspace{1cm} (30)

If the equations of motion (18) have the free vibration (26), the following determinant consisting of the coefficients of \( \text{Re}[A], \text{Im}[A], \text{Re}[B] \) and \( \text{Im}[B] \) must satisfy the following relation:

\[ \begin{vmatrix}
1 - A_{1} \cos 2\gamma - \omega^{2} + m^{2} & -A_{1} \sin 2\gamma - 2\omega \text{Im} \sin 2\gamma & \gamma (1 - A_{1} \cos 2\gamma) & -\gamma A_{1} \sin 2\gamma \\
-A_{1} \sin 2\gamma - 2\omega \text{Im} \sin 2\gamma & 1 + A_{1} \cos 2\gamma - \omega^{2} + m^{2} & -\gamma A_{1} \sin 2\gamma & \gamma (1 + A_{1} \cos 2\gamma) \\
\gamma (1 - A_{1} \cos 2\gamma) & -\gamma A_{1} \sin 2\gamma & (i_{1} - i_{2}) \omega^{2} + (1 - A_{1}) m^{2} & -\gamma A_{1} \sin 2\gamma - (2 - i_{1}) \omega \text{Im} \\
-\gamma A_{1} \sin 2\gamma & \gamma (1 + A_{1} \cos 2\gamma) & -\gamma A_{1} \sin 2\gamma + (2 - i_{2}) \omega \text{Im} & (i_{1} - i_{2}) \omega^{2} + (1 + A_{1}) m^{2}
\end{vmatrix} = 0 \] \hspace{1cm} (31)

The cofactor of each row (\( i = 1, 2, 3, 4 \)) of the determinant (31) has the following relation:

\[ A_{i} : A_{1} : A_{2} : A_{3} = \text{Re}[A] : \text{Im}[A] : \text{Re}[B] : \text{Im}[B] \] \hspace{1cm} (32)

Thus, the absolute value of amplitude ratio \( |A/B| \) and arguments \( \arg A \) and \( \arg B \) in equation (30) can be calculated as follows:

\[ \frac{A}{B} = \sqrt{\frac{A_{0}^{2} + A_{3}^{2}}{A_{1}^{2} + A_{2}^{2}}}, \quad \arg A = \tan^{-1} \frac{A_{0}}{A_{1}}, \quad \arg B = \tan^{-1} \frac{A_{0}}{A_{1}} \] \hspace{1cm} (33)

The necessary condition for occurrence of unstable vibration coincides with the condition under which the right-hand side of equation (30) is negative.

An appropriate combination of inertia asymmetry \( A_{1} \) and stiffness asymmetry \( A_{2} \) makes the imaginary part of a complex number \( f \) of equation (28) zero, and the unstable region may almost vanish.
3.3.2 Necessary condition for occurrence of dynamically unstable vibration
When whirl natural frequencies of a shaft system are put as
\[ p = P_1 + im, \quad 2\omega - p = P_2 + im, \quad (0 < \omega < \omega_P) \]  
(34a)
a dynamically unstable vibration is considered which certainly satisfies the relation
\[ p_1 + p_2 = 2\omega \]  
(34b)
and in which both amplitudes of frequencies \( p_1 \) and \( p_2 \) increase exponentially as \( e^{at} \). In this unstable region, the solutions of free vibration (19) are rewritten as follows:
\[ z = A_1 e^{(\alpha_1 + \beta_1) t} + A_2 e^{(\alpha_2 + \beta_2) t} + A_3 e^{(\alpha_3 + \beta_3) t} + A_4 e^{(\alpha_4 + \beta_4) t} \]
\[ \theta = B_1 e^{(\alpha_1 + \beta_1) t} + B_2 e^{(\alpha_2 + \beta_2) t} + B_3 e^{(\alpha_3 + \beta_3) t} + B_4 e^{(\alpha_4 + \beta_4) t} \]
(35)
As with equation (25), the first and third terms in the right-hand side of equation (35) may be negligible. Thus, the second and fourth terms of equation (35) are adopted as the solutions of equation (13) and the subscript 2 is omitted. Thus,
\[ z = A_2 e^{(\alpha_2 t)} + \theta = B_2 e^{(\alpha_2 t)} \]
(36)
When equation (36) is transformed into equation (13) and introduced into equation (24), the increase in rate of total energy is given as follows:
\[ \omega T_r = \dot{J} + \dot{Y} = - \omega e^{at} \text{Im} [\alpha_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + 2\alpha_2 P_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + B_2 e^{(\alpha_2 t)} + \delta_2 (P_2 - im) B_2 e^{(\alpha_2 t)}] \]
\[ + 2\alpha_2 P_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + 2\alpha_2 P_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + 2\alpha_2 P_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + \delta_2 (P_2 - im) B_2 e^{(\alpha_2 t)} \]
\[ + R_2 (P_2 - im) B_2 e^{(\alpha_2 t)} + (P_2 - im) B_2 e^{(\alpha_2 t)} + (P_2 - im) B_2 e^{(\alpha_2 t)} \]
(37)
The sum of the constant term is included in equation (37) is defined by \( J \), and a time average of torque \( T_r \) is expressed by \( T_m \). The condition under which an unstable vibration occurs is reduced as follows:
\[ \text{Im}[J] = - T_m e^{im} < 0 \]
(38)
By using an argument of a complex number and equation (34b), \( \text{Im}[J] \) in equation (38) is obtained in the following form:
\[ \text{Im}[J]/BB' = \alpha_2 (P_2 - im) \sin(\arg B + \arg B') - 2\alpha_2 \text{Im}[\alpha_2 (P_2 - im) \sin(\arg B + \arg B')]
\[ + \alpha_2 (P_2 - im) \sin(\arg A + \arg A') - 2\alpha_2 \sin(\arg A + \arg B' - 2\alpha_2) \]
\[ + \sin(\arg A + \arg B - 2\alpha_2) + \delta_2 \sin(\arg B + \arg B' - 2\alpha_2) + (i\gamma_2) m (\arg B + \arg B'/\arg B) \]
(39)
Amplitude ratio and argument of complex amplitudes \( A, B, A', \) and \( B' \) are necessary in order to calculate the right-hand side of equation (39). For the whirling solutions of free vibration (36), the cofactors of determinant (20) into which \( P = P_2 - im \) and \( 2\omega - p = 2\omega - P_1 + im = P_1 + im \) are inserted have the following relation:
\[ A_1 : A_2 : A_3 : A_4 = A : B : B'/B' \]
(40)
which gives absolute values of amplitude ratio and arguments regarding the complex amplitudes \( A, B, A', \) and \( B' \) as follows:
\[ \begin{align*}
  &A = |A| e^{i\theta A}, \quad A' = |A'| e^{i\theta A'}, \quad B = |B| e^{i\theta B}, \quad B' = |B'| e^{i\theta B'} \\
  &\arg A = \arg A_1, \quad \arg A' = \arg A_3, \quad \arg B = \arg A_2, \quad \arg B' = \arg A_4
\end{align*} \]
(41)
Because the right-hand side of equation (39) includes the orientation angle \( \zeta \), the condition under which a dynamically unstable vibration occurs changes remarkably with the angle \( \zeta \). The right-hand side of equation (39) can be made zero with an appropriate combination of inertia asymmetry \( A_1 \) and stiffness asymmetry \( A_2 \), and thus, the unstable region may almost vanish.

4. Solutions for free vibration obtained by analog computer

Substitution of equation (13) into equation (18) yields the following equations of motion regarding \( x' \) and \( \theta' \):
\[ x' = -(1 - D_1 \cos 2\zeta - \omega^2 m) x' + 2\omega y' + D_1 \sin 2\zeta \]
\[ y' = -(1 + D_1 \cos 2\zeta + \omega^2 m) y' - 2\omega x' + D_1 \sin 2\zeta \]
\[ \phi' = -(\delta_1 - D_1 \sin 2\zeta) \theta' + D_1 \sin 2\zeta \theta' \]
\[ \psi' = -(\delta_1 - D_1 \sin 2\zeta) \phi' + D_1 \sin 2\zeta \phi' \]
\[ + (2 - i\gamma_2) \sin \theta' + \delta_1 \sin 2\zeta \theta' \]
\[ + (2 - i\gamma_2) \sin \phi' \]
(42)
Torque \( T_r \) of equation (24) is given by using equation (13) as follows:
\[ T_r = (D_1 (x'^2 - y'^2) + 2\delta_1 (x' \theta' - y' \phi')) \]
\[ + D_1 (x'^2 - y'^2) \sin 2\zeta - 2\delta_1 (x' \theta' + y' \phi') \]
\[ + \delta_1 (x' \theta' + y' \phi') \cos 2\zeta \]
\[ - 2\delta_1 (x' \theta' + y' \phi') \theta' \]
\[ - (i\gamma_2) (\theta' \phi' - \theta' \phi') + 2\omega (\theta' \phi' + \theta' \phi') \]
\[ + (2 - i\gamma_2) (\theta' \phi' + \theta' \phi') \]
(43)
Figures 2(a) and (b) show the simulation circuits for an analog computer which satisfy equations (43) and (44). A time average of the fourth term \(-i\gamma_2/2 (\theta' \phi' - \theta' \phi')\) in equation (44) becomes zero in a steady-state vibration or as small as the negative damping coefficient \( m \) in a nonsteady-state vibration. Thus, this term is omitted in Fig. 2(b).
4.1 Solutions for statically unstable vibration

The vibratory waves \( x', y', \theta', \phi' \) obtained by an analog computer ALS-200X are shown in Fig. 3 when the orientation angle \( \zeta \) is changed to \( 0, \pi/4 \) and \( \pi/2 \) in the other parameters\(^{[1]} \) fixed as \( M=1.993 \), \( \delta=1.797 \), \( \beta=-1.050 \), \( \Delta_1=0.304 \), \( \Delta_2=0.068 \), \( \Delta_3=0.051 \), \( \Delta_4=0.069 \) and \( \omega=0.725 \). Because the number of potentiometers and multipliers of an analog computer is insufficient for \( \zeta=\pi/4 \), torque \( T \) is not shown in Fig. 3. A torque applied to the shaft end and the negative damping coefficient decrease as the orientation angle \( \zeta \) increases. The tendency of Fig. 3 agrees with that of the experimental results\(^{[1]} \) for the same parameters.

The negative damping coefficient \( m \) and \( \text{Im}[J]/|J|^2 \) calculated by using equations (20) and (33), are plotted against the shaft speed \( \omega \) in Figs. 4(a) and (b) for the same parameters as Fig. 3, but \( \zeta=0, \pi/8, \pi/4, 3\pi/8, \pi/2 \). The circles in Fig. 4(a) indicate the negative damping coefficient \( m \) measured from the vibratory solution of an analog computer, and the solid lines represent the imaginary part \( m \) of the exact complex root \( p \) calculated from equation (22). When the angle \( \zeta \) increases from 0 to \( \pi/2 \), the negative damping coefficient \( m \) decreases, and the width of the unstable region is also reduced. Because \( \text{Im}[J] \) has a negative value in the statically unstable region, it satisfies the condition (28) under which a statically unstable vibration occurs.

When inertia asymmetry \( A_0 \) is changed
from 0 to 0.5 and the orientation angle $\zeta$ is fixed as $\pi/2$ with the other parameters the same as Fig. 3, the negative damping coefficient $m$ is as shown in Fig. 5. The circles and the solid lines in Fig. 3 have the same meanings as those in Fig. 4(a). From Fig. 5, it is clear that the unstable region is narrowest in the neighborhood of $A_s = 0.34$.

### 4.2 Solutions for dynamically unstable vibration

The vibratory waves $x'$, $y'$, $\theta'$ and $\theta''$ for dynamically unstable vibration obtained with an analog computer are shown in Fig. 6. When the orientation angle $\zeta$ is changed to 0, $\pi/4$ and $\pi/2$, with the other parameters fixed as $i_p = 0.7536$, $\beta = 14.179$, $\gamma = -3.253$, $A_{10} = 0.003$, $A_{11} = 0.1032$, $A_{12} = 0.0880$, $A_{13} = 0.0780$ and $\omega = 2.760$. Torque $T_\varphi$ for $\zeta = \pi/4$ is not shown because the number of potentiometers and multipliers is insufficient. A torque applied to the shaft end is equal to zero when $\zeta = 0$, and the vibration is always stable. On the other hand, the torque $T_\varphi$ increases rapidly with time when $\zeta = \pi/2$, and an unstable vibration occurs. The tendency of Fig. 6 agrees with that of the experimental results\(^{(1)}\) for the same parameters.

Figures 7(a) and (b) for the same parameters as Fig. 6 except $\zeta$ show the measured results and exact solutions of the negative damping coefficient $m$, and $\text{Im} [\mathcal{F}] / 2|\mathcal{B}|^2$ calculated from equation (39). The circles in Fig. 7(a) indicate the negative damping coefficient $m$ measured from vibratory solutions of an analog computer, and the solid lines correspond to the imaginary part $m$ of exact complex root $\mathcal{F}$ calculated from a frequency equation (22). An unstable vibration does not occur when $\zeta = 0$. As $\zeta$ increases to $\pi/2$, the negative damping coefficient $m$ increases, and also the width of the unstable region becomes greater. Because $\text{Im} [\mathcal{F}]$ has a negative value in the unstable region, it satisfies the condition under which a dynamically unstable vibration occurs.

Figure 8 shows the negative damping coefficient $m$, which changes with the magnitude of $A_s$ for the same parameters as Fig. 6 except $\zeta = 0$. A dynamically unstable vibration does not occur in the neighborhood of $A_s = 0.09$ as shown in Figs. 6 and 8.
5. Conclusions

The obtained conclusions may be summarized as follows:

1. In a rotating asymmetrical shaft carrying an asymmetrical rotor, the increase in rate of the total energy of the shaft system is identified with the time rate of work, and it is given by equation (24).

2. The condition under which a statically unstable vibration occurs means that \( \text{Im}[J] \) of equation (30) is negative, and this condition depends on the orientation angle \( \zeta \) between stiffness asymmetry and inertia asymmetry. As \( \zeta \) increases from 0 to \( \pi/2 \), the statically unstable region becomes narrow in the present study, and the negative damping coefficient \( m \) decreases.

3. The condition under which a dynamically unstable vibration occurs means that \( \text{Im}[J] \) of equation (39) is negative, and this condition also depends on the orientation angle \( \zeta \). When \( \zeta = 0 \), an unstable vibration does not occur in the present study. Width of the dynamically unstable region becomes greater and the negative damping coefficient \( m \) increases as \( \zeta \) increases to \( \pi/2 \).

4. \( \text{Im}[J] \) of equations (30) and (39) becomes negative in the region in which the vibratory solutions obtained with an analog computer are unstable. Thus, it is apparent that the necessary conditions for instability (29) and (38) are correct.

5. An appropriate combination of inertia asymmetry \( \Delta_d \) and stiffness asymmetry \( \Delta_k \) may give the condition under which an unstable vibration does not occur. This condition holds in each case of the statically unstable vibration and the dynamically unstable one, which is ascertained by the vibration solutions obtained with an analog computer.

References


