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Nonlinear Forced Oscillations of a Rotating Shaft Carrying an Unsymmetrical Rotor at the Major Critical Speed*

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This paper deals with the motion of an unsymmetrical rotor possessing nonlinear spring characteristics, due to the angular clearance of a single-row deep groove ball bearing, in the vicinity of the major critical speed. When this system is operated near the major critical speed, no unstable vibration occurs, but we obtain peculiar resonance curves of hard spring type which have three stable amplitudes at most against the rotating speed. The resonance curves are classified into two types. They are the cases where the jump phenomenon appears once or twice during deceleration of the rotating speed. It is clarified that the type of the resonance curve depends on the angular position of rotor unbalance.

Key Words: Vibration of Rotating Body, Bearing, Nonlinear Forced Oscillation, Major Critical Speed, Resonance Curve, Jump Phenomenon

1. Introduction

There are many reports concerning unstable vibrations in unstable regions at the major critical speeds of unsymmetrical rotor systems and unsymmetrical shaft systems⁽¹⁾⁻⁽⁹⁾ with linear spring characteristics.

In the present paper, it is clarified both experimentally and theoretically that particular vibration phenomena appear in harmonic oscillations (that is, forced oscillations which have the same frequency as the angular velocity of the shaft) in the neighborhood of the major critical speed when the restoring force of a vertical shaft carrying an unsymmetrical rotor has nonlinear spring characteristics.

2. Equations of motion

Experiments were performed in a four-degree-of-freedom system where the deflection r and inclination θ of the rotor are coupled with each other. If we suppose the case in which such coupling does not exist, the inclination oscillation and the deflection oscillation can be treated independently. Because, as explained later, this two-degree-of-freedom system for inclination oscillation has the same vibration characteristics as those of a four-degree-of-freedom system of the present experimental apparatus, we treat a two-degree-of-freedom system in theoretical analysis for simplicity.

We consider a stationary rectangular coordinate system $O-xyz$ where the origin O is selected at the equilibrium point of the center of gravity of the rotor and the z -axis coincides with the bearing center line (that is, the line which connects the centers of the upper and the lower bearings). We let the projectional angles of θ to the xz - and yz -planes be θ_x and θ_y , respectively; the polar moment of inertia of the unsymmetrical rotor be I_p ; the diametral moments of inertia of it be I_1 and I_2 ($I_1 > I_2$); the mean and the difference of these diametral moments of inertia be $I = (I_1 + I_2)/2$, $\Delta I = (I_1 - I_2)/2$; the dynamic unbalance of the rotor be τ ; the bending stiffness of the shaft be δ ; the damping coefficient be c ; the rotating speed of the shaft be ω ; the time be t ; and the coefficients of the unsymmetrical nonlinear terms (the second power terms) and the symmetrical nonlinear terms (the third power terms) in the restoring moment be ϵ_{ij} ($i+j=3$) and β_{ij} ($i+j=4$), respectively. The time when the principal axis of moment of inertia I_1 passes the xz -plane is taken as $t=0$. The angular position of the dynamic unbalance measured from the principal axis of moment of inertia I_1 in the direction of rotation is designated by α_i . Introducing the dimensionless quantities

$$\left. \begin{aligned} i_p &= I_p/I, & t' &= t\sqrt{\delta/I}, & \omega' &= \omega\sqrt{I/\delta}, \\ c' &= c/\sqrt{\delta I}, & \Delta_i &= \Delta I/I, & N_{\theta x}' &= N_{\theta x}/\delta, \\ N_{\theta y}' &= N_{\theta y}/\delta, & \epsilon_{ij}' &= \epsilon_{ij}/\delta, & \beta_{ij}' &= \beta_{ij}/\delta \end{aligned} \right\} \dots (1)$$

and supposing that the quantities Δ_i and τ are the quantities of the same order as a small parameter ϵ (we express this order by the symbol $O(\epsilon)$), we obtain the following equations of motion for inclination oscillations represented by dimensionless quantities within the accuracy of $O(\epsilon)^{11}$. If we express the nonlinear terms in restoring moment by the symbols $N_{\theta x}$ and $N_{\theta y}$, the equations of motion become

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$$\left. \begin{aligned} \ddot{\theta}_x + i_p \omega \dot{\theta}_y + c \dot{\theta}_x + \theta_x - \Delta_i \frac{d}{dt} (\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) + N_{\theta x} &= F \cos(\omega t + \alpha_i) \\ \ddot{\theta}_y - i_p \omega \dot{\theta}_x + c \dot{\theta}_y + \theta_y - \Delta_i \frac{d}{dt} (\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t) + N_{\theta y} &= F \sin(\omega t + \alpha_i) \end{aligned} \right\} \dots\dots\dots (2)$$

where $F = \tau \omega^2 (1 - i_p)$. The primes of the dimensionless quantities are omitted. We consider the nonlinear terms up to the third power terms of coordinates, as mentioned above, and they are expressed as follows⁽¹⁰⁾:

$$\left. \begin{aligned} N_{\theta x} &= (3\epsilon_{30}\theta_x^3 + 2\epsilon_{21}\theta_x\theta_y + \epsilon_{12}\theta_y^3) + (4\beta_{40}\theta_x^4 + 3\beta_{31}\theta_x^3\theta_y + 2\beta_{22}\theta_x^2\theta_y^2 + \beta_{13}\theta_y^4) \\ N_{\theta y} &= (\epsilon_{21}\theta_x^2 + 2\epsilon_{12}\theta_x\theta_y + 3\epsilon_{03}\theta_y^3) + (\beta_{31}\theta_x^3 + 2\beta_{22}\theta_x^2\theta_y + 3\beta_{13}\theta_x\theta_y^2 + 4\beta_{04}\theta_y^4) \end{aligned} \right\} \dots\dots\dots (3)$$

The quantities c , Δ_i , $N_{\theta x}$, $N_{\theta y}$, and F in Eq.(2) are $O(\epsilon)$.

We denote the natural frequency of the system by p , and put $\bar{p} = 2\omega - p$. The frequency equation of Eq.(2) is obtained as follows:

$$\left. \begin{aligned} G(p)G(\bar{p}) - (\Delta_i p \bar{p})^2 &\equiv (1 + i_p \omega p - p^2)(1 + i_p \omega \bar{p} - \bar{p}^2) - (\Delta_i p \bar{p})^2 = 0 \\ G(p) &= 1 + i_p \omega p - p^2, \quad G(\bar{p}) = 1 + i_p \omega \bar{p} - \bar{p}^2 \end{aligned} \right\} \dots\dots\dots (4)$$

This equation has at most four real roots for a given value of ω . Among these four roots, we represent the roots which have the corresponding natural frequencies in a symmetrical system (that is, the system where a symmetrical rotor (a rotor where $I_1 = I_2$ and $\Delta I = 0$) is mounted on a rotating shaft with circular cross section) by $p_f (>0)$ and $p_b (<0)$, and the other two roots by $\bar{p}_f (=2\omega - p_f)$ and $\bar{p}_b (=2\omega - p_b)$. A p - ω diagram is shown in Fig.1. This diagram has been calculated for $i_p = 0.7$ and $\Delta_i = 0.015$. In a rotating shaft system with linear spring characteristics, unstable vibrations with the frequency ω appear in the region between $\omega_{c1} (=1/\sqrt{1-i_p+\Delta_i})$ and $\omega_{c2} (=1/\sqrt{1-i_p-\Delta_i})$ which are derived by putting $p=\omega$ in Eq.(4). In the following chapters, we investigate the vibratory characteristics of a harmonic oscillation (that is, the oscillation with the frequency ω) in this speed range in the case where nonlinear spring characteristics exist.

Fig.1 p - ω diagram (A two-degree-of-freedom system)

3. On the harmonic oscillation at the major critical speed

We put the solution with frequency ω at the major critical speed in the following form.

$$\left. \begin{aligned} \theta_x &= P \cos(\omega t + \beta) + \epsilon \{a_1 \cos(\omega t + \beta) + b_1 \sin(\omega t + \beta)\} \\ \theta_y &= P \sin(\omega t + \beta) + \epsilon \{a_1' \sin(\omega t + \beta) + b_1' \cos(\omega t + \beta)\} \end{aligned} \right\} \dots\dots\dots (5)$$

This solution is considered within the accuracy of $O(\epsilon)$. The amplitudes P , a_1 , a_1' , b_1 , b_1' and phase angle β are quantities of $O(\epsilon^0)$ which vary slowly with time. Because the nonlinear terms $N_{\theta x}$, $N_{\theta y}$ in the x - and y -directions in Eq.(2) are different as shown in Eq.(3), the whirling orbit of the shaft shifts slightly from a circle. The second terms in the right sides of Eq.(5) represent the degree of this discrepancy. Substituting Eq.(5) into Eq.(2) and using the principle of harmonic balance concerning $\cos(\omega t + \beta)$ and $\sin(\omega t + \beta)$ in the accuracy of $O(\epsilon)$, we obtain the following equations after some calculations. Here, for convenience sake, the variables u and v ($u = P \cos \beta$, $v = P \sin \beta$) are used.

$$\left. \begin{aligned} (2 - i_p) \omega \dot{u} &= -c \omega u - (G + 4\beta^{(0)} P^2 + \Delta_i \omega^2) v + F \sin \alpha_i \\ (2 - i_p) \omega \dot{v} &= (G + 4\beta^{(0)} P^2 - \Delta_i \omega^2) u - c \omega v - F \cos \alpha_i \end{aligned} \right\} \dots\dots\dots (6)$$

where

$$G = G(\omega) = 1 + i_p \omega^2 - \omega^2, \quad \beta^{(0)} = (3\beta_{40} + \beta_{22} + 3\beta_{04})/8 \dots\dots\dots (7)$$

As mentioned above, $\beta^{(0)}$ is the coefficient of an isotropic symmetrical nonlinear term. Namely, it is known from Eq.(6) that only the nonlinear term which is uniform directionally has influence on this kind of oscillation. Expressing the stationary solutions of Eq.(6) by $u = u_0$, $v = v_0$, $P_0^2 = u_0^2 + v_0^2$, we obtain the following equations:

$$\left. \begin{aligned} c \omega u_0 + (G + 4\beta^{(0)} P_0^2 + \Delta_i \omega^2) v_0 &= F \sin \alpha_i \\ (G + 4\beta^{(0)} P_0^2 - \Delta_i \omega^2) u_0 - c \omega v_0 &= F \cos \alpha_i \end{aligned} \right\} \dots\dots\dots (8)$$

By solving the above equations for u_0 and v_0 , we get the following equations which give the stationary solution $P = P_0 (= \sqrt{u_0^2 + v_0^2})$ and $\beta = \beta_0 (= \tan^{-1}(v_0/u_0))$:

$$\left. \begin{aligned} & \{k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2\}^2 P_0^2 - \{k^2 + (\Delta_i \omega^2)^2 + (c\omega)^2\} F^2 \\ & - 2\Delta_i \omega^2 F^2 (k \cos 2\alpha_i + c\omega \sin 2\alpha_i) = 0 \\ & \tan \beta_0 = \frac{(k - \Delta_i \omega^2) \sin \alpha_i - c\omega \cos \alpha_i}{(k + \Delta_i \omega^2) \cos \alpha_i + c\omega \sin \alpha_i} \end{aligned} \right\} \dots\dots\dots (9)$$

where $k = G + 4\beta^{(0)}P_0^2$. Considering small deviations from the stationary solutions and substituting $u = u_0 + \xi$ and $v = v_0 + \eta$ into Eq.(6), we obtain the differential equations for ξ and η . From the characteristic equation of this equation, we get the following condition for a stable stationary solution:

$$k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2 + 8\beta^{(0)}P_0^2(k + \Delta_i \omega^2 \cos 2\beta_0) > 0 \dots\dots\dots (10)$$

Differentiating the first equation of Eq.(9) partially with respect to P_0 and putting $\partial\omega/\partial P_0 = 0$, we obtain

$$P_0 \{k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2\} \{k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2 + 8\beta^{(0)}P_0^2(k + \Delta_i \omega^2 \cos 2\beta_0)\} = 0 \dots\dots\dots (11)$$

The relation $P_0 \{k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2\} \neq 0$ holds in general, and we get

$$k^2 - (\Delta_i \omega^2)^2 + (c\omega)^2 + 8\beta^{(0)}P_0^2(k + \Delta_i \omega^2 \cos 2\beta_0) = 0 \dots\dots\dots (12)$$

Therefore, from Eqs.(10) and (12), we know that the point where the tangent line becomes vertical is consistent with a boundary between a stable and an unstable resonance curves.

(a) Resonance curves
(A case of jumping to P_{02})

(b) A phase angle diagram
(A case of jumping to β_{02})

(c) Solution curves ($\omega = 1.935$)
Fig.2

(a) Resonance curves
(A case of jumping to P_{01})

(b) A phase angle diagram
(A case of jumping to β_{01})

(c) Solution curves ($\omega = 1.935$)
Fig.3

Examples of resonance curves calculated by Eq.(9) are shown in Figs.2a and 3a. These examples have been calculated for a positive value of $\beta^{(0)}$ and the resonance curves belong to a hard spring type. The ordinate represents the amplitude P_0 , and the abscissa represents the angular velocity ω of the shaft. The full and the broken lines represent the stable and the unstable solutions, respectively. The resonance curves have at most five values of amplitude for a given value of ω , and the stable solutions are named P_{01}, P_{02}, P_{05} while the unstable solutions are named P_{03}, P_{04} in order of magnitude. When the angular velocity ω decreases from the higher speed side, the amplitude changes along the stable curve P_{05} and jumps from the point 0 to the other point 1 or 2 on the stable resonance curves P_{01} and P_{02} at the speed marked A in the figure. By integrating Eq.(6) numerically and investigating the transient solution in jump phenomena, we find the point to which it jumps. Figure 2a and Fig.3a are examples of the jump phenomena to the points 2 and 1, respectively, and the jumps are indicated by arrows. Figure 2b and Fig.3b are phase angle diagrams corresponding to Fig.2a and Fig.3a, respectively, and $\beta_{01}, \beta_{02}, \dots$ are phase angles corresponding to the amplitudes $P_{01}, P_{02}, \dots$. When the amplitude changes, the phase angle β_0 also jumps as indicated by arrows.

In Figs.2c and 3c, the transient solution jumping from the point 0 is shown by the broken line in a ωv -plane. The points 1, 2, and 3 in the figures correspond to the

Fig.4 A relation between the type of jump and the magnitude τ and phase angle α of the dynamic unbalance

Fig.5 An effect of the unsymmetry Δ_i

points 1, 2, and 3 in Figs.2a, 2b, 3a, and 3b, and these points are singular points (corresponding to stationary solutions) in the ωv -plane. The points 1 and 2 are stable spiral points and the point 3 is an unstable saddle point. The solution curves (shown by full lines) passing the point 3 divides the phase plane into two regions and all the solution curves in each region arrive at the stable equilibrium point in respective regions. The point 0 is the starting point in jump and the locus during jump is shown by a broken line with an arrow. The amplitude and phase angle jump to the point 2 in Fig.2c and to the point 1 in Fig.3c.

We shall investigate which type of jump phenomena between Fig.2 and Fig.3 occurs depending on the angular position α_i of dynamic unbalance. The result is shown in Fig.4 where the magnitude and the angular position of dynamic unbalance τ are expressed by a polar coordinate system (τ, α_i) . It jumps to the point 2 as in Fig.2a in the shaded region and jumps to the point 3 as in Fig.3a in the other region. The angles $\angle AOA'$ and $\angle BOB'$ in the figure are zero when $c = 0$ and they increase as the damping coefficient c increases.

Figure 5 shows an effect of the unsymmetry Δ_i . In the case of $\Delta_i = 0$ where a symmetrical rotor is mounted on a shaft with circular cross section, the same type of resonance curve having at most three stationary solutions as that of the ordinary symmetrical nonlinear system is obtained. As Δ_i increases, the width of the resonance curves expands and five stationary solutions come into existence.

Figure 6 shows an effect of the dynamic unbalance τ . The one dot chain lines (the curves A and A') in the figure are obtained by putting $\tau = 0$ in the first equation in

Fig.6 An effect of the dynamic unbalance τ

Fig.7 An effect of the coefficient of nonlinear terms $\beta^{(0)}$

Eq.(9). As τ increases, the resonance curves become wide, leaving the curves A and A'. The two dots chain lines obtained by Eq.(12) are the boundary lines dividing into the stable and the unstable regions.

Figure 7 shows an effect of the coefficient $\beta^{(0)}$ of nonlinear term. The resonance curves become a hard spring type when $\beta^{(0)}$ is positive, they become a soft spring type when $\beta^{(0)}$ is negative; and they become a type

of a linear system and unstable region appears when $\beta^{(0)}=0$. When $\beta^{(0)}=0$, at least one stable solution exists and the unstable region disappears because the resonance curves bend toward higher speed side or lower speed side.

The amplitude P_0 and phase angle β_0 of the oscillation with frequency ω in the case where $\tau=0$ are expressed as follows:

$$\left. \begin{aligned} (G+4\beta^{(0)}P_0^2)^2 - (\Delta_1\omega^2)^2 + (c\omega)^2 &= 0 \\ \tan \beta_0 &= -\frac{c\omega}{G+4\beta^{(0)}P_0^2+\Delta_1\omega^2} = \frac{G+4\beta^{(0)}P_0^2-\Delta_1\omega^2}{c\omega} \end{aligned} \right\} \dots\dots\dots(13.a)$$

$$P_0=0 \dots\dots\dots(13.b)$$

The equations for the A- and A'-curves in Fig.6 are obtained from Eq.(13.a). It is also proved that A is stable and A' is unstable. The curves with zero amplitude expressed by Eq.(13.b) coincide with the abscissa, and this solution $P_0=0$ is unstable in the region $\omega_{c1}' < \omega < \omega_{c2}'$ in Fig.6 and stable outside of it. The ω_{c1}' and ω_{c2}' in the figure are given by the following equations:

$$\omega_{c1}' = \frac{1}{(1-i_p)^2 - \Delta_1^2} \left\{ 1-i_p - \frac{c^2}{2} \mp \sqrt{\Delta_1^2 + \frac{c^4}{4} - (1-i_p)c^2} \right\} \dots\dots\dots(14)$$

4. Theoretical analysis in a four-degree-of-freedom system

In this chapter, we shall analyze the same four-degree-of-freedom system where the deflection r and the inclination θ of the rotor are coupled like the experimental apparatus. From the results of this analysis, it follows that the constitution of equations in this system is the same as the corresponding equations in the two-degree-of-freedom system and consequently the vibratory characteristics are the same qualitatively in these two systems.

Let the deflections of the rotor be x and y , the mass of the rotor be m , the spring constants of the shaft be α , γ , and δ , the static and dynamic unbalances of the rotor be e and τ , the angular positions of these unbalances measured from the direction of the principal moment of inertia I , be α_e and $\alpha_\tau + \pi$, the damping coefficients be c_{11} , $c_{21}(=c_{12})$ and c_{22} , and the nonlinear parts in restoring forces be N_x , N_y , $N_{\theta x}$, and $N_{\theta y}$. For simplicity, we shall adopt the following dimensionless quantities:

$$\left. \begin{aligned} I_p/I &= i_p, & \Delta I/I &= \Delta_1, & x/\sqrt{I/m} &= x', & y/\sqrt{I/m} &= y', & e/\sqrt{I/m} &= e' \\ t\sqrt{\alpha/m} &= t', & \omega\sqrt{m/\alpha} &= \omega', & \gamma/(\alpha\sqrt{I/m}) &= \gamma', & m\delta/(\alpha I) &= \delta' \\ c_{11}/\sqrt{m\alpha} &= c_{11}', & c_{12}/\sqrt{\alpha I} &= c_{12}', & c_{22}/(I\sqrt{\alpha/m}) &= c_{22}', & N_x/(\alpha\sqrt{I/m}) &= N_x' \\ N_y/(\alpha\sqrt{I/m}) &= N_y', & N_{\theta x}/(\alpha I/m) &= N_{\theta x}', & N_{\theta y}/(\alpha I/m) &= N_{\theta y}' \end{aligned} \right\} \dots\dots\dots(15)$$

The dimensionless equations of motion are obtained as follows within the accuracy of $O(\epsilon)^{1)}$

$$\left. \begin{aligned} \ddot{x} + c_{11}\dot{x} + c_{12}\dot{y} + x + \gamma\theta_x + N_x &= e\omega^2 \cos(\omega t + \alpha_e) \\ \ddot{y} + c_{11}\dot{y} + c_{12}\dot{x} + y + \gamma\theta_y + N_y &= e\omega^2 \sin(\omega t + \alpha_e) \\ \ddot{\theta}_x + i_p\omega\dot{\theta}_y + c_{21}\dot{x} + c_{22}\dot{y} + \gamma x + \delta\theta_x - \Delta_1 \frac{d}{dt}(\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) \\ &+ N_{\theta x} = (i_p - 1)\tau\omega^2 \cos(\omega t + \alpha_\tau) \\ \ddot{\theta}_y - i_p\omega\dot{\theta}_x + c_{21}\dot{y} + c_{22}\dot{x} + \gamma y + \delta\theta_y - \Delta_1 \frac{d}{dt}(\dot{\theta}_x \sin 2\omega t - \dot{\theta}_y \cos 2\omega t) \\ &+ N_{\theta y} = (i_p - 1)\tau\omega^2 \sin(\omega t + \alpha_\tau) \end{aligned} \right\} \dots\dots\dots(16)$$

The primes of the dimensionless quantities are omitted in these equations. The solutions of Eq.(16) in the neighborhood of the major critical speed can be expressed as follows within the accuracy of $O(\epsilon)$.

$$\left. \begin{aligned} x &= u_1 \cos \omega t - v_1 \sin \omega t + \epsilon(d_1 \cos \omega t + e_1 \sin \omega t) \\ y &= u_1 \sin \omega t + v_1 \cos \omega t + \epsilon(d_1' \sin \omega t + e_1' \cos \omega t) \\ \theta_x &= u_2 \cos \omega t - v_2 \sin \omega t + \epsilon(d_2 \cos \omega t + e_2 \sin \omega t) \\ \theta_y &= u_2 \sin \omega t + v_2 \cos \omega t + \epsilon(d_2' \sin \omega t + e_2' \cos \omega t) \end{aligned} \right\} \dots\dots\dots(17)$$

The amplitudes u_1 , u_2 , v_1 , v_2 , d_1 , d_1' , d_2 , d_2' , e_1 , e_1' , e_2 , and e_2' are quantities of $O(\epsilon^0)$ varying slowly with time. By substituting Eq.(17) into Eq.(16), we get the equations corresponding to Eq.(6). And from these equations, we obtain the following relations within the accuracy of $O(\epsilon^0)$:

$$u_2/u_1 = v_2/v_1 = u_{20}/u_{10} = v_{20}/v_{10} = -(1-\omega^2)/\gamma = -\gamma/(\delta + i_p\omega^2 - \omega^2) = \kappa \dots\dots\dots(18)$$

where u_{10} and v_{20} represent stationary solutions. With κ of Eq.(18), the equations corresponding to Eq.(6) are

$$\left. \begin{aligned} \omega \dot{u}_1 &= -c' \omega u_1 - (f + \beta^{(0)'} R^2 + \Delta_i' \omega^2) v_1 + F' \sin \alpha_i' \\ \omega \dot{v}_1 &= (f + \beta^{(0)'} R^2 - \Delta_i' \omega^2) u_1 - c' \omega v_1 - F' \cos \alpha_i' \end{aligned} \right\} \dots\dots\dots (19)$$

where

$$\left. \begin{aligned} A &= 2G' - \gamma \kappa (2 - i_p), & c' &= \{G'(c_{11} + \kappa c_{12}) - \gamma(c_{21} + \kappa c_{22})\} / A \\ F' &= (\omega^2 / A) \sqrt{(G'e)^2 + \gamma^2 \tau^2 (i_p - 1)^2 - 2G'\gamma(i_p - 1)e\tau \cos(\alpha_e - \alpha_\tau)} \\ f &= \{(1 - \omega^2)G' - \gamma^2\} / A, & G' &= \delta + i_p \omega^2 - \omega^2, & R^2 &= u_1^2 + v_1^2 \\ \tan \alpha_i' &= \{G'e \sin \alpha_e - \gamma \tau (i_p - 1) \sin \alpha_\tau\} / \{G'e \cos \alpha_e - \gamma \tau (i_p - 1) \cos \alpha_\tau\} \\ \beta_{1(0)} &= 4\beta_{40(0)} + 3\kappa\beta_{31(0)} + 2\kappa^2\beta_{22(0)} + 2\kappa^2\beta_{22(0)} + \kappa^3\beta_{18(0)} \\ \beta_{2(0)} &= \beta_{31(0)} + 2\kappa\beta_{22(0)} + 2\kappa\beta_{22(0)} + 3\kappa^2\beta_{18(0)} + 4\kappa^3\beta_{04(0)} \\ \beta^{(0)'} &= (G'\beta_{1(0)} - \gamma\beta_{2(0)}) / A, & \Delta_i' &= \Delta_i(1 - \omega^2) / A \end{aligned} \right\} \dots\dots\dots (20)$$

The coefficients $\beta_{40}^{(0)}$, $\beta_{31}^{(0)}$, etc., and the coefficient $\beta^{(0)'}$ in Eqs.(19) and (20) are coefficients of isotropic symmetrical nonlinear terms⁽¹⁰⁾. They are the same kind of coefficients as $\beta^{(0)}$ in Eqs.(6)~(9).

By comparing Eq.(19) for a four-degree-of-freedom system with Eq.(6) for a two-degree-of-freedom system, we know that they are the same in construction and have the same dynamical characteristics of their coefficients. Therefore, it is clear that Eq.(19) gives qualitatively the same results as those of inclination oscillation in a two-degree-of-freedom system mentioned above.

5. Experimental apparatus and experimental results

The experimental apparatus is shown in Fig.8. An unsymmetrical rotor R was mounted on a vertical elastic shaft with circular cross section at the position of the ratio $a:b=1:4$. The shaft was 700mm ($=l$) in length, 12mm ($=d$) in diameter, and it was supported by a self-aligning double-row ball bearing (#1200) at the upper end and by a single-row deep groove ball bearing (#6204) at the lower end. The dimensions of the unsymmetrical rotor were 9.746kg in mass,

Fig.8 An experimental apparatus

$I_p=0.2342\text{kg}\cdot\text{m}^2$, $I_1=0.1558\text{kg}\cdot\text{m}^2$ (about x' -axis), and $I_2=0.0799\text{kg}\cdot\text{m}^2$ (about y' -axis). The deflections of the rotor were optically recorded in x and y directions which were rectangular in the horizontal plane. A light source SA with a band-shaped filament was set under the rotor and a real image was produced at the rotor edge (the point A). This image was partly obscured by the rotor before passing through the lens L_2 , then was reflected away by the mirror M and focused again on the film F⁽¹¹⁾. The deflection in the y -direction was also measured at the point B in the same way.

When the experimental apparatus was assembled carefully to attain good alignment of the center lines of the upper and the lower bearing pedestals and the bearing center line was situated at the middle of the angular clearance of the lower bearing, the restoring force of the shaft had strong symmetrical nonlinear spring characteristics⁽¹²⁾.

(a) Resonance curves (A case of jumping to P_{02})

(b) Phase angle curves (A case of jumping to β_{02})

Fig.9

After balancing the rotor sufficiently, we performed many experiments for various magnitudes and directions of the static unbalance by changing the small correction weight.

Typical examples of experimental results on resonance curves and phase angle curves are shown in Figs. 9a, 9b, 10a, and 10b. Due to the strong symmetrical nonlinear spring characteristics, the resonance curves are a hard spring type⁽²⁾. In Fig. 9a, when the shaft was decelerated from the higher speed side, the oscillation jumped to the curve P_{02} at $\omega = 1111$ rpm, passed along the curve P_{02} and jumped again to the curve P_{01} at $\omega = 920$ rpm as shown by arrows. This figure has qualitatively the same characteristics as Fig. 2a. Figure 10 shows the case where it jumped not to the curve P_{02} (shown by the symbols Φ and Θ) but to the curve P_{01} at $\omega = 1029$ rpm. Although the amplitudes P_{01} and P_{02} are almost the same in magnitude as shown in Fig. 10a, the values of phase angles β_{01} and β_{02} are distinctly different from each other as shown in Fig. 10b. Therefore, we can make a clear distinction between the P_{01} curve and the P_{02} curve in recorded waves and can easily determine that the oscillation jumped to P_{01} in this case. The curve P_{02} in Fig. 10a was obtained by forcing a jump to it with impulse. Figure 10a is qualitatively the same as Fig. 3a. The curves P_{01} in Figs. 9a and 10a were measured up to the amplitude about 1.5 mm for safety, and no jump phenomenon was observed in acceleration.

The relation between the type of jump and the magnitude and phase angle of unbalance of the rotor is shown in Fig. 11. As the discrepancy between the direction of the unbalance and the direction of the shaft deflection became large when the shaft rotating speed approached the resonance point,

(a) Resonance curves (A case of jumping to P_{01})

(b) Phase angle curves (A case of jumping to β_{01})
Fig. 10

we measured the amplitude r and the phase angle β at $\omega = 1800$ rpm where the rotating speed was at some distance from the resonance point and the discrepancy is almost negligible. The symbol $\bullet$ and $\circ$ in the figure correspond to the resonance curves of the types in Fig. 9a and Fig. 10a, respectively. These experimental results agree with the theoretical results shown in Fig. 4. In the above-mentioned examples, $(r, \beta) = (0.176 \text{ mm}, 155.5^\circ)$ in Fig. 9a and $(r, \beta) = (0.156 \text{ mm}, 19.5^\circ)$ in Fig. 10a.

We also experimented on the case of linear spring characteristics by changing the lower bearing to a self-aligning double-row ball bearing (#1204). The width of the unstable region was about 110 rpm.

6. Conclusions

The theoretical and the experimental conclusions about the harmonic oscillations at the major critical speed in an unsymmetrical rotor system with symmetrical nonlinear spring characteristics are summarized as follows:

- (1) The unstable region appearing in a linear system disappears in a nonlinear system and stationary resonance curves of a hard or soft spring type are obtained.
- (2) The resonance curves can be divided into three regions concerning the rotating speed, that is, the regions where one, three, and five stationary amplitudes exist, and jump phenomena occur at the rotating speeds of their boundaries.
- (3) Among various kinds of nonlinear spring characteristics, only the isotropic nonlinear component has influence on this oscillation.
- (4) The four-degree-of-freedom system where the deflection and the inclination of the shaft couple with each other and the two-degree-of-freedom system of inclination oscillation which does not couple with deflection oscillation have qualitatively the same vibratory characteristics.
- (5) In the experiments with the above-mentioned unsymmetrical rotor, no jump phenomena appeared in acceleration processes.
- (6) When the shaft was decelerated from the higher speed side, two types of resonance curves where one or two jumps appears were

Fig. 11 The relation between the type of jump and the magnitude and phase angle of the unbalance (r and β are the amplitude and phase angle of the static deflection at $\omega = 800$ rpm. r and β correspond to r and α in Fig. 4.)

obtained.

(7) It depends on the angular position of the unbalance which type of these two kinds of vibration phenomena appears. The boundaries of these regions concerning angular position coincide approximately with the main axis of inertia of the rotor.

References

- (1) Yamamoto, T. and Ota, H., *Bull. JSME*, Vol.6, No.21 (1963), p.29.
- (2) Yamamoto, T. and Ota, H., *Bull. JSME*, Vol.6, No.23 (1963), p.404.
- (3) Yamamoto, T. and Ota, H., *Trans. Jpn. Soc. Mech. Eng.* (in Japanese), Vol.33, No.247 (1967), p.365.
Yamamoto, T. and Ota, H., *Memoirs of the Faculty of Engineering, Nagoya Univ.*, Vol.19, No.2 (1967), p.197.
- (4) Brosens, P.J. and Crandall, S.H., *Trans. ASME, Ser.E*, Vol.28, No.3 (1961), p.355.
- (5) Smith, D.M., *Proc. R. Soc. Lond., Ser. A*, Vol.142 (1933), p.92.
- (6) Den Hartog, J.P., *Mechanical Vibrations*, (1956), p.336, McGraw-Hill.
- (7) Taylor, H.D., *Trans. ASME, Ser.E*, Vol.62 (1940), p.A71.
- (8) Hull, E. H., *Trans. ASME, Ser.B*, Vol.83, No.2 (1961), p.219.
- (9) Ariaratnum, S. T., *Trans. ASME, Ser.E*, Vol.87, No.1 (1965), p.157.
- (10) Yamamoto, T. and Ishida, Y., *Trans. Jpn. Soc. Mech. Eng.* (in Japanese), Vol.41, No.345 (1975), p.1374.
Yamamoto, T. and Ishida, Y., *Ing.-Arch.*, Band 46, Heft 2 (1977), p.125.
- (11) Yamamoto, T., *Memoirs of the Faculty of Engineering, Nagoya University*, Vol.6, No.2 (1954), p.110.
- (12) Yamamoto, T. et al., *Bull. JSME*, Vol.18, No.123 (1975), p.965.