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Unstable Vibrations of an Unsymmetrical Shaft at the Secondary Critical Speed due to Ball Bearings*

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It is clarified theoretically and experimentally that, when an unsymmetrical shaft is supported by single-row deep groove ball bearings, an unstable vibration with frequency $+2\omega$ (ω : rotating speed of a shaft) appears at the secondary critical speed due to the coexistence of a stationary directional difference and a rotating directional difference of stiffness. The stationary directional difference is due to the directional difference of support condition by means of a single-row deep groove ball bearing which has angular clearance, and the rotating directional difference is due to the flatness of the shaft. The width of this unstable region depends on the assembly of the system. No unstable vibration appears at the secondary critical speed in the system where an unsymmetrical shaft is supported by self-aligning double-row ball bearings and in an unsymmetrical rotor system.

Key Words: Vibration of Rotating Body, Bearing, Nonlinear Vibration, Unstable Vibration, Unsymmetrical Shaft, Secondary Critical Speed

1. Introduction

The dynamic phenomena of unsymmetrical rotating shafts with unequal bending stiffness have been studied mainly for unstable vibrations at the major critical speed ω_c and for vibrations at the secondary critical speed. The latter is due to the unsymmetry of the shaft and the gravity, and its mode is a stationary whirling motion with a frequency $+2\omega$ (twice the frequency of the shaft rotating speed ω), where the positive sign $+$ represents a forward precessional whirling motion. This secondary critical speed has been studied by many researchers including Soderberg⁽¹⁾, Smith⁽²⁾, and Taylor⁽³⁾. Kellenberger⁽⁴⁾, and Bishop & Parkinson⁽⁵⁾ discussed these phenomena on an unsymmetrical shaft with distributed mass.

The present paper deals with an unstable vibration with a frequency $+2\omega$ which occurs at the secondary critical speed owing to a single-row deep groove ball bearing in an unsymmetrical shaft system. Some researchers indicated the possibility of unstable vibrations at the secondary critical speed in a system where an unsymmetrical shaft was supported by flexible bearing pedestals with unsymmetrical stiffness. For example, Foote et al.⁽⁶⁾ and Tondl⁽⁷⁾ have theoretically dealt with this sort of system. They concluded that unstable regions appear at speeds of $1/2$, $1/3$, $1/4$, ... times the major critical speed but their widths are narrow and

easily disappear due to a small damping. Moreover a few experiments have been made by Hull⁽⁸⁾ and Messal & Bonthron⁽⁹⁾, but it is not clear whether unstable regions exist or not.

In the present paper, both the theoretical and the experimental discussions are carried out about the unstable vibration which appears owing to coexistence of the unsymmetry of the shaft and the directional difference of the support condition⁽¹⁰⁾ (that is, the unsymmetrical boundary condition for inclination at the shaft end). This directional difference is due to the angular clearance of a single-row deep groove ball

Fig.1 Coordinate systems.

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bearing. In our experiments on an unsymmetrical shaft supported by single-row deep groove ball bearings, even when rigid bearing pedestals were used, we observed a wide unstable region in which the unstable vibration of the frequency $+2\omega$ occurred. In addition, it is concluded that this kind of unstable vibration doesn't appear in the unsymmetrical shaft supported by self-aligning double-row ball bearings, nor in the unsymmetrical rotor system which has similar vibratory characteristics to those of the unsymmetrical shaft system in general.

2. Restoring forces of the shaft and equations of motion

First, we derive expressions for the restoring forces and equations of motion in the unsymmetrical rotating shaft system, when there exists a directional difference in the support condition at the lower end of the shaft.

As shown in Fig.1, a rotor is mounted on a vertical unsymmetrical shaft (its length is $l=a+b$) at the position of the ratio $a:b$. The boundary conditions are a simple support at the upper end for simplicity, and an elastic support having a directional difference at the lower end. The elastic support means that the inclination angle θ_0 of the shaft at the lower end is proportional to the bending moment M_0 at the end, namely the relation

$$M_0 = K\theta_0 \quad (1)$$

holds. The proportional constant K is independent of the dimensions of a shaft and its configuration, and it is a sort of a spring constant (whose unit is $\text{N}\cdot\text{m}/\text{rad}$) determined by only the condition of bearings. The case $K=0$ represents a simple support, and $K=\infty$ a fixed support. In the present paper, we regard all kinds of boundary conditions as an elastic support in a wide sense involving these particular cases, and assume that K can take values from 0 to ∞ . The shaft shows a linear spring characteristic when it is supported elastically at both shaft ends. When the boundary condition is not expressed by Eq.(1), a nonlinear spring characteristic appears in the shaft restoring force.

As shown in Fig.1, let $O-xyz$ be a stationary rectangular coordinate system whose origin O coincides with the equilibrium position of the point S (the center of the rotor),

and where the z -axis is coincident with the bearing center line (that is, the line connecting the centers of the upper and lower bearings). Now, we suppose that the strength of the elastic support has a directional difference. If we suppose that the strength is the greatest in the yz -plane, it is the smallest in the xz -plane. Putting the spring constants in Eq.(1) in the xz - and yz -planes as K_x and K_y respectively, the relation

$$K_y > K_x \quad (2)$$

holds. We rewrite them as

$$K = (K_x + K_y)/2, \quad \Delta K = (K_y - K_x)/2 \quad (3)$$

where K represents the average spring constant, and ΔK the directional difference of them.

Let $O-x'y'z'$ be a rotating rectangular coordinate system where the x' - and y' -axes are taken in the directions of the principal axes of moment of inertia of area of the unsymmetrical shaft, and where the z' -axis coincides with the direction of the principal axis of polar moment of inertia of area of the shaft. We designate the moments of inertia of area of the unsymmetrical shaft about the x' - and y' -axes as $I_{x'}$ and $I_{y'}$ respectively (where $I_{x'} > I_{y'}$ is assumed), Young's modulus as E . We put

$$EI_{x'} = B + \Delta B, \quad EI_{y'} = B - \Delta B \quad (4)$$

In Eq.(4), B represents the average bending stiffness of the unsymmetrical shaft, and ΔB the magnitude of the shaft unsymmetry.

Let r (having components x, y) be a deflection of the rotor, $\theta(\theta_x, \theta_y)$ be an inclination angle, and $-F(-F_x, -F_y)$ and $-M(-M_x, -M_y)$ be the restoring force and the moment which act from the shaft to the rotor. We put

$$\left. \begin{aligned} r &= x + iy, \quad \bar{r} = x - iy, \quad \theta = \theta_x + i\theta_y \\ \bar{\theta} &= \theta_x - i\theta_y, \quad F = F_x + iF_y, \quad M = M_x + iM_y \end{aligned} \right\} \quad (5)$$

where $i = \sqrt{-1}$.

Since the motion of the rotor is an infinitesimal vibration, we assume the deflection r and the inclination θ to be small, and also the twist angle of the shaft during running to be small. Neglecting the higher order terms of these small quantities, the restoring forces in the case of the elastic support having the directional difference as above mentioned are represented in the following forms after some calculation⁽¹⁾;

$$\left. \begin{aligned} F &= ar + \gamma\theta + \sum_{n=0}^{\infty} (\Delta\alpha_{2n}\bar{r} + \Delta\gamma_{2n}\bar{\theta})e^{2in\omega t} + \sum_{n=1}^{\infty} \{(\Delta\alpha_{2n}'r + \Delta\gamma_{2n}'\theta)e^{-2in\omega t} \\ &\quad + (\Delta\alpha_{2n}r + \Delta\gamma_{2n}\theta)e^{2in\omega t} + (\Delta\alpha_{2n}'\bar{r} + \Delta\gamma_{2n}'\bar{\theta})e^{-2in\omega t}\} \\ M &= \gamma r + \delta\theta + \sum_{n=0}^{\infty} (\Delta\gamma_{2n}\bar{r} + \Delta\delta_{2n}\bar{\theta})e^{2in\omega t} + \sum_{n=1}^{\infty} \{(\Delta\gamma_{2n}'r + \Delta\delta_{2n}'\theta)e^{-2in\omega t} \\ &\quad + (\Delta\gamma_{2n}r + \Delta\delta_{2n}\theta)e^{2in\omega t} + (\Delta\gamma_{2n}'\bar{r} + \Delta\delta_{2n}'\bar{\theta})e^{-2in\omega t}\} \end{aligned} \right\} \quad (6)$$

where ω is the rotating speed of a shaft, and t denotes the time. The moment when the x' -axis becomes parallel to the xz -plane is taken as $t=0$. If the unsymmetry $\Delta B/B$ is a small quantity, the magnitudes of the coefficients such as $\Delta\alpha_{2n}$ and others in Eq.(6) become smaller as n ($=1, 2, \dots$) increases.

Let m be the mass of a rotor, I_p be the polar moment of inertia, I be the diametral

moment of inertia, and $c_{11}, c_{12}, c_{21}(=c_{12}), c_{22}$ be the damping coefficients. Using F and M in Eq.(6), we then get the following equations of motion;

$$\left. \begin{aligned} m\ddot{r} + c_{11}\dot{r} + c_{12}\dot{\theta} + F &= 0 \\ I\ddot{\theta} - iI_p\omega\dot{\theta} + c_{21}\dot{r} + c_{22}\dot{\theta} + M &= 0 \end{aligned} \right\} \quad (7)$$

In the experimental apparatus mentioned later, there appear the nonlinear spring characteristics⁽¹⁰⁾ in the restoring force in addition to the directional difference of support condition at the shaft end due to the ball bearing. But, for simplicity, we consider the linear system represented by Eq.(7), because the nonlinearity has no substantial influences in our discussion. Though forced oscillations $[+\omega]$ and $[+2\omega]$ ⁽¹²⁾ also appear due to the unbalances of a rotor and the variation of the static deflection of a shaft, respectively, we neglect the external forces which cause these oscillations. This is because they have no direct influence on the unstable vibration treated here.

Using the symbol $[p]$, we express the forward and backward precessional whirling motions of the frequency $[p]$ corresponding to the sign of p . On account of the terms of the coefficients of $\Delta\alpha_{2n}$, $\Delta\alpha_{2n}'$, $V\alpha_{2n}$, $V\alpha_{2n}'$ and so on in Eq.(6), the free vibration $[p]$ produces the whirling motions of $[2n\omega-p]$, $[-2n\omega+p]$, $[2n\omega+p]$, and $[-2n\omega-p]$ respectively⁽¹³⁾⁽¹⁴⁾.

3. Unstable vibration at the secondary critical speed

We designate the natural frequency of the system by p . We shall obtain an approximate solution of Eq.(7) near the rotating speed satisfying $p \approx 2\omega$. The forms of the solution are put in the following;

$$r = Ae^{i(2\omega t + \varphi)}, \quad \theta = A'e^{i(2\omega t + \varphi')} \quad \dots\dots\dots(8)$$

Here, it is assumed that $A(>0)$, A' , φ , and φ' are slowly varying functions of time, and the first and the second time derivatives of them have the magnitudes of the small quantity ϵ (labelled $O(\epsilon)$) and $O(\epsilon^2)$ respectively. Also, the damping coefficients in Eq.(7) and the third and the fourth terms in the right sides of Eq.(6) are assumed to have the magnitude of $O(\epsilon)$. Putting the coefficients of $e^{i(2\omega t + \varphi)}$ to zero after the substitution of Eq.(8) into Eq.(7), and neglecting the terms of $O(\epsilon^2)$ and below, we obtain the following equations;

$$4m\omega A\dot{\varphi} = H(2\omega)A + \gamma A' \cos(\varphi - \varphi') + \{\Delta\alpha_4 A \cos 2\varphi + \Delta\gamma_4 A' \cos(\varphi + \varphi')\} - 2\omega c_{12} A' \sin(\varphi - \varphi') \quad \dots\dots\dots(9.a)$$

$$(4I - I_p)\omega A' \dot{\varphi}' \cos(\varphi - \varphi') = G(2\omega)A' \cos(\varphi - \varphi') + \gamma A + \{\Delta\gamma_4 A \cos 2\varphi + \Delta\delta_4 A' \cos(\varphi + \varphi')\} + (4I - I_p)\omega A' \sin(\varphi - \varphi') + 2\omega c_{22} A' \sin(\varphi - \varphi') \quad \dots\dots\dots(9.b)$$

$$4m\omega \dot{A} = \gamma A' \sin(\varphi - \varphi') + \{\Delta\alpha_4 A \sin 2\varphi + \Delta\gamma_4 A' \sin(\varphi + \varphi')\} - 2\omega\{c_{11}A + c_{12}A' \cos(\varphi - \varphi')\} \quad \dots\dots\dots(9.c)$$

$$(4I - I_p)\omega A' \dot{A}' \cos(\varphi - \varphi') = G(2\omega)A' \sin(\varphi - \varphi') + \{\Delta\gamma_4 A \sin 2\varphi + \Delta\delta_4 A' \sin(\varphi + \varphi')\} + (4I - I_p)\omega A' \dot{\varphi}' \sin(\varphi - \varphi') - 2\omega\{c_{12}A + c_{22}A' \cos(\varphi - \varphi')\} \quad \dots\dots\dots(9.d)$$

Here,

$$H(2\omega) = \alpha - m(2\omega)^2, \quad G(2\omega) = \delta + I_p\omega(2\omega) - I(2\omega)^2 \quad \dots\dots\dots(10)$$

Neglecting the small terms in Eq.(9), we get the following relations satisfied within the accuracy of $O(\epsilon^0)$;

$$\left. \begin{aligned} \frac{A'}{A} &= -\frac{H(2\omega)}{\gamma \cos(\varphi - \varphi')} = -\frac{\gamma}{G(2\omega) \cos(\varphi - \varphi')} \\ \sin(\varphi - \varphi') &= 0 \quad \therefore \varphi' = \varphi, \text{ or } \varphi' = \varphi + \pi \end{aligned} \right\} \quad \dots\dots\dots(11)$$

Neglecting the small terms in Eq.(7), we obtain the frequency equation as follows;

$$\left. \begin{aligned} f(p) &= (\alpha - mp^2)(\delta + I_p\omega p - Ip^2) - \gamma^2 = H(p)G(p) - \gamma^2 = 0 \\ [H(p) &= \alpha - mp^2, \quad G(p) = \delta + I_p\omega p - Ip^2] \end{aligned} \right\} \quad \dots\dots\dots(12)$$

After putting $2\omega = p$ in Eq.(12), we designate the root of this equation by p_0 . Using the symbols $H(p_0) = H_0$ and $G(p_0) = G_0$, we obtain the following relation near the rotating speed $p \approx 2\omega$ within the accuracy of $O(\epsilon^0)$;

$$H(2\omega) \approx H(p) \approx H_0, \quad G(2\omega) \approx G(p) \approx G_0 \quad \dots\dots\dots(13)$$

If the amplitude A' is assumed to take positive and negative values, we can adopt only the relation $\varphi' = \varphi$. Consequently, from Eq.(11), it follows that

$$A'/A = -H_0/\gamma = -\gamma/G_0 \quad \dots\dots\dots(14)$$

Thus, it is known that the ratio of the amplitudes of inclination and deflection is constant within the accuracy of $O(\epsilon^0)$. Referring to Eqs.(13), (14) and the relation $\varphi' = \varphi$, and after the calculations of (9.a) $\times G(2\omega) - (9.b) \times \gamma$, and (9.c) $\times G(2\omega) - (9.d) \times \gamma$ in Eq.(9), we obtain the equations within the accuracy of $O(\epsilon)$ as follows;

$$\dot{\varphi} = -\sigma - \Delta \cos 2\varphi = -\kappa \left(\omega - \frac{p_0}{2} \right) - \Delta \cos 2\varphi, \quad \dot{A} = -cA - \Delta A \sin 2\varphi \quad \dots\dots\dots(15)$$

here

$$\left. \begin{aligned} D &= 4mG_0 + H_0(4I - I_p), \quad \sigma = -2f(2\omega)/(p_0D) = \kappa \left(\omega - \frac{p_0}{2} \right) \\ \kappa &= 4\{2G_0m + H_0(2I - I_p)\}/D \quad (>0), \quad \Delta = -2(G_0\Delta\alpha_4 - 2\gamma\Delta\gamma_4 + H_0\Delta\delta_4)/(p_0D) \quad (>0) \\ c &= 2(G_0c_{11} - 2\gamma c_{12} + H_0c_{22})/D \quad (>0) \end{aligned} \right\} \quad \dots\dots\dots(16)$$

In Eq.(15), σ is the so-called detuning term, Δ the coefficient due to the coexistence of the shaft unsymmetry and the directional difference of bearing support, and c the damping

coefficient. Also it is easily verified that κ , Δ , and c are positive constants.

Equation (15) is able to be solved in nearly the same manner as that of the previous paper⁽¹⁵⁾, and has the solutions as mentioned below.

The values of ω satisfying the relation $\Delta^2 = \sigma^2$ are designated by ω_1' and ω_2' ($\omega_1' < \omega_2'$). From Eq.(15), it follows that

$$\omega_1' = p_0/2 - \Delta/\kappa, \quad \omega_2' = p_0/2 + \Delta/\kappa \quad \dots\dots\dots(17)$$

(I) The case of $\Delta^2 > \sigma^2$ (that is, $\omega_1' < \omega < \omega_2'$): When ω approaches the value of $p_0/2$ and the relation $\Delta^2 > \sigma^2$ holds, the solution is given as follows;

$$r = e^{-ct} (A_1 e^{\mu t} e^{i(2\omega t + \varphi)} + A_2 e^{-\mu t} e^{i(2\omega t - \varphi)}) \quad \dots\dots\dots(18)$$

where A_1 and A_2 are arbitrary constants and

$$\mu = \sqrt{\Delta^2 - \sigma^2} \quad (>0) \quad \dots\dots\dots(19)$$

If μ is larger than the damping coefficient c , the first term in the right side of Eq. (18) represents an unstable vibration of the frequency 2ω whose amplitude increases exponentially with time. The boundary frequencies ω_1 and ω_2 ($>\omega_1$) of the unstable region in which the unstable vibration occurs are given by the equation $\mu - c = 0$ as follows;

$$\omega_1 = p_0/2 - \sqrt{\Delta^2 - c^2}/\kappa, \quad \omega_2 = p_0/2 + \sqrt{\Delta^2 - c^2}/\kappa \quad \dots\dots\dots(20)$$

The unstable region is $\omega_1 < \omega < \omega_2$. When $\sigma = 0$, that is, $\omega = p_0/2$, μ takes the maximum value as follows;

$$\mu_{\max} = \Delta \quad \dots\dots\dots(21)$$

The phase angle φ is determined by the following equations.

$$\left. \begin{aligned} \cos 2\varphi &= -\sigma/\Delta = -\kappa(\omega - p_0/2)/\Delta \\ \sin 2\varphi &= -\mu/\Delta = -\sqrt{\Delta^2 - \sigma^2}/\Delta \end{aligned} \right\} \quad \dots\dots\dots(22)$$

From Eq.(22), $\varphi = 0$ for $\omega = \omega_1'$, $\varphi = -\pi/4$ for $\omega = p_0/2$, and $\varphi = -\pi/2$ for $\omega = \omega_2'$.

(II) The case of $\Delta^2 < \sigma^2$ (that is, $\omega < \omega_1'$ and $\omega_2' < \omega$): When ω is slightly different from the value $p_0/2$, and $\Delta^2 < \sigma^2$, the solution is given as follows;

$$\left. \begin{aligned} \text{For } \omega < \omega_1', \\ r &= e^{-ct} [A e^{i\{2(\omega + \nu)t + \beta\}} + \lambda A e^{i\{2(\omega - \nu)t - \beta\}}] \\ \text{For } \omega > \omega_2', \\ r &= e^{-ct} [A e^{i\{2(\omega - \nu)t + \beta\}} + \lambda A e^{i\{2(\omega + \nu)t - \beta\}}] \end{aligned} \right\} \quad \dots\dots\dots(23)$$

Fig.2 Frequencies near the secondary critical speed and the unstable region.

Here, Δ and β are arbitrary constants, and ν and λ are as follows;

$$\left. \begin{aligned} \nu &= \sqrt{\sigma^2 - \Delta^2} \quad (>0) \\ \lambda &= \frac{\sqrt{|\sigma| + \Delta} - \sqrt{|\sigma| - \Delta}}{\sqrt{|\sigma| + \Delta} + \sqrt{|\sigma| - \Delta}} < 1 \end{aligned} \right\} \quad \dots\dots\dots(24)$$

Consequently, the amplitude of the whirling motion $[2\omega + \nu]$ becomes predominant in the lower side ($\omega < \omega_1'$) and that of $[2\omega - \nu]$ becomes predominant in the higher side ($\omega_2' < \omega$). But, it is seen from Eq.(23) that they diminish with time.

Figure 2 shows the frequencies of the vibration near $\omega = p_0/2$ and the unstable region. Broken lines represent the frequencies of decreasing vibrations with smaller amplitudes. When $\omega_1' < \omega < \omega_2'$ the vibration $[+2\omega]$ appears, and this vibration becomes unstable when $\omega_1 < \omega < \omega_2$.

4. On the value of the coefficient Δ

We appraise the value of the coefficient Δ , which is a main factor causing the unstable vibration $[+2\omega]$, when the upper shaft end is simply supported and the lower end is under an elastic support condition with a directional difference. The spring constants K_x and K_y ($K_y > K_x$) may take values from 0 to ∞ . We introduce the small quantity ζ as a parameter available to appraise the magnitudes of quantities. For example, $K_x = 0$ represents a simple support, $K_x = \infty$ a fixed one, $K_x = O(\zeta)$ an elastic one close to the simple one, $K_x = O(\zeta^0)$ a medium-sized elastic one (namely, K_x is neither large nor small), and $K_x = O(\zeta^{-1})$ an elastic one close to the fixed one. Though even in the above examples of bearing supports we can consider over twenty combinations depending on the magnitudes of K_x and K_y , the following relation always holds

$$\Delta\alpha_i : \Delta\gamma_i : \Delta\delta_i = \left(-\frac{3}{a}\right) : 1 : \left(-\frac{a}{3}\right) \quad \dots\dots\dots(25)$$

where a is a distance from the lower shaft end to the rotor. Substituting Eq.(25) into Eq.(16), we get

$$\Delta = \left\{ \frac{2}{p_0 D} \left(\frac{3G_0}{a} + 2\gamma + \frac{H_0 a}{3} \right) \right\} \Delta\gamma_i \quad \dots\dots\dots(26)$$

Since the term in the bracket $\{ \}$ in Eq.(26) is a constant with the magnitude of $O(\zeta^0)$, we may appraise the value of $\Delta\gamma_i$ instead of the coefficient Δ . Among the above-mentioned combinations, we show the calculated values of $\Delta\gamma_i$ for several cases which are expected to be realized practically in experiments. The values of $\Delta\gamma_i$ are described only by the largest term with the smaller terms neglected.

(a) In the case of the medium-sized elastic support in the y -direction and the simple support or the elastic one close to it in the x -direction (that is, $K_x = 0$ or $O(\zeta)$, and $K_y = O(\zeta^0)$):

$$\Delta\gamma_i = \frac{3aK_y^3(\Delta B)^2}{8B(4B + aK_y)^3} \quad \dots\dots\dots(27.a)$$

(b) In the case of the medium-sized elastic supports both in the x - and y -directions, and the directional difference ΔK with the medium magnitude (that is, $K_x = O(\zeta^0)$, $K_y = O(\zeta^0)$, and $\Delta K = O(\zeta^0)$):

$$\Delta\gamma_4 = \frac{6a(K_y - K_x)(2B(K_x + K_y) + aK_xK_y)^2}{(4B + aK_x)^3(4B + aK_y)^3} (\Delta B)^2$$

$$= \frac{12a\Delta K(4BK + aK^2 - a(\Delta K)^2)(\Delta B)^2}{(4B + aK + a\Delta K)^3(4B + aK - a\Delta K)^3} \dots\dots\dots(27.b)$$

(c) In the case of the medium-sized elastic supports both in the x - and y -directions, and the small ΔK (that is, $K_x = O(\zeta^0)$, $K_y = O(\zeta^0)$, and $\Delta K = O(\zeta)$):

$$\Delta\gamma_4 = \frac{12aK^2\Delta K(\Delta B)^2}{(4B + aK)^4} \dots\dots\dots(27.c)$$

If $\Delta B/B = O(\zeta)$, we know that $\Delta\gamma_4$ has the magnitude of $O(\zeta^2)$ in (a) and (b), and $O(\zeta^3)$ in (c). In the theoretical analysis in Chapter 3, $\Delta\gamma_4$ is supposed as the magnitude of $O(\epsilon)$ for any case (that is, $O(\zeta^2) = O(\epsilon)$ for (a) and (b), and $O(\zeta^3) = O(\epsilon)$ for (c)).

For comparison, we show the expressions for Δ and $\Delta\gamma_4$ in a system where an unsymmetrical shaft is supported by a flexible bearing pedestal having a directional difference (that is, a system having an unsymmetrical flexibility of the bearing support for deflection) at the lower shaft end. In this system, it is assumed that both of the shaft ends are simply supported. We denote the spring constants for deflection in the x - and y -directions of the lower bearing pedestal by $k - \Delta k$ and $k + \Delta k$ respectively, and put $K_0 = 3B(a^2 + ab + b^2)/(\alpha^3 b l) + k$ and $\Delta K_0 = 3\Delta B(a^2 + ab + b^2)/(\alpha^3 b l)$. In the case of small $\Delta k/k$, we get the following equations.

$$\left. \begin{aligned} \Delta &= 2\{(G_0/a + 2\gamma + H_0\alpha)/(p_0 D)\} \Delta\gamma_4 \\ \Delta\gamma_4 &= \frac{9\Delta k}{a^5 K_0^2} \left\{ \frac{B^2(\Delta K_0)^2}{K_0^2} \right. \\ &\quad \left. - \frac{B\Delta K_0\Delta B(l+b)}{lK_0} + \frac{b(\Delta B)^2}{l} \right\} \end{aligned} \right\} \dots\dots(28)$$

When $\Delta k = O(\zeta)$, $\Delta\gamma_4$ in Eq.(28) has the magnitude of $O(\zeta^3)$ and it is the same order as that of $\Delta\gamma_4$ in Eq.(27.c).

Incidentally, the coefficients $\Delta\alpha_2$, $\Delta\gamma_2$ and $\Delta\delta_2$, factors causing the unstable region at the major critical speed, have the magnitude of $O(\zeta)$ proportional to the unsymmetry of the shaft $\Delta B/B$ in all cases in the system treated here where there exists a directional difference of the bearing support at the shaft end.

5. Experimental apparatus ⁽¹⁶⁾

Experiments were performed in a rotating shaft system where a rotor was mounted at the position below the middle of an unsymmetrical shaft (that is, $a:b=1:4$ in Fig.1). The shaft was arranged vertically to avoid the effect of gravity. The length of the shaft (that is, the distance between the upper and the lower bearings) was $l=700$ mm. We made an unsymmetrical shaft of 11 mm in thickness by cutting both sides of a round shaft of $d=16$ mm in diameter. The dimensions of a rotor were 481.3 mm in diameter, 5.55 mm in thickness, 7.87 kg in mass, the polar moment of inertia $I_p=0.2280$ kg·m², and the diametral moment of inertia $I=0.1141$ kg·m². The lower shaft end was supported by a single-row deep groove ball bearing (#6200), and the upper end was simply supported by

a self-aligning double-row ball bearing (#1200). Consequently, when the bearing center line deviates slightly from the center of the angular clearance in the lower bearing, there appears the strongest elastic support in the direction of this deviation, and the weakest one in the normal direction to that. Consequently, the shaft end has a support condition having a directional difference for inclination⁽¹⁰⁾. Deflections of the rotor were measured in mutually perpendicular x - and y -directions (in Fig.1).

6. Experimental results

6.1 Natural frequencies in the non-rotating state

In the experimental apparatus, the deflections were measured in two mutually perpendicular directions. When the shaft was hammered in the state of $\omega=0$, two free vibrations occurred predominantly, and beat phenomena were observed. These vibrations correspond to the modes of vibrations with the lowest frequencies in the directions of the maximum and minimum flexibilities, respectively. We designate these two frequencies by p_{01} and p_{02} ($p_{01} > p_{02}$). Measuring the frequencies for various angular positions θ of the rotor when $\omega=0$, we know that the values of p_{01} and p_{02} change twice periodically during one revolution of the rotor (from $\theta=0$ to $\theta=2\pi$). When the direction of the rotating directional difference (due to the shaft unsymmetry) coincides with that of the stationary directional difference (due to the difference of the boundary condition at the shaft end), namely when the x -direction coincides with the x' - or $-x'$ -direction in Fig.1, $p_{01}-p_{02}$ takes a maximum value. When these directions are at right angles to each other (namely, when the x -direction coincides with the y' - or $-y'$ -direction), $p_{01}-p_{02}$ takes a minimum value. The results of measurement can be expressed approximately by $p_{01} = \bar{p}_{01} + \Delta p_{01} \sin 2\theta$ and $p_{02} = \bar{p}_{02} + \Delta p_{02} \sin (2\theta + \pi)$ where $\bar{p}_{01}=1361.5$ rpm, $\Delta p_{01}=36.5$ rpm, $\bar{p}_{02}=1104$ rpm, and $\Delta p_{02}=39$ rpm. Here, $p_{01}-p_{02}$ means the difference between the natural frequencies due to the unsymmetry of the shaft, and Δp_{01} (or Δp_{02}) does the magnitude of variation of the natural frequency due to the directional difference of the support condition at the shaft end. As the minimum values of p_{01} and p_{02} are not so different from the natural frequencies of the system where the unsymmetrical shaft having the bending stiffnesses of $B+\Delta B$ and $B-\Delta B$ was simply supported at both of the shaft ends, we can consider that the direction of the weak elastic support (that is, the x -direction in Fig.1) is approximately under the simple support condition. Therefore, the state of our experimental apparatus corresponds to the case (a) in Chapter 4.

6.2 Resonance phenomena near the resonance point $p_2 = +2\omega$

In this experimental apparatus, in addition to the natural frequencies p_i ($i=1, 2, 3, 4$, $p_1 > p_2 > 0 > p_3 > p_4$) corresponding to those in the symmetrical system (that is, the system carrying a disc mounted on a round shaft), the frequencies $\bar{p}_i$ ($=2\omega - p_i$), $-p_i$ and others are brought about by the unsymmetry of the shaft, the unsymmetrical

support condition at the shaft end, or the coexistence of them⁽¹²⁾⁽¹³⁾. We measured the resonance curve in the region of the rotating speeds less than $\omega=2700$ rpm in various assemblies of the experimental apparatus, and we observed unstable vibrations near the major critical speed satisfying $p_2=\omega$ and near the secondary critical speed satisfying $p_2=2\omega$. In addition, on account of use of a disc ($I_p/I \neq 2$) as a rotor, no resonance points satisfying $p_1=\omega$ and $p_1=2\omega$ exist. Near the major critical speed satisfying $p_2=\omega$, an unstable region exists in a broad range of $\omega=1530 \sim 1790$ rpm. This corresponds to the unstable region which is generally observed in an unsymmetrical shaft system or in an unsymmetrical rotor system. Near the secondary critical speed satisfying $p_2=2\omega$, we obtained the resonance curve shown in Fig.3. In Fig.3, the unstable region exists in the range of $\omega=705 \sim 750$ rpm. It was already reported that the peak of the steady oscillation $[+2\omega]$ appeared near the secondary critical speed owing to the gravity in a horizontal unsymmetrical shaft system. From Fig.3, however, it is found that the unstable region of the unstable vibration $[+2\omega]$ occurs when there exists a directional difference of the support condition in the vertical unsymmetrical shaft system. The steady vibration $[+2\omega]$ also appears in both sides of this unstable

Fig.3 Experimental resonance curve near the secondary critical speed satisfying $p_2=2\omega$ (in the case of presence of the unstable region).

Fig.4 A waveform of the unstable vibration $[+2\omega]$ (at $\omega=746$ rpm).

region. This vibration is caused by an external force of frequency 2ω which is due to the fluctuation of the static deflection of the shaft. The static deflection changes even in a vertical shaft system when the unsymmetrical shaft is supported elastically at the shaft end or when the shaft curvature and the angular clearance of a bearing co-exist⁽¹⁴⁾.

Figure 4 shows an example of a waveform of the unstable vibration in the unstable region. It was recorded at $\omega=746$ rpm and shows how the unstable vibration grows from the state of the zero amplitude. From this picture, in which vertical white lines are recorded with each rotation of the shaft, it follows that the amplitude of a vibration with twice the frequency of the rotating speed ω grows very rapidly.

We can verify the existence of this unstable region also by measuring the natural frequencies of this system. The experimental values of p_2 near the resonance point are shown in Fig.5. If there are no unstable regions, p_2 will take the values of the curve shown in a broken line in the figure. But the experimental p_2 curve is the same in shape as the full line in Fig.2 owing to the existence of the unstable region. We couldn't observe the vibrations with smaller amplitudes shown in broken lines in Fig.2.

This unstable region occurs owing to the coexistence of the shaft unsymmetry and

Fig.5 Natural frequencies near the secondary critical speed ($p_2=2\omega$).

Fig.6 Experimental resonance curve near the secondary critical speed satisfying $p_2=2\omega$ (in the case of absence of the unstable region).

the directional difference of the bearing support. Therefore, when the bearing center line shifts its position against the angular clearance of the bearing, the magnitude of the directional difference of the support condition at the shaft end varies, and thus the width of the unstable region changes. When the center lines of the upper and the lower bearings are in good alignment and there is little directional difference of the bearing support, $\Delta\gamma_4$ and others in Chapter 4 become zero approximately. Then, as shown in Fig.6, the steady vibration $[+2\omega]$ appears with no unstable region. Furthermore, as the center lines of the upper and the lower bearings are in good alignment in Fig.6, the resonance point is lower than that in Fig.3 by about 35 rpm.

In the experiments where we used the same unsymmetrical shaft and the same bearing pedestals as those in the above experiments but exchanged only the lower bearing for a self-aligning double-row ball bearing, both the shaft ends were under the simple support condition, and no unstable vibration of $[+2\omega]$ occurred. Therefore, we conclude that the unsymmetry of the flexible bearing pedestals represented by Eq.(28) in Chapter 4 has no effect on the unstable vibration treated here.

7. An unsymmetrical rotor system

An unsymmetrical rotor system often shows the same vibratory characteristics as those of an unsymmetrical shaft system, but no unstable vibrations $[+2\omega]$ occur in the former system. Figure 7 shows an example of the experimental resonance curve near the secondary critical speed in the unsymmetrical rotor system. This experimental apparatus is designed as follows: an unsymmetrical rotor is mounted at the position of the ratio 1:4 from the end of a round shaft which has a diameter of 12 mm, and a length of $l=700$ mm (that is, the distance between the centers of the upper and the lower bearings). The shaft is supported by a self-aligning double-row ball bearing at the

Fig.7 Experimental resonance curve near the secondary critical speed satisfying $p_2=+2\omega$ in the unsymmetrical rotor system.

upper end, and a single-row deep groove ball bearing at the lower end. The dimensions of the unsymmetrical rotor are 8,791 kg in mass, $I_p=0.204$ kg·m², $I_1=0.159$ kg·m², and $I_2=0.048$ kg·m². Though experiments were performed for various assemblies, the resonance curves were all the same in shape as the curve of Fig.7, and no unstable region appeared. The steady vibration $[+2\omega]$ in Fig.7 appeared as a result of resonance to the external force caused by variation of the static deflection as mentioned in the previous paper⁽¹⁴⁾.

In the unsymmetrical rotor system, the restoring force and moment working on the rotor are not expressed by Eq.(6), but the following equations.

$$\left. \begin{aligned} F &= ar + \gamma\theta + \Delta\alpha_0\bar{r} + \Delta\gamma_0\bar{\theta} \\ M &= r\gamma + \delta\theta + \Delta\gamma_0\bar{r} + \Delta\delta_0\bar{\theta} \end{aligned} \right\} \dots\dots\dots(29)$$

Therefore, Eq.(29) doesn't involve any terms for the coefficients $\Delta\alpha_4$, $\Delta\gamma_4$, and $\Delta\delta_4$ in Eq.(6) which are the factors causing unstable vibration near $p_2=2\omega$. Consequently, no unstable vibrations $[+2\omega]$ can appear in the unsymmetrical rotor system.

8. Conclusions

On the vibration near the secondary critical speed in a rotating shaft system where an unsymmetrical shaft is supported by a single-row deep groove ball bearing, the results of the theoretical and the experimental analyses are summarized as follows:

(1) Near the rotating speed satisfying $p_2=+2\omega$, there exists an unstable region where the unstable vibration of the frequency 2ω appears.

(2) The cause of this unstable region is the coexistence of the rotating directional difference due to the shaft unsymmetry and the stationary directional difference due to the angular clearance of the bearing.

(3) Change of the assembly of the experimental apparatus gives rise to variations in the bearing support condition. Consequently, the width of the unstable region varies, and sometimes the unstable region vanishes.

(4) In the system where the elastic shaft with the circular cross section carrying an unsymmetrical rotor is supported by a single-row deep groove ball bearing, no unstable vibration $[+2\omega]$ occurs.

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