Extended Perfect Model Following*

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The perfect model following (PMF) is an established technique in control system synthesis which has such a characteristic that the state variables of plant perfectly follow those of model. However, the condition for PMF to be able to synthesize is rather severe. In order to relax this condition and enable easy synthesis of PMF system, extended PMF (EPMF) is proposed. EPMF is a PMF where the output (not state variables) of plant perfectly follows that of model. A condition of EPMF and design procedure of EPMF are derived. Sensitivity reduction in EPMF is analyzed. Design examples for some systems and an aircraft control system are performed and simulated. Possibility of wide application and desirable characteristics of EPMF are shown.

Key Words: Control Theory, Application of Control, Perfect Model Following, Output Control, Sensitivity Reduction, Aircraft

1. Introduction

Model following technique has been considered a method to give desired properties to control systems. Especially, the perfect model following (PMF) system with structure shown in Fig. 1 has such a characteristic that state variable of plant perfectly follows that of model (1), (2). For this system, it is necessary to have the following relations among feedback gain K_p , feedforward gains K_m and K_v in the figure.

$$B_p K_m = B_p K_p + (A_m - A_p) \quad (1)$$

$$B_p K_v = B_m \quad (2)$$

In order to satisfy these relations, however, the following condition has to be satisfied:

$$\begin{aligned} \text{rank } B_p &= \text{rank } [B_p, B_m] \\ &= \text{rank } [B_p, (A_m - A_p)]. \end{aligned} \quad (3)$$

This is rather a severe condition and properties of models which can be followed are extremely limited and therefore the applicable field of this technique is much restricted. That is, the model and the plant must have the same dimensions and zeros of transfer function of the model must all coincide with those of the plant and moreover the form of system matrix A_m of the model is severely restricted by the condition of Eq. (3). In order to remove these defects and to facilitate the designing of a model following system, this extended PMF

system is proposed.

The extended PMF system is different from the conventional PMF system in two points. That is, the output of the plant Y_p is made to perfectly coincide with that of the model Y_m using output matrices C_p and C_m , and K_m may have some elements of functions of s . In this system, Y_p and Y_m should have the same dimensions while dimensions of state variables X_p and X_m need not coincide. Therefore a plant can follow a model with dimension and zeros of transfer function different from those of the plant. When all state variables of a plant need not coincide with those of a model, it is possible to select some state variables of a plant which follow a model. For these reasons, a set of models which can be followed is enlarged and special consideration is not necessary in the selection of model and consequently area of the application is broadened.

On the other hand, it is well-known that sensitivity of a plant to parameter perturbation can be more reduced in closed loop system with optimal feedback gain than in open loop system (3), (4). Because PMF system corresponds to a nominally equivalent closed loop system, sensitivity of PMF system is reduced more than that of a system compensated by inserting an element in series to the plant. By using this extended PMF system, it is also possible to reduce sensitivity for selected state variables.

Fig. 1 Structure of PMF system

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2. Extended PMF

Fig. 2 shows a block diagram of the extended PMF system. System equations of the plant and the model are written as follows:

$$\text{plant } \dot{X}_p(t) = A_p X_p(t) + B_p U_p(t) \quad (4)$$

$$Y_p(t) = C_p X_p(t) \quad (5)$$

$$\text{model } \dot{X}_m(t) = A_m X_m(t) + B_m U_m(t) \quad (6)$$

$$Y_m(t) = C_m X_m(t), \quad (7)$$

where $X_p(t) \in R^n$ is a state variable of the plant, $Y_p(t) \in R^m$ is the plant output, $U_p(t) \in R^r$ is the plant input, $X_m(t) \in R^p$ is a state variable of the model, $Y_m(t) \in R^m$ is the model output and $U_m(t) \in R^q$ is the model input. It is supposed that both systems are controllable and observable.

From Laplace transform of Eqs. (6) and (7) with all initial conditions set equal to zero, output of the model is expressed by using transfer function matrix as follows:

$$Y_m(s) = C_m(sI - A_m)^{-1} B_m U_m(s), \quad (8)$$

where s denotes Laplace operator. On the other hand, the plant input U_p is expressed from Fig. 2 as follows:

$$U_p(s) = -K_p X_p(s) + K_m(s) X_m(s) + K_v U_m(s), \quad (9)$$

where $K_m(s)$ is a constant gain or an element with dynamics. From Laplace transform of Eqs. (4) and (5) with all initial conditions set equal to zero, using Eq. (9), output of the plant for the model input is expressed as follows:

$$Y_p(s) = C_p(sI - A_p + B_p K_p)^{-1} B_p \times \{K_m(s)(sI - A_m)^{-1} B_m + K_v\} U_m(s). \quad (10)$$

That the output of the plant perfectly follows the output of the model for the same input U_m means that $Y_p(s) = Y_m(s)$, which requires that the following relation hold.

$$C_p(sI - A_p + B_p K_p)^{-1} \{B_p K_m(s)(sI - A_m)^{-1} B_m + B_p K_v\} = C_m(sI - A_m)^{-1} B_m \quad (11)$$

Now, put

$$C_p L = C_m. \quad (12)$$

If the following relation is satisfied, L exists.

$$\text{rank } C_p = \text{rank } [C_p, C_m] \quad (13)$$

Fig. 2 Structure of extended PMF system

Generally L is not unique when C_p is not square. L is supposed to be a constant matrix. Put

$$B_p K_v = L B_m. \quad (14)$$

If L which satisfies the following relation can be chosen, K_v exists.

$$\text{rank } B_p = \text{rank } [B_p, L B_m] \quad (15)$$

By substituting Eq. (14) into Eq. (11) and manipulating, the following equation results.

$$C_p(sI - A_p + B_p K_p)^{-1} \{sL - L A_m + B_p K_m(s)\} \times (sI - A_m)^{-1} B_m = C_m(sI - A_m)^{-1} B_m \quad (16)$$

Hence if the following relation is satisfied, Eq. (16) is satisfied.

$$C_p(sI - A_p + B_p K_p)^{-1} \{sL - L A_m + B_p K_m(s)\} = C_m \quad (17)$$

Moreover, put

$$C_p M(s) = C_m. \quad (18)$$

This $M(s)$ also exists if Eq. (13) is satisfied. $M(s)$ is not unique and generally supposed to have elements which are functions of s . Even if $M(s)$ is a constant matrix, it is not necessarily equal to L . From relations of (17) and (18), we have

$$(sI - A_p + B_p K_p)^{-1} \{sL - L A_m + B_p K_m(s)\} = M(s). \quad (19)$$

Rewriting this equation, we obtain

$$B_p K_m(s) = s\{M(s) - L\} + \{L A_m - A_p M(s)\} + B_p K_p M(s). \quad (20)$$

Define the right hand side of the above equation as $\Gamma(s)$, i.e.,

$$\Gamma(s) = s\{M(s) - L\} + \{L A_m - A_p M(s)\} + B_p K_p M(s) \quad (21)$$

and if L and $M(s)$ can be selected so as to satisfy the following relation, $K_m(s)$ exists.

$$\text{rank } B_p = \text{rank } [B_p, \Gamma(s)] \quad (22)$$

In this context, K_p is determined in advance so as for the closed loop system to be stable or to be optimal as mentioned later.

K_v and K_m , which are determined in this manner using L and $M(s)$ satisfying the rank condition of Eqs. (13), (15) and (22), satisfy Eq. (11) and therefore the extended PMF system is realized.

That is, using Eqs. (4), (9), (14) and (20), we obtain

$$sX_p(s) = A_p X_p(s) - B_p K_p X_p(s) + \{s(M(s) - L) + LA_m - A_p M(s) + B_p K_p M(s)\} X_m(s) + LB_m U_m(s) \quad (23)$$

while Eq. (6) becomes

$$sX_m(s) = A_m X_m(s) + B_m U_m(s). \quad (24)$$

Premultiplying $M(s)$ to this equation and subtracting it from Eq. (23), we obtain

$$\begin{aligned} s\{X_p(s) - M(s)X_m(s)\} &= (A_p - B_p K_p)\{X_p(s) - \\ &- M(s)X_m(s)\} + s(M(s) - L)X_m(s) + (L - M(s)) \\ &\times A_m X_m(s) + (L - M(s))B_m U_m(s). \end{aligned} \quad (25)$$

Premultiplying C_p to this equation and considering that $C_p\{L - M(s)\} = C_m - C_m = 0$, we obtain

$$C_p(sI - A_p + B_p K_p)\{X_p(s) - M(s)X_m(s)\} = 0. \quad (26)$$

Defining

$$X_p(s) - M(s)X_m(s) \stackrel{\Delta}{=} Z(s) \quad (27)$$

and substituting this equation into (26) and converting the resulting equation into time domain expression, we obtain

$$C_p \dot{Z}(t) = C_p(A_p - B_p K_p)Z(t). \quad (28)$$

Now, the solution to equation

$$\dot{Z}(t) = (A_p - B_p K_p)Z(t) \quad (29)$$

for initial value $Z(0)$ becomes

$$Z(t) = \exp(-A_p + B_p K_p)t \cdot Z(0). \quad (30)$$

Denoting the error between $Y_p(t)$ and $Y_m(t)$ by $E(t)$, the following relation is obtained.

$$\begin{aligned} E(t) &= Y_p(t) - Y_m(t) = C_p Z(t) \\ &= C_p \exp(-A_p + B_p K_p)t \cdot Z(0) \end{aligned} \quad (31)$$

Hence if $Z(0) = 0$, the error $E(t)$ remains zero for all control intervals and Y_p coincides with Y_m . When $Z(0) \neq 0$, the error at initial value attenuates and Y_p tends to Y_m because K_p is selected as the system becomes stable. Therefore the extended PMF is achieved.

3. Design procedure

As a design procedure, examine first the condition of Eq. (13) and if it is satisfied, determine a part of elements in L from Eq. (12). If $\text{rank } C_p = n$, L is unique and $L = M$, hence M becomes constant matrix and

$$L = M = C_p^{-1} C_m. \quad (32)$$

Therefore examine the conditions of Eqs. (15) and (22), which are equivalent to

$$\begin{aligned} \text{rank } B_p &= \text{rank } [B_p, (C_p^{-1} C_m B_m)] \\ &= \text{rank } [B_p, (C_p^{-1} C_m A_m - A_p C_p^{-1} C_m)]. \end{aligned} \quad (33)$$

If they are satisfied, K_v and K_m are determined from the following relations:

$$\begin{cases} B_p K_v = C_p^{-1} C_m B_m \\ B_p K_m = C_p^{-1} C_m A_m - A_p C_p^{-1} C_m + B_p K_p C_p^{-1} C_m. \end{cases} \quad (34)$$

If $\text{rank } C_p \neq n$, L is not unique. Hence determine a part of undetermined elements in L so as to satisfy Eq. (15). Then determine a part of elements in $M(s)$ from Eq. (18). Using these L and $M(s)$, obtain $\Gamma(s)$ and determine the remaining elements so as to satisfy Eq. (22) and make $K_m(s)$ proper. When undetermined elements are still left, they are selected as the form of $K_m(s)$ becomes as simple as possible, for example as constant gain. When L and $M(s)$ are determined, obtain K_v and $K_m(s)$ from Eqs. (14) and (20).

[Example 1] Following a model with different dimension.

When

$$\begin{aligned} A_p &= \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} & B_p &= \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} & C_p &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ A_m &= \begin{bmatrix} -49 & 4 & 14 \\ -146 & 12 & 39 \\ -98 & 9 & 26 \end{bmatrix} & B_m &= \begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 6 & 1 \end{bmatrix} \\ C_m &= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{aligned}$$

relation

$$L = M = C_p^{-1} C_m = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

is satisfied because $\text{rank } C_p = n$, and the following K_v is obtained.

$$K_v = B_p^{-1} L B_m = \begin{bmatrix} 3 & 7.11 \\ 1 & -0.22 \end{bmatrix}$$

Put

$$K_p = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then the closed loop is stable because its roots are $-1 \pm j2\sqrt{5}$. Since Eq. (33) is satisfied, the following K_m is obtained from Eq. (34).

$$K_m = \begin{bmatrix} -124.11 & 8.22 & 30.78 \\ -0.78 & 2.56 & 2.44 \end{bmatrix}$$

Using above K_v , K_p and K_m , it is shown that Y_p coincides with Y_m as follows:

$$Y_p(s) = \frac{1}{(s+5)(s^2+6s+25)} \times \begin{bmatrix} 8s^2+73s-237 & 6s^2-577s-617 \\ 7s^2+51s-724 & 14s^2-919s-109 \end{bmatrix} U_m(s) \\ = Y_m(s).$$

From this example, we know that extended PMF can be achieved for a model with different dimension from the plant.

[Example 2] Elimination of zero.

Consider a case where a plant with transfer function $(s+2)/(s+1)(s+3)$, which has a zero, follows a model with transfer function $1/(s+1)(s+3)$ which has not zero. Then

$$A_p = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \quad B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A_m = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \quad B_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If PMF of the two state variables is intended, $C_p = C_m = I$ is required and the following comes out.

$$\text{rank } B_p = 1$$

$$\text{rank } [B_p, B_m] = 2,$$

which means that PMF is not achievable.

Next, consider a case where PMF of only x_{p1} is intended. Then, it is required that

$$C_p = [1 \ 0], \quad C_m = [1 \ 0].$$

In this case, $\text{rank } C_p = \text{rank } [C_p, C_m] = 1$, and hence we obtain from Eq. (12)

$$L = \begin{bmatrix} 1 & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}.$$

To satisfy Eq. (15), L becomes

$$L = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 0 \end{bmatrix},$$

while from Eq. (18), $M(s)$ becomes as follows:

$$M(s) = \begin{bmatrix} 1 & 0 \\ m_{21}(s) & m_{22}(s) \end{bmatrix}.$$

Hence

$$s(M(s) - L) = \begin{bmatrix} 0 & 0 \\ s(m_{21}(s) - \ell_{21}) & sm_{22}(s) \end{bmatrix}.$$

Now, put $K_p = [-2 \ 7]$. Then the closed loop of the plant becomes stable because its roots are -4 and -5. $\Gamma(s)$ of Eq. (21) is obtained as

$$\Gamma(s) = \begin{bmatrix} -2+6m_{21}(s) & 1+6m_{22}(s) \\ (s+8)m_{21}(s) & (s+8)m_{22}(s)+\ell_{21} \\ -(s+3)\ell_{21}-2 & \end{bmatrix}.$$

To satisfy Eq. (22), it is required that

$$-2+6m_{21}(s) = (s+8)m_{21}(s) - (s+3)\ell_{21} - 2$$

$$1+6m_{22}(s) = (s+8)m_{22}(s) + \ell_{21}.$$

From these equations, the following relations are obtained.

$$m_{21}(s) = \frac{(s+3)\ell_{21}}{s+2}, \quad m_{22}(s) = \frac{1-\ell_{21}}{s+2}$$

Then from Eq. (20), $K_m(s)$ is represented as follows:

$$K_m(s) = \begin{bmatrix} -2 + \frac{6(s+3)\ell_{21}}{s+2} & 1 + 6 \frac{1-\ell_{21}}{s+2} \end{bmatrix}.$$

Since it is impossible to make both elements of $K_m(s)$ constant, choose $\ell_{21}=0$ and we obtain

$$K_m(s) = \begin{bmatrix} -2 & \frac{s+8}{s+2} \end{bmatrix}.$$

On the other hand, since the right hand side of Eq. (14) becomes zero, we obtain

$$K_v = 0.$$

Then it follows that

$$R_p(s) = \frac{s^2+9s+20}{(s+1)(s+3)(s+2)} u_m(s)$$

$$Y_p(s) = \frac{s+2}{s^2+9s+20} R_p(s)$$

and therefore the following relation is obtained:

$$Y_p(s) = \frac{1}{(s+1)(s+3)} u_m(s),$$

which shows that properties of the designed system coincide with those of the model which has no zeros. Incidentally, if zero of plant has a positive real part, then the positive real part appears at poles of some elements of $K_m(s)$. Therefore in this case, since the system becomes unstable, elimination of zero is not achieved.

4. Sensitivity reduction by extended PMF

In the case of the last example, elimination of zero is possible by inserting an element of $1/(s+2)$ in series instead of using PMF. However, since this is an open loop system, effect of parameter perturbation, if it exists, appears directly in output. On the other hand, since PMF system corresponds to a nominally equivalent closed loop system, the effect of parameter perturbation can be reduced.

Now denoting output of the former system by $Y_{po}(t)$ and output of the latter system by $Y_{pc}(t)$, sensitivity is more reduced in the latter than in the former if the following inequality is satisfied for all $t > 0$.

$$\int_0^{t'} \{ \delta Y_{po}^T(t) W \delta Y_{po}(t) - \delta Y_{pc}^T(t) W \delta Y_{pc}(t) \} dt > 0, \quad (35)$$

where $\delta Y_{po}(t)$ and $\delta Y_{pc}(t)$ denote the variations of $Y_{po}(t)$ and $Y_{pc}(t)$ respectively which are caused by parameter perturbation, and W is a positive semidefinite symmetric weighting matrix. Eq. (35) can be rewritten by Parseval's theorem as follows:

$$\int_{-\infty}^{\infty} \{ \delta Y_{po}^T(-j\omega) W \delta Y_{po}(j\omega) - \delta Y_{pc}^T(-j\omega) W \delta Y_{pc}(j\omega) \} d\omega \geq 0. \quad (36)$$

We show that in extended PMF system, Eq. (36) can be satisfied for small perturbation of parameters by selecting K_p as the optimal feedback. In Eq. (4), the state variable $X_p(t)$ is transformed into phase variable $\bar{X}_p(t)$ and the following relation is assumed.

$$X_p(t) = T_p \bar{X}_p(t) \quad (37)$$

Then Eq. (4) becomes

$$\dot{\bar{X}}_p(t) = \bar{A}_p \bar{X}_p(t) + \bar{B}_p U_p(t) \quad (38)$$

$$Y_p(t) = \bar{C}_p \bar{X}_p(t), \quad (39)$$

where $\bar{A}_p = T_p^{-1} A_p T_p$, $\bar{B}_p = T_p^{-1} B_p$ and $\bar{C}_p = C_p T_p$.

Feedback gain K_p is determined so that the plant output minimizes the following performance index:

$$J = \int_0^{\infty} \{ Y_p^T(t) Q Y_p(t) + U_p^T(t) R U_p(t) \} dt \\ = \int_0^{\infty} \{ \bar{X}_p^T(t) \bar{Q} \bar{X}_p(t) + U_p^T(t) R U_p(t) \} dt, \quad (40)$$

where Q is a positive semidefinite symmetric weighting matrix and R is a positive definite symmetric weighting matrix and $\bar{Q} = \bar{C}_p^T Q \bar{C}_p$. K_p is obtained from positive definite solution P of Riccati equation

$$\bar{A}_p^T P + P \bar{A}_p - P \bar{B}_p R^{-1} \bar{B}_p^T P + \bar{Q} = 0 \quad (41)$$

as

$$\bar{K}_p = R^{-1} \bar{B}_p^T P \quad (42)$$

$$K_p = \bar{K}_p T_p^{-1}. \quad (43)$$

Meanwhile transfer function matrix of the plant is written as follows:

$$G_p(s) = \bar{C}_p (sI - \bar{A}_p)^{-1} \bar{B}_p = \bar{C}_p (sI - \bar{A}_p)^{-1} \bar{B}_p. \quad (44)$$

Now, suppose that parameter A_p^* and B_p^* at nominal values are perturbed to A_p and B_p . Hereafter a quantity with * denotes its nominal value. Then $\bar{A}_p^*$, $\bar{C}_p^*$, T_p^* and $\bar{K}_p^*$ are supposed to perturb to $\bar{A}_p$, $\bar{C}_p$, T_p and $\bar{K}_p$ respectively. Usually $\bar{B}_p^* = \bar{B}_p$ holds. Output variation by this perturbation in

open loop system is

$$\delta Y_{po}(s) = Y_{po}(s) - Y_{po}^*(s) \\ = \{ G_p(s) - G_p^*(s) \} U_{po}^*(s) \quad (45)$$

and that in nominally equivalent PMF system is

$$\delta Y_{pc}(s) = Y_{pc}(s) - Y_{pc}^*(s) \\ = G_p(s) U_{pc}(s) - G_p^*(s) U_{pc}^*(s). \quad (46)$$

Substituting a relation

$$U_{pc}(s) = R_p(s) - \bar{K}_p \bar{X}_{pc}(s) \\ = R_p(s) - \bar{K}_p \bar{X}_{pc}(s) + \bar{K}_p^* \bar{X}_{pc}^*(s) - \bar{K}_p^* \bar{X}_{pc}^*(s) \\ = -\{ \bar{K}_p \bar{X}_{pc}(s) - \bar{K}_p^* \bar{X}_{pc}^*(s) \} + U_{pc}^*(s) \quad (47)$$

into Eq. (46), we obtain

$$\delta Y_{pc}(s) = -G_p(s) \{ \bar{K}_p \bar{X}_{pc}(s) - \bar{K}_p^* \bar{X}_{pc}^*(s) \} \\ + \{ G_p(s) - G_p^*(s) \} U_{pc}^*(s). \quad (48)$$

Noting that $U_{po}^*(s) = U_{pc}^*(s)$, and substituting Eq. (45) into Eq. (48), the following relation is obtained.

$$\delta Y_{pc}(s) + G_p(s) \{ \bar{K}_p \bar{X}_{pc}(s) - \bar{K}_p^* \bar{X}_{pc}^*(s) \} \\ = \delta Y_{pc}(s) + G_p(s) \bar{K}_p \{ \bar{X}_{pc}(s) - T_p^{-1} T_p^* \bar{X}_{pc}^*(s) \} \\ = \delta Y_{po}(s) \quad (49)$$

Now writing

$$\delta \bar{X}_{pc}(s) = \bar{X}_{pc}(s) - T_p^{-1} T_p^* \bar{X}_{pc}^*(s), \quad (50)$$

the following relation is also obtained.

$$\delta Y_{pc}(s) = \bar{C}_p \bar{X}_{pc}(s) - \bar{C}_p^* \bar{X}_{pc}^*(s) \\ = \bar{C}_p \delta \bar{X}_{pc}(s) \quad (51)$$

Hence, Eq. (49) can be rewritten by using Eqs. (44) and (51) as follows:

$$\bar{C}_p \{ I + (sI - \bar{A}_p)^{-1} \bar{B}_p \bar{K}_p \} \delta \bar{X}_{pc}(s) \\ = \delta Y_{po}(s). \quad (52)$$

Suppose the parameter perturbation is as small as $T_p^* = T_p$, and $\delta \bar{X}_{pc}(s)$ remains phase variable. Then since a relation

$$\{ I + (sI - \bar{A}_p)^{-1} \bar{B}_p \bar{K}_p \} \delta \bar{X}_{pc}(s) \\ = \{ I + \bar{K}_p (sI - \bar{A}_p)^{-1} \bar{B}_p \} \delta \bar{X}_{pc}(s) \quad (53)$$

holds for case of single input(3), the following relation is obtained.

$$\begin{aligned}
& \{1 + \bar{K}_p (sI - \bar{A}_p)^{-1} \bar{B}_p\} \bar{C}_p \delta \bar{X}_{pc}(s) \\
&= \{1 + \bar{K}_p (sI - \bar{A}_p)^{-1} \bar{B}_p\} \delta Y_{pc}(s) \\
&= \delta Y_{po}(s)
\end{aligned} \quad (54)$$

Substituting Eq. (54) into Eq. (36), we obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} \delta Y_{pc}^T(-j\omega) [\{1 + \bar{K}_p (-j\omega I - \bar{A}_p)^{-1} \bar{B}_p\} W \\
& \times \{1 + \bar{K}_p (j\omega I - \bar{A}_p)^{-1} \bar{B}_p\} - W] \delta Y_{pc}(j\omega) d\omega \geq 0.
\end{aligned} \quad (55)$$

A sufficient condition for this inequality to hold is that the following relation is satisfied for all ω .

$$\begin{aligned}
& \{1 + \bar{K}_p (-j\omega I - \bar{A}_p)^{-1} \bar{B}_p\} \\
& \times \{1 + \bar{K}_p (j\omega I - \bar{A}_p)^{-1} \bar{B}_p\} \geq 1
\end{aligned} \quad (56)$$

On the other hand, the following relation is obtained by modifying Eq. (40) (3).

$$\begin{aligned}
& \{I + R^{1/2} \bar{K}_p (-sI - \bar{A}_p)^{-1} \bar{B}_p R^{-1/2}\}^T \\
& \times \{I + R^{1/2} \bar{K}_p (sI - \bar{A}_p)^{-1} \bar{B}_p R^{-1/2}\} \\
&= I + R^{-1/2} \bar{B}_p^T (-sI - \bar{A}_p)^{-1} \bar{Q} (sI - \bar{A}_p)^{-1} \bar{B}_p R^{-1/2}
\end{aligned} \quad (57)$$

Substituting $s = j\omega$ into this equation and assuming single input and $R = 1$, we obtain for nominal value

$$\begin{aligned}
& \{1 + \bar{K}_p^* (-j\omega I - \bar{A}_p^*)^{-1} \bar{B}_p^*\} \{1 + \bar{K}_p^* (j\omega I - \bar{A}_p^*)^{-1} \bar{B}_p^*\} \\
&= 1 + \bar{B}_p^{*T} (-j\omega I - \bar{A}_p^*)^{-1} \bar{Q} (j\omega I - \bar{A}_p^*)^{-1} \bar{B}_p^* \geq 1,
\end{aligned} \quad (58)$$

which is satisfied identically and hence Eq. (56) is always satisfied for small perturbation of parameters. Therefore it is seen that sensitivity to parameter perturbation can be reduced by using the extended PMF system with optimal feedback.

However, though it is assumed in the above theory that parameter perturbation is small, it is not evident to what extent of variation this theory is applicable. Therefore it is necessary to verify whether sensitivity of the extended PMF system is reduced when specific, bounded perturbed parameter values are given.

State variables of open loop system are written as follows:

$$\bar{X}_{po}^*(s) = (sI - \bar{A}_p^*)^{-1} \bar{B}_p^* U_{po}^*(s) \quad (59)$$

$$\bar{X}_{po}(s) = (sI - \bar{A}_p)^{-1} \bar{B}_p U_{po}^*(s) \quad (60)$$

while those of closed loop (PMF) system are written as follows:

$$\bar{X}_{pc}^*(s) = (sI - \bar{A}_p^* + \bar{B}_p^* \bar{K}_p^*)^{-1} \bar{B}_p^* R_{pc}^*(s) \quad (61)$$

$$\bar{X}_{pc}(s) = (sI - \bar{A}_p + \bar{B}_p \bar{K}_p)^{-1} \bar{B}_p R_{pc}^*(s). \quad (62)$$

Since cases of phase variables are considered, the following relations hold.

$$\bar{X}_{po}^*(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \bar{x}_{p01}^*(s),$$

$$\bar{X}_{po}(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \bar{x}_{p01}(s)$$

$$\bar{X}_{pc}^*(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \bar{x}_{pc1}^*(s),$$

$$\bar{X}_{pc}(s) = \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} \bar{x}_{pc1}(s) \quad (63)$$

Now we consider a case of single input. Defining $D = [1 \ 0 \ \dots \ 0]$, the following are obtained from Eqs. (59) and (60),

$$\bar{x}_{p01}^*(s) = [D(sI - \bar{A}_p^*)^{-1} \bar{B}_p^*] U_{po}^*(s) \quad (64)$$

$$\bar{x}_{p01}(s) = [D(sI - \bar{A}_p)^{-1} \bar{B}_p] U_{po}^*(s) \quad (65)$$

and also

$$\bar{x}_{p01}(s) = \frac{\det(sI - \bar{A}_p^*)}{\det(sI - \bar{A}_p)} \bar{x}_{p01}^*(s). \quad (66)$$

Therefore, the following relation results from Eqs. (63) and (66).

$$\begin{aligned}
\bar{X}_{po}(s) &= \frac{\det(sI - \bar{A}_p^*)}{\det(sI - \bar{A}_p)} \bar{X}_{po}^*(s) \\
&= \bar{H}_O^{\Delta}(s) \bar{X}_{po}^*(s)
\end{aligned} \quad (67)$$

Similarly from Eqs. (61) and (62), the following relation results.

$$\begin{aligned}
\bar{X}_{pc}(s) &= \frac{\det(sI - \bar{A}_p^* + \bar{B}_p^* \bar{K}_p^*)}{\det(sI - \bar{A}_p + \bar{B}_p \bar{K}_p)} \bar{X}_{pc}^*(s) \\
&= \bar{H}_C^{\Delta}(s) \bar{X}_{pc}^*(s)
\end{aligned} \quad (68)$$

On the other hand, since output variations are expressed by

$$\delta Y_{po}(s) = Y_{po}(s) - Y_{po}^*(s) = \bar{C}_p \bar{X}_{po}(s) - \bar{C}_p^* \bar{X}_{po}^*(s) \quad (69)$$

$$\delta Y_{pc}(s) = Y_{pc}(s) - Y_{pc}^*(s) = \bar{C}_p \bar{X}_{pc}(s) - \bar{C}_p^* \bar{X}_{pc}^*(s) \quad (70)$$

respectively, the following relations are obtained by substituting Eqs. (67) and (68) into these equations.

$$\delta Y_{po}(s) = \{H_o(s) \bar{C}_p - \bar{C}_p^*\} \bar{X}_{po}^*(s) \quad (71)$$

$$\delta Y_{pc}(s) = \{H_c(s) \bar{C}_p - \bar{C}_p^*\} \bar{X}_{pc}^*(s) \quad (72)$$

Since both systems are nominally equivalent and $\bar{X}_{po}^*(s)$ and $\bar{X}_{pc}^*(s)$ are represented in phase variables, the following relation holds.

$$\bar{X}_{pc}^*(s) = \bar{X}_{po}^*(s) \quad (73)$$

Substituting Eqs. (71), (72) and (73) into Eq. (36), we obtain

$$\int_{-\infty}^{\infty} \bar{X}_{po}^{*T}(-j\omega) [\{H_o(-j\omega) \bar{C}_p - \bar{C}_p^*\}^T W \{H_o(j\omega) \bar{C}_p - \bar{C}_p^*\} - \{H_c(-j\omega) \bar{C}_p - \bar{C}_p^*\}^T W \{H_c(j\omega) \bar{C}_p - \bar{C}_p^*\}] \bar{X}_{po}^*(j\omega) d\omega \geq 0. \quad (74)$$

Defining

$$\bar{C}_p \cdot [1 \ s \ \dots \ s^{n-1}]^T \triangleq \bar{C}_p(s), \quad \bar{C}_p^* \cdot [1 \ s \ \dots \ s^{n-1}]^T \triangleq \bar{C}_p^*(s) \quad (75)$$

and considering Eq. (63), a sufficient condition for Eq. (74) to be satisfied becomes as follows:

$$\{H_o(-j\omega) \bar{C}_p(-j\omega) - \bar{C}_p^*(-j\omega)\}^T W \{H_o(j\omega) \bar{C}_p(j\omega) - \bar{C}_p^*(j\omega)\} - \{H_c(-j\omega) \bar{C}_p(-j\omega) - \bar{C}_p^*(-j\omega)\}^T W \{H_c(j\omega) \bar{C}_p(j\omega) - \bar{C}_p^*(j\omega)\} \geq 0. \quad (76)$$

If Eq. (76) holds for all ω when perturbed parameter values are substituted, it can be said that sensitivity is more reduced in PMF system than in open loop system at these parameter values.

Defining elements of $\bar{C}_p^*(s)$ and $\bar{C}_p(s)$ by $\bar{c}_{pi}^*(s)$ and $\bar{c}_{pi}(s)$ respectively where $i=1, \dots, m$, the following relation is obtained for each element of output.

$$\left| \frac{H_o(j\omega) \bar{c}_{pi}(j\omega) - \bar{c}_{pi}^*(j\omega)}{H_c(j\omega) \bar{c}_{pi}(j\omega) - \bar{c}_{pi}^*(j\omega)} \right|^2 = \left| 1 + \frac{\{H_o(j\omega) - H_c(j\omega)\} \bar{c}_{pi}(j\omega)}{H_c(j\omega) \bar{c}_{pi}(j\omega) - \bar{c}_{pi}^*(j\omega)} \right|^2 \geq 1 \quad (77)$$

On the other hand, defining $\bar{K}_p(s) \triangleq \bar{K}_p \cdot [1 \ s \ \dots \ s^{n-1}]^T$ and $\bar{K}_p^*(s) \triangleq \bar{K}_p^* \cdot [1 \ s \ \dots \ s^{n-1}]^T$, we obtain

$$\det(sI - \bar{A}_p^* + \bar{B}_p \bar{K}_p^*) = \det(sI - \bar{A}_p) + \bar{K}_p^*(s) \quad (78)$$

$$\det(sI - \bar{A}_p + \bar{B}_p \bar{K}_p) = \det(sI - \bar{A}_p) + \bar{K}_p(s). \quad (79)$$

Therefore from Eqs. (67), (68), (75), (78) and (79), we obtain

$$\begin{aligned} & \frac{\{H_o(s) - H_c(s)\} \bar{c}_{pi}(s)}{H_c(s) \bar{c}_{pi}(s) - \bar{c}_{pi}^*(s)} \\ &= \frac{\{\bar{K}_p(s) \det(sI - \bar{A}_p^*) - \bar{K}_p^*(s) \det(sI - \bar{A}_p)\} \bar{c}_{pi}(s)}{\det(sI - \bar{A}_p) \{\det(sI - \bar{A}_p^*) \bar{c}_{pi}(s) - \det(sI - \bar{A}_p) \bar{c}_{pi}^*(s) + \bar{K}_p^*(s) \bar{c}_{pi}(s) - \bar{K}_p(s) \bar{c}_{pi}^*(s)\}} \\ &\triangleq \Sigma_n(s) / \Sigma_d(s) \end{aligned} \quad (80)$$

and therefore substituting this relation into (77), we also obtain

$$\left| 1 + \frac{\Sigma_n(j\omega)}{\Sigma_d(j\omega)} \right|^2 = \left\{ 1 + \frac{\Sigma_n(j\omega) \Sigma_d(-j\omega) + \Sigma_n(-j\omega) \Sigma_d(j\omega) + \Sigma_n(j\omega) \Sigma_n(-j\omega)}{\Sigma_d(j\omega) \Sigma_d(-j\omega)} \right\} \geq 1. \quad (81)$$

Therefore if an inequality

$$\begin{aligned} & \Sigma_n(j\omega)\Sigma_d(-j\omega) + \Sigma_n(-j\omega)\Sigma_d(j\omega) \\ & + \Sigma_n(j\omega)\Sigma_n(-j\omega) \geq 0 \end{aligned} \quad (82)$$

holds for all ω , we can say that sensitivity of the corresponding component is reduced. Even if Eq. (82) does not hold for all ω , its sensitivity may well be regarded as practically reduced if Eq. (82) holds in the effective frequency range of the given system.

5. Application to an aircraft control system

The equation of longitudinal short period dynamics of a jet fighter which is flying at altitude of 9000 m and speed of 0.6 Mach is given as follows:

$$\dot{x}_p = \begin{bmatrix} -0.78 & -3.482 & -7.483 \\ 1 & -0.439 & -0.064 \\ 0 & 0 & -5 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u_p$$

where $x_{p1} = \dot{\theta}$ = pitch rate (deg/sec), $x_{p2} = \alpha$ = angle of attack (deg), $x_{p3} = \delta_e$ = elevator deflection angle (deg) and u_p is input to elevator.

In order for an aircraft to have good handling quality, it is said that pitch rate should have property of second order lag system such as $[\omega_n^2 / (s^2 + 2\zeta\omega_n s + \omega_n^2)]$, where especially $\zeta = 0.7$ and $\omega_n = 4.4$ rad/sec are best. Therefore the following model is selected to achieve good handling quality.

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -19.36 & -6.16 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ 19.36 \end{bmatrix} u_m$$

Then we design the system such that x_{p1} perfectly follows x_{m1} which requires $C_p = [1 \ 0 \ 0]$ and $C_m = [1 \ 0]$. By the previous procedure L satisfying Eqs. (12) and (15), and M(s) satisfying Eq. (18) are obtained as follows:

$$L = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 0 \\ \ell_{31} & \ell_{32} \end{bmatrix}, \quad M(s) = \begin{bmatrix} 1 & 0 \\ m_{21}(s) & m_{22}(s) \\ m_{31}(s) & m_{32}(s) \end{bmatrix}.$$

Putting $K_p = [k_{p1} \ k_{p2} \ k_{p3}]$, $\Gamma(s)$ of Eq. (21) is obtained as follows:

$$\Gamma(s) = \begin{bmatrix} 0.78 + 3.482m_{21}(s) + 7.483m_{31}(s) \\ (s + 0.439)m_{21}(s) - s\ell_{21} - 1 + 0.064m_{31}(s) \\ (s + 5 + 5k_{p3})m_{31}(s) - s\ell_{31} - 19.36\ell_{32} \\ \quad + 5k_{p1} + 5k_{p2}m_{21}(s) \\ 1 + 3.482m_{22}(s) + 7.483m_{32}(s) \\ (s + 0.439)m_{22}(s) + \ell_{21} + 0.064m_{32}(s) \\ (s + 5 + 5k_{p3})m_{32}(s) - (s + 6.16)\ell_{32} \\ \quad + \ell_{31} + 5k_{p2}m_{22}(s) \end{bmatrix}.$$

To satisfy the condition of Eq. (22), it is required that all elements in the first and second rows in the above equations vanish.

From this requirement we obtain

$$\begin{aligned} m_{21}(s) &= \frac{7.483s + 7.533}{7.483s + 3.062}, \\ m_{22}(s) &= \frac{0.064 - 7.483\ell_{21}}{7.483s + 3.062}, \\ m_{31}(s) &= \frac{-\{(0.78 + 3.482\ell_{21})s + 3.824\}}{7.483s + 3.062}, \\ m_{32}(s) &= \frac{-\{s + (0.439 - 3.482\ell_{21})\}}{7.483s + 3.062}. \end{aligned}$$

These relations are substituted into the third row in the above $\Gamma(s)$.

On the other hand, T_p of Eq. (36) is obtained from A_p and B_p , then $\bar{A}_p$, $\bar{B}_p$ and $\bar{C}_p$ are calculated. Using $Q=100$ and $R=1$ in the performance index of Eq. (39), we obtain K_p from Eqs. (40)~(42) as follows:

$$K_p = [-9.2845 \quad 2.3685 \quad 4.3637].$$

By substituting this K_p into the third row in the above $\Gamma(s)$ and assuming $\ell_{31} = -(3.482\ell_{12} + 0.78)/7.483$ and $\ell_{32} = -1/7.483$ so that the result becomes simple and further assuming $\ell_{21} = 2.45$ so that k_{m1} becomes a constant, $K_m(s)$ is obtained from Eq. (20) as follows:

$$\begin{aligned} K_m(s) &= [-9.639 \\ &\quad -0.574(s + 0.0278)/(s + 0.4092)]. \end{aligned}$$

It also follows from Eq. (14) that

$$K_v = -0.5174.$$

The solid line in Fig. 3 shows a calculated step response of the extended PMF system using K_p , K_v and $K_m(s)$ designed above, which coincides with that of model.

When flying condition of the aircraft changes to altitude of 4000 m and speed of 0.4 Mach, the parameters change as follows:

$$A_p = \begin{bmatrix} -0.89 & -2.752 & -6.276 \\ 1 & -0.539 & -0.074 \\ 0 & 0 & -5 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

The dotted line in Fig. 3 shows the step response for these parameter values. Though a little difference appears, this response is very close to that of model.

On the other hand, though a series compensated system has, of course, the same response as the model at the nominal parameter value, it is seen that variation of the flying condition causes a large variation of response as dashed line shown in Fig. 3.

Next we verify Eq. (81) for the above perturbed values. From above values, we have

$$\det(sI - \bar{A}) = s^3 + 6.22s^2 + 9.92s + 19.123,$$

$$\det(sI - \bar{A}_p) = s^3 + 6.429s^2 + 10.377s + 16.159$$

Fig. 3 Step responses of an aircraft control system using extended PMF (simulation)

Fig. 4 Frequency characteristic of Eq. (81) which represents the condition of sensitivity reduction

$$\bar{C}_p^*(s) = -37.19s - 15.212,$$

$$\bar{C}_p(s) = -31.38s - 15.895$$

$$\bar{K}_p^*(s) = 21.82s^2 + 373.22s + 136.39,$$

$$\bar{K}_p(s) = 21.82s^2 + 321.63s + 142.99.$$

Using these relations, expression in Eq. (81) can be calculated as follows:

$$\begin{aligned} \Sigma_n(j\omega)\Sigma_d(-j\omega) + \Sigma_n(-j\omega)\Sigma_d(j\omega) + \Sigma_n(j\omega)\Sigma_n(-j\omega) \\ = -2.0474 \times 10^4 \omega^{12} + 2.9437 \times 10^6 \omega^{10} \\ + 1.0468 \times 10^8 \omega^8 + 4.9918 \times 10^8 \omega^6 \\ + 9.6226 \times 10^8 \omega^4 + 5.5088 \times 10^8 \omega^2 + 8.4948 \times 10^7 \\ \Sigma_d(j\omega)\Sigma_d(-j\omega) = 33.756 \omega^{14} + 2.5989 \times 10^4 \omega^{12} \\ + 6.1157 \times 10^5 \omega^{10} - 5.4782 \times 10^5 \omega^8 \\ - 2.5113 \times 10^6 \omega^6 + 2.1589 \times 10^7 \omega^4 \\ + 7.9126 \times 10^6 \omega^2 + 6.7384 \times 10^5 \end{aligned}$$

From these two equations, we calculated frequency characteristic of Eq. (81) or Eq. (77) and showed the result in Fig. 4. Its

value is far larger than unity in the angular frequency range below $\omega = 13.1$ rad/sec and is a little smaller than unity in the higher frequency range and tends to unity as ω approaches infinity. Since frequency band of the model is 4.4 rad/sec, it is seen that within this frequency band the sensitivity is far more reduced in closed loop (PMF) system than in open loop system. Therefore we can state that this fact explains the differences of step response errors in Fig. 3.

6. Conclusions

The extended PMF made it possible for a plant to follow a multiple model which had different dimension or different zeros and to specify variables for following, which were not possible by the conventional PMF. Thus selection of model was facilitated and special consideration in model determination became needless with consequent broadening of the range of applications.

It was also shown that sensitivity of specified variables to parameter perturbation could be reduced by the extended PMF with the optimal feedback. Moreover, a method of verifying the effectiveness of the sensitivity reduction for given perturbed parameter values was devised. An application to an aircraft control system as practical problem was demonstrated with good results.

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