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A Numerical Analysis of Flow using Streamline Coordinates*

(The Case of Two-dimensional Steady Incompressible Flow)

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Basic equations for a fluid motion are derived using streamline coordinates (ϕ, ψ) . Streamlines are taken as the coordinate curves of ϕ , and the coordinate curves of ψ are so chosen that they are orthogonal with the streamlines. The quantity ϕ is related to the length s of the streamline by $\partial\phi/\partial s = \lambda$, where λ is a new variable. The quantity ψ is related to the length n of the orthogonal curve by $\partial\psi/\partial n = q$, where q is flow velocity.

A two-dimensional liquid jet from a channel with parallel walls into the atmosphere is taken as an example of numerical calculation. The basic equations are transformed into difference equations. The flow field is solved numerically, and this method is proved to be useful, especially in the case where the flow has free surfaces. There is also an important merit that the basic equations can be considerably simplified when a general picture of streamlines is predicted in advance.

Key Words: Numerical Analysis, Fluid Mechanics, Flow Net, Streamline Coordinates, Two-dimensional Jet, Free Surface.

1. Introduction

In analyzing fluid motions theoretically, cartesian coordinates are commonly adopted. However, cartesian coordinates are not always the best choice. For instance, it is better to use cylindrical coordinates for the flow around a circular cylinder or the flow in a circular pipe and to use spherical coordinates for the flow around a sphere. This is because proper coordinates can be chosen corresponding to the shape of boundaries.

Since a boundary coincides with only one streamline as far as separation does not occur, it can be deduced from the above that the theoretical treatment will become easier when choosing streamlines as coordinate curves. On the contrary, the basic equations will become more complicated, which makes it more difficult to solve them. It can be easily seen, however, that this method has great merits for some types of flow, for instance, a flow with free surfaces.

The idea of streamline coordinates already appears in the two-dimensional flow theory. It has been adopted also for flows inside turbomachines. Uchida⁽¹⁾ developed a graphical solution which is very effective for elliptic basic equations and called it the "streamline analysis method". Hamrick applied the streamline coordinates to axisymmetric flows and designed the hub and the shroud of a centrifugal compressor.⁽²⁾

Although in the above mentioned methods the streamlines and the curves orthogonal to them are chosen as coordinate curves, Katsanis⁽³⁾, Senoo & Nakase⁽⁴⁾ and Daiguji & Nozaki⁽⁵⁾ used non-orthogonal curves as coordinate ones and analyzed the internal flows inside turbomachines. Duda and Vrentas⁽⁶⁾ also obtained the free surface of a liquid jet from a pipe into the atmosphere using similar coordinates.

In the present paper the author tries to derive the basic equations for two-dimensional steady incompressible flows using the streamline coordinates. The streamlines and the curves orthogonal to them are chosen as coordinate curves. It is effective to use these curves because the streamlines correspond to the curves along which stream functions are constant and the orthogonal ones to the curves of equi-velocity potential.

It must be noted that, if the length s along a streamline and the length n along an orthogonal curve were chosen as coordinate curves, it would be difficult to determine the unit of length. This will be explained in detail later.

In this study, this difficulty is solved by transforming the coordinates (s, n) into alternative coordinates (ϕ, ψ) with which the unit of quantity can be determined.

The liquid jet from a two-dimensional parallel walled channel into the atmosphere is taken as a numerical example, and this flow is analyzed using the new coordinates (ϕ, ψ) .

Nomenclature

e_n, e_s : non-dimensional unit vectors in the n - and s -directions respectively,
 $e_n = e_n^*/L^*$, $e_s = e_s^*/L^*$
 L^* : reference length

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- n : non-dimensional coordinate normal to the streamline $= n^*/L^*$
 p : non-dimensional pressure $= q^*/(\rho^* q_0^{*2})$
 q : non-dimensional velocity $= q^*/q_0^*$
 q_0^* : reference velocity
 q_∞ : non-dimensional velocity at an infinite distance downstream
 Re : Reynolds number $= L^* q_0^*/\nu^*$
 r_n, r_s : non-dimensional radii of curvature of the coordinate curves n and s , respectively, $r_n = r_n^*/L^*$, $r_s = r_s^*/L^*$
 s : non-dimensional coordinate along a streamline $= s^*/L^*$
 x, y : non-dimensional cartesian coordinates, $x = x^*/L^*$, $y = y^*/L^*$
 θ : angle between the unit vector e_s and x -axis
 λ : $\partial\phi/\partial s$
 ν^* : kinematic viscosity of the liquid
 ρ^* : density of the liquid
 τ : non-dimensional shear stress $= \tau^*/(\rho^* q_0^{*2})$
 ϕ, ψ : new coordinates in the s - and n -directions, respectively
 Ω : non-dimensional vorticity $= \Omega^*/(q_0^*/L^*)$

Superscript

* : dimensional quantities

2. Basic theory

In this chapter, the basic equations for a two-dimensional steady incompressible flow are derived, and their applicability to numerical calculation is investigated.

2.1 Unit vectors

We take the s -coordinate curve along a streamline and the n -coordinate curve normal to the streamline. The unit vectors in the s - and n -directions are denoted by e_s and e_n , respectively.

The derivatives of the unit vectors are represented as follows:

$$\frac{\partial e_s}{\partial s} = -\frac{1}{r_s} e_n \quad (1)$$

$$\frac{\partial e_s}{\partial n} = \frac{1}{r_n} e_n \quad (2)$$

$$\frac{\partial e_n}{\partial s} = \frac{1}{r_s} e_s \quad (3)$$

$$\frac{\partial e_n}{\partial n} = -\frac{1}{r_n} e_s \quad (4)$$

where r_s is the radius of curvature of the s -curve and r_n that of the n -curve; r_s is positive when its center of curvature lies on the opposite side of the vector e_n and r_n is positive when the center lies on the opposite side of the vector e_s (c.f. Fig.1)

The radii of curvature are written as

$$\frac{1}{r_s} = -\frac{\partial\theta}{\partial s} \quad (5)$$

$$\frac{1}{r_n} = \frac{\partial\theta}{\partial n} \quad (6)$$

where θ is the angle between e_s and x -axis.

2.2 $\phi\psi$ coordinates

In Fig.1, the length $\widehat{AB}$ is not equal to length $\widehat{DC}$, where $\widehat{AB}$ and $\widehat{DC}$ are the lengths of the coordinate curves s cut by two coordinate curves n which are drawn through points $A(s_1, n_1)$ and $B(s_2, n_2)$ respectively. In other words, $s_2 - s_1$ is not equal to $s_2' - s_1'$. Consequently if we use the sn coordinates, the unit of length becomes indefinite. For example, the second derivatives of q depend on the order of differentiations, namely $\partial^2 q / \partial n \partial s \neq \partial^2 q / \partial s \partial n$.

Therefore, it is necessary to transform ns coordinates into $\phi\psi$ coordinates defined by the following equations:

$$\frac{\partial\phi}{\partial s} = \lambda \quad \frac{\partial\phi}{\partial n} = 0 \quad (7)$$

$$\frac{\partial\psi}{\partial n} = q \quad \frac{\partial\psi}{\partial s} = 0 \quad (8)$$

where both λ and q are functions of ϕ and ψ .

ψ is constant along a streamline, and ϕ is constant along a curve orthogonal to the streamline.

Since ψ is constant along the curves $\widehat{AB}$ and $\widehat{DC}$ in Fig.1, we have

$$\frac{\partial\theta}{\partial n} = -\frac{1}{q} \frac{\partial q}{\partial s} \quad (9)$$

which represents the continuity condition.

Similarly, since ϕ is constant along the curves $\widehat{AD}$ and $\widehat{BC}$, the following relation is obtained:

$$\frac{\partial\theta}{\partial s} = \frac{1}{\lambda} \frac{\partial\lambda}{\partial n} \quad (10)$$

Fig.1 Streamline coordinates

The derivatives of a quantity w are rewritten using Eqs.(7) and (8) as follows:

$$\frac{\partial w}{\partial s} = \lambda \frac{\partial w}{\partial \phi} \quad \text{.....(11)}$$

$$\frac{\partial w}{\partial n} = q \frac{\partial w}{\partial \psi} \quad \text{.....(12)}$$

Since the scales of ϕ and ψ are independent of the position, it is possible to exchange the order of differentiations, for example $\partial^2 w / \partial \phi \partial \psi = \partial^2 w / \partial \psi \partial \phi$.

Rewriting Eqs.(6), (9) and (5), (10) in the form of differentiation with respect to ϕ and ψ , we have

$$q \frac{\partial \theta}{\partial \psi} = -\lambda \frac{\partial q}{\partial \phi} = \frac{1}{r_n} \quad \text{.....(13)}$$

$$\lambda \frac{\partial \theta}{\partial \phi} = \frac{q}{\lambda} \frac{\partial \lambda}{\partial \psi} = -\frac{1}{r_s} \quad \text{.....(14)}$$

Elimination of the variable θ from Eqs.(13) and (14) yields the following equations:

$$\frac{\partial}{\partial \phi} \left[\lambda \frac{\partial}{\partial \psi} \left(\frac{1}{q} \right) \right] + \frac{\partial}{\partial \psi} \left[q \frac{\partial}{\partial \phi} \left(\frac{1}{\lambda} \right) \right] = 0 \quad \text{.....(15)}$$

2.3 Equations of motion

The Navier-Stokes equation of motion in vector form is expressed as

$$(\mathbf{q} \cdot \nabla) \mathbf{q} = -\nabla p + (1/R_e) \nabla^2 \mathbf{q} \quad \text{.....(16)}$$

where body forces are neglected; ∇ denotes a non-dimensional vector differential operator which is defined by

$$\nabla = \mathbf{e}_s \lambda \frac{\partial}{\partial \phi} + \mathbf{e}_n q \frac{\partial}{\partial \psi} \quad \text{.....(17)}$$

Since $\mathbf{q} = q \mathbf{e}_s$, substituting this into Eq.(16) and rearranging, the following equations in both $\mathbf{e}_s$ - and $\mathbf{e}_n$ -directions are obtained: ⁽¹⁵⁾

$$-\frac{q^2}{r_n} = -\lambda \frac{\partial p}{\partial \phi} + \frac{1}{R_e} \left[\lambda \frac{\partial}{\partial \phi} \left(-\frac{q}{r_n} \right) + q \frac{\partial}{\partial \phi} \left(q \frac{\partial q}{\partial \psi} \right) + \frac{q}{r_s} \frac{\partial q}{\partial \psi} - \frac{q}{r_s^2} - \frac{2q}{r_n^2} \right] \quad \text{.....(18)}$$

$$-\frac{q^2}{r_s} = -q \frac{\partial p}{\partial \psi} + \frac{1}{R_e} \left[q^2 \frac{\partial}{\partial \psi} \left(\frac{1}{r_n} \right) + q \lambda \frac{\partial}{\partial \psi} \left(-\frac{1}{r_s} \right) + \frac{2q}{r_n} \frac{\partial q}{\partial \phi} + \frac{2q}{r_s r_n} \right] \quad \text{.....(19)}$$

2.4 Vorticity transport equation

The vorticity vector is defined by

$$\boldsymbol{\Omega} = \boldsymbol{\Omega} \mathbf{e}_b = \nabla \times \mathbf{q} = -\mathbf{e}_s \left(\frac{q}{r_s} + q \frac{\partial q}{\partial \phi} \right) \quad \text{.....(20)}$$

where $\mathbf{e}_b (= \mathbf{e}_s \times \mathbf{e}_n)$ is a non-dimensional unit vector perpendicular to sn plane.

Taking the curl of both sides of Eq.(16)

$$(\mathbf{q} \cdot \nabla) \boldsymbol{\Omega} = (1/R_e) \nabla^2 \boldsymbol{\Omega} \quad \text{.....(21)}$$

and taking the $\mathbf{e}_b$ -component out of this equation, finally we obtain

$$q \lambda \frac{\partial \Omega}{\partial \phi} = \frac{1}{R_e} \left[\lambda \frac{\partial}{\partial \phi} \left(\lambda \frac{\partial \Omega}{\partial \psi} \right) + q \frac{\partial}{\partial \psi} \left(q \frac{\partial \Omega}{\partial \phi} \right) + \frac{\lambda}{r_n} \frac{\partial \Omega}{\partial \psi} + \frac{q}{r_s} \frac{\partial \Omega}{\partial \phi} \right] \quad \text{.....(22)}$$

2.5 Some notes on the application to numerical calculations

Equations (13), (14), (18) and (19) are approximated by difference equations, from which the flow velocity q , the flow angle θ , the pressure p and $\lambda = \partial \phi / \partial s$ are obtained by numerical calculation. Alternatively, the vorticity Ω , the flow velocity q , λ and the angle θ may be obtained from Eqs.(13), (14), (20) and (22) in difference form as functions of ϕ and ψ .

The curves $\psi = \text{const}$ are coincident with the streamlines on the physical plane. A diagram which consists of the curves $\psi = \text{const}$ and $\phi = \text{const}$ is often called "flow net".

Now, we try to study the property of λ in more detail. While the gradient of λ in the ψ -direction is determined from

Eq.(14), it is sufficient for the gradient of λ in the ϕ -direction only to satisfy the boundary conditions, because it has no governing equation. For instance, if the vorticity along the streamline of a boundary equals zero, the value of λ along the line can be set equal to q , and ϕ is referred to as the velocity potential. If all streamlines are parallel to each other, λ can be set equal to $1/r_s$. In this case $d\phi$ is proportional to the angle corresponding with ds .

When configuration of streamlines is visualized or estimated by any other procedures, the basic equations can greatly be simplified, and in some cases even analytical solutions would be found. Some simplified cases are shown below:-

- (a) The case of large Reynolds number:

For the flow with a large Reynolds number, the first term of [ ] on the right-hand side of Eq.(18) can usually be neglected compared to a parabolic equation with respect to q ,⁽⁸⁾ so that the necessary storage locations can greatly be reduced.⁽⁹⁾

- (b) The case of small streamline curvature: For a small boundary curvature this type of flow is usually obtained. From Eq.(14) we can write $\lambda \partial / \partial \phi = \partial / \partial s$. Furthermore, since Eq.(20) can be written as $\Omega = -q \partial q / \partial \psi$, from both these equations and Eq.(22) q and Ω can be calculated. In this case, however, since Eq.(22) is elliptic, iterative calculation is needed.

- (c) The case of large Reynolds number and small streamline curvature: For a small boundary curvature and large Reynolds number, Eq.(19) reduces to $\partial p / \partial \psi = 0$, and Eq.(14) to $\partial \lambda / \partial \psi = 0$. Thus from Eqs.(13) and (18) we have

$$q \frac{\partial q}{\partial s} = -\frac{dp}{ds} + \frac{1}{R_*} q \frac{\partial}{\partial \phi} \left(q \frac{\partial q}{\partial \phi} \right) \dots\dots\dots(23)$$

which is obviously a parabolic equation. Therefore, when dp/ds is given along a certain streamline, the solution can easily be obtained. An example for this type of flow is a boundary layer flow near the boundary wall with a small curvature.

- (d) The case of small value of $1/r_n$: This type of flow is obtained when the convective acceleration is small. In this case, since the first term of [ ] on the right-hand side is small, Eq.(18) is assumed to be parabolic and the necessary memory capacity can greatly be reduced. Furthermore, $\partial r_g / \partial n$ becomes nearly equal to unity, which results in $\lambda \approx 1/r_g$. The quantity $d\phi$ is proportional to the angle corresponding with an elemental displacement ds . An example for this type of flow is a concentric flow around a point.

Fig.2 Jet flow from the two-dimensional channel with parallel walls

3. Numerical example

3.1 Calculation procedure

As an application example for the basic equations derived in Chap.2, a liquid flow issuing from the two-dimensional channel with parallel walls into the atmosphere will be solved numerically. Body forces such as gravitational force are neglected.

This problem was already investigated by Tillet⁽¹⁰⁾ and Miyake et al.⁽¹¹⁾ They assumed, however, that the velocity distribution at the outlet section of the channel is same as that at an infinite distance upstream.

Here, this flow problem will be solved by numerical method without such assumption, with Eqs.(15), (20) and (22) approximated in the form of difference equations. The simplifications mentioned in 2.5 are not made in this case.

Since the flow is symmetrical with respect to the streamline OAB, it is sufficient to calculate only the flow on the upper half plane, namely OABCDEO, in Fig.2.

Quantities at point P are denoted by subscripts i and j . The distance OE is chosen as a reference length L^* and the average velocity along OE as a reference velocity q_0^* .

The velocity distribution q_{1j} at section OE is expressed as a function of ψ_{1j} with y_{1j} as a parameter.

$$q_{1j} = (3/2)(1 - y_{1j}^2) \dots\dots\dots(24)$$

$$\psi_{1j} = (3y_{1j}/2)(1 - y_{1j}^2/3) \dots\dots\dots(25)$$

Dividing the length OE by $(n-1)$, we have $y_{1j} = (j-1)/(n-1)$, and the step width $\Delta\psi_j$ in the ψ -direction becomes

$$\Delta\psi_j = \psi_{j+1} - \psi_j = (3/2)(y_{1,j+1} - y_{1j}) \times \{1 - (y_{1,j+1}^2 + y_{1,j+1}y_{1j} + y_{1j}^2)/3\} \dots\dots\dots(26)$$

When we transform the variable ϕ into a new variable ξ by $\xi = \tanh\phi$, $\phi = -\infty$ corresponds with $\xi = -1$ and $\phi = +\infty$ with $\xi = +1$. Putting $\Delta\xi = 1/(m-1)$, we obtain $\xi_i = (i-1)/(m-1) - 1$, $i = 1, 2, \dots, m$, and the step width $\Delta\phi_i$ in the ϕ -direction becomes

$$\Delta\phi_i = \phi_{i+1} - \phi_i = -\frac{1}{2} \log \left(\frac{1 - \xi_{i+1}}{1 + \xi_{i+1}} \right) \left(\frac{1 + \xi_i}{1 - \xi_i} \right) \dots\dots\dots(27)$$

In calculating the vorticity Ω at the internal points of the flow field, the first-order differential term of Eq.(22) was expressed as either forward difference or backward difference.⁽¹²⁾ This satisfies the sufficient condition for convergence in the case where a set of first-order simultaneous equations is solved by relaxation method.

Denoting the values of quantities at each point by the subscripts shown in Fig. 3, we obtain the following:- from Eq.(22)

$$\Omega_P = A_u \Omega_U + A_d \Omega_D + A_r \Omega_R + A_l \Omega_L \dots\dots\dots(28)$$

from Eq. (20)

$$q_P = B_r q_R + B_r' q_R' + C_r \dots (29)$$

and from Eq. (15)

$$1/\lambda_P = D_i/\lambda_L + D_r/\lambda_R \dots (30)$$

where

$$\left. \begin{aligned} A_u &= \frac{1}{K_P \Delta s_u} \left(\frac{1}{R_e \Delta s_P} + \frac{|L_P| - L_P}{2} \right), & A_d &= \frac{1}{K_P \Delta s_d} \left(\frac{1}{R_e \Delta s_P} + \frac{|L_P| + L_P}{2} \right) \\ A_r &= \frac{1}{K_P \Delta n_r} \left(\frac{1}{R_e \Delta n_P} + \frac{|M_P| + M_P}{2} \right), & A_i &= \frac{1}{K_P \Delta n_i} \left(\frac{1}{R_e \Delta n_P} + \frac{|M_P| - M_P}{2} \right) \\ B_r &= 1 - \frac{\Delta n_r}{\Delta n_r'} + \frac{\Delta n_r}{\lambda_R} \left(\frac{\lambda_P - \lambda_R}{\Delta n_r} + \frac{\lambda_R - \lambda_R'}{\Delta n_r'} \right), & B_r' &= \frac{\Delta n_r}{\Delta n_r'}, & C_r &= -2 \Delta n_r \Omega_R \\ D_i &= \frac{1}{N_P \Delta n_i}, & D_r &= \frac{1}{N_P \Delta n_r} \\ K_P &= \frac{1}{R_e} \left[\frac{1}{\Delta n_P} \left(\frac{1}{\Delta n_i} + \frac{1}{\Delta n_r} \right) + \frac{1}{\Delta s_P} \left(\frac{1}{\Delta s_d} + \frac{1}{\Delta s_u} \right) \right] + \frac{|M_P| + M_P}{2 \Delta n_r} \\ &\quad + \frac{|M_P| - M_P}{2 \Delta n_i} + \frac{|L_P| + L_P}{2 \Delta s_d} + \frac{|L_P| - L_P}{2 \Delta s_u} \\ L_P &= -q_P - \frac{1}{2 R_e q_P} \left(\frac{q_D - q_P}{\Delta s_d} + \frac{q_P - q_U}{\Delta s_u} \right), & M_P &= \frac{1}{2 R_e \lambda_P} \left(\frac{\lambda_L - \lambda_P}{\Delta n_i} + \frac{\lambda_P - \lambda_R}{\Delta n_r} \right) \\ N_P &= \frac{1}{\Delta n_i} + \frac{1}{\Delta n_r} - \frac{q_P \Delta n_P}{\Delta s_P} \left[\left(\frac{1}{q_D} - \frac{1}{q_P} \right) / \Delta s_d - \left(\frac{1}{q_P} - \frac{1}{q_U} \right) / \Delta s_u \right] \\ \Delta s_P &= (\Delta s_u + \Delta s_d)/2, & \Delta d_u &= 2 \Delta \phi_{i-1} / (\lambda_U + \lambda_P), & \Delta s_d &= 2 \Delta \phi_i / (\lambda_P + \lambda_D) \\ \Delta n_P &= (\Delta n_r + \Delta n_i)/2, & \Delta n_r &= 2 \Delta \phi_{j-1} / (q_R + q_P), & \Delta n_i &= 2 \Delta \phi_j / (q_P + q_L) \end{aligned} \right\} \dots (31)$$

Boundary conditions were determined as follows:

At section $\overline{OE}$ at an infinite distance upstream: the relation between q_{1j} and ψ_{1j} was obtained from Eqs. (24) and (25) with y_{1j} as parameters. λ_{1j} was set equal to q_{11} ($= 1.5$). Furthermore,

from Eqs. (20), (24) and (25) $\Omega_{1j} = 3y_{1j}$ was obtained.

On the symmetry line $\overline{OAB}$: $q_{i1} = q_{i2}$, $\lambda_{i1} = \lambda_{i2} = 1.5$, $\Omega_{i1} = 0$.

On the solid wall $\overline{ED}$: $q_{in} = 0$, $\Omega_{in} = q_{i,n-1}^2 / (2 \Delta \psi_{n-1})$

On the free surface $\overline{DC}$: $q_{in} = q_{i,n-1}$, $\partial p / \partial \phi = 0$

At the section $\overline{BC}$ at an infinite distance downstream:

$$q_{mj} = q_\infty, \lambda_{mj} = \lambda_{m-1,j} = 1.5, \Omega_{mj} = 0$$

The flow chart for calculation is shown in Fig. 4. First, we assume the velocity $q_\infty^{(1)}$ at an infinite distance downstream. Next, Eqs. (28)~(30) are calculated by relaxation method iteratively. When the sum S of the maximum values of the differences between the k th and the $(n-1)$ th calculated values, i.e.,

$$S = |\Omega_{ij}^{(k)} - \Omega_{ij}^{(k-1)}|_{\max} + |q_{ij}^{(k)} - q_{ij}^{(k-1)}|_{\max} + |\lambda_{ij}^{(k)} - \lambda_{ij}^{(k-1)}|_{\max}$$

is less than a certain value, calculation is considered convergent.

It is still unclear, however, whether the obtained values satisfy the boundary condition $\partial p / \partial \phi = 0$ on the free surface $\overline{DC}$.

One method to check this would be to calculate the pressure distribution on $\overline{DC}$ from Eq. (18). This method yields, however, considerable calculation errors near the free surface, because the variation in q in the ψ -direction is so large that the derivatives $\partial q / \partial \psi$ and $\partial^2 q / \partial \psi^2$ would cause large errors.

Instead of this method, we applied the momentum theory to the fluid particles in the control volume $\overline{OABCEO}$ and modified $q_\infty^{(k)}$ to make the pressure gradient zero on the free surface.

Fig. 3 Grid points

From the momentum theory we have

$$\int_{BC} q_{\infty} d\phi - \int_{OE} q d\phi = \int_{OE} p \frac{d\phi}{q} - \int_{EO} \tau \frac{d\phi}{\lambda} \quad \dots\dots\dots(32)$$

The pressure at point E is calculated from Eqs.(18), (13) and (14). Since the pressure is uniform along OE, we can calculate the value of the first term on the right-hand side of Eq.(32). The shear force along ED is calculated from the following equation:

$$\tau = \frac{q}{R_e} \frac{\partial q}{\partial \phi} \Big|_{\phi=1} \quad \dots\dots\dots(33)$$

from which the second term on the right-hand side of Eq.(32) is obtained.

Thus q_{∞} was calculated from the first term on the left-hand side of Eq.(32) and was checked whether it coincided with the assumed value of q_{∞} . Since the curvature $1/r_s$ of the streamlines within the channel is very small, this calculation seems more accurate. Iteration was made until $\epsilon = |q_{\infty}^{(k)} - q_{\infty}^{(k-1)}|$ became less than a limited value.

In execution, the following values were used:

$$m = 61, m_0 = 30, n = 31, S < 0.001, \epsilon < 0.01$$

3.2 Calculated results and discussion

Figure 5 shows the flow pattern at a Reynolds number of 100. Streamlines are generally smooth over the most parts of the flow field, except near the outlet of the channel where the streamlines curve sharply.

Figure 6 shows the velocity distribution at the outlet section of the channel. Although for $Re > 1000$ the velocity distribution at the outlet section is very similar to that at upstream infinity, the effect of downstream flow conditions gradually appears as Re decreases.⁽¹³⁾ This may be explained as follows: for a large Reynolds number the basic equations are considered parabolic, so that the effect of downstream conditions can be neglected.

Figure 7 shows the relation between shapes of free surfaces and Reynolds numbers. The free surfaces curve sharply near the outlet of the channel, and for $Re > 100$ the jet widths are kept almost constant with $x > 0.5$. If Re is small enough, however, there is a minimum cross section of the jet so called vena contracta, and then the jet width gradually increases toward downstream.⁽¹⁴⁾

Figure 8 shows the relation between jet width at downstream infinity and Reynolds number. For $Re > 1000$ the jet width takes a value of $5/6$ ⁽¹⁰⁾ and increases as Reynolds number decreases.

Fig.4 Flow chart

Fig.5 Shape of streamlines

Fig.6 Velocity distribution at the outlet section of the channel

The effect of surface tension appears more or less on a flow with free surfaces. The pressure difference Δp between the outside and inside of the free surface due to surface tension is calculated by

$$\Delta p = 1/(We, r_s) \dots \dots \dots (34)$$

where Weber number We is

$$We = \rho_0^* L^* \rho^* / \sigma^*$$

σ^* being surface tension. The curvature $1/r_s$ of the free surface directly downstream of the channel outlet takes a very large value. For instance, the curvature becomes about -4.95 , with $Re = 100$ and $\Delta s = 0.44$ where Δs is the length from point D along the free surface. For such a large curvature and small Weber number, the effect of surface tension must be taken into account in calculating Eq.(32).

4. Conclusions

For a two-dimensional steady incompressible flow, using the coordinate curve ϕ along a streamline and the curve ψ perpendicular to the ϕ -curve, the basic equations (13), (14), (18), (20) and (22) are derived with ϕ and ψ as independent variables.

A two-dimensional liquid jet flow from a channel with parallel walls was taken as a numerical example. The basic equations were solved by a finite-difference method. The results showed this method to be very effective.

Based on the work reported in this paper, the following conclusions appear to be valid:

- (1) This calculation method is very useful for the flows with free surfaces.
- (2) When the configuration of streamlines can roughly be predicted before calculation, the basic equations are simplified, and calculation becomes considerably easy.
- (3) For the jet flow from the channel with parallel walls:
  - (a) The flow is not affected by downstream conditions for a Reynolds number greater than 1000. This fact shows that the basic equations may be approximated by parabolic ones.
  - (b) For a Reynolds number greater than 100, the width of the jet becomes constant farther downstream than at a distance equal to a quarter of the channel width. However, for a Reynolds number less than 100 a minimum width of the jet exists.

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Fig.7 Shape of free surfaces

Fig.8 Jet width at an infinite distance downstream

References

- (1) Homma, H. and Uchida, S., Nomographs and Graphical calculation methods (in Japanese), (1965), p.189, Korona Co.
- (2) Smith, K.J. and Hamrick, J.T., NACA TN 3399, (1965)
- (3) Katsanis, T., NASA TN D-2546, (1964)
- (4) Senoo, Y. and Nakase, T., Trans. Japan Soc. Mech. Engrs. (in Japanese), Vol.37, No.302 (1971-10), p.1927
- (5) Daiguji, H. and Nozaki, A., Preprints Japan Soc. Mech. Engrs. (in Japanese), No.790-7 (1979-4), p.119
- (6) Duda, J.L. and Vrentas, J.S., Chem. Eng. Sci., Vol.22 (1967), p.855
- (7) Kondo, J., High Speed Aerodynamics (in Japanese), (1977), p.138, Korona Co.
- (8) Pratap, V.S. and Spalding, D.B., Int. J. Heat Mass Transfer, Vol.19 (1976), p.1183
- (9) Moore, J. and Moore, J.G., Trans. ASME, J. Fluids Eng., Vol.104, No.4 (1979-12), p.415
- (10) Tillet, J.P.K., J. Fluid Mech., Vol. 32, Part 2 (1968-5), p.273
- (11) Miyake, H. et al., Trans. Japan Soc. Mech. Engrs. (in Japanese), Vol.45, No.390 (1979-2), p.180
- (12) Gosman, A.D. et al., Heat and Mass Transfer in Recirculating Flow, (1969), p.94, Academic Press
- (13) Yamaguchi, R. and Takahashi, K., Bull. Japan Soc. Mech. Engrs., Vol.23, No. 185 (1980-11), p.1798
- (14) Goren, S.L. and Wronski, S.J., J. Fluid Mech., Vol.25, Part 1 (1966-5), p.185
- (15) Milne-Thomson, L.M., Theoretical Hydrodynamics, (1968), p.649, Macmillan Co.