Dynamic Response Analysis of Structures with Large Degrees of Freedom by Step-by-step Transfer Matrix Method*

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In this study, a new method of dynamic response analysis, termed "step-by-step transfer matrix method" is presented, which can be applied to general dynamic problems. This method needs only small degrees of freedom, nearly equal to those of elements, for dynamic response analysis of structures even though the structures have large degrees of freedom and any algorithm of existing step-by-step integration techniques can be used jointly with much of concepts of the standard transfer matrix method. Some significant relations :

- (a) a driving point stiffness matrix,
 - (b) an absorbing matrix and
 - (c) a recurrence formula for a driving point stiffness matrix
- are developed to treat various types of structural systems.

Several numerical examples are obtained and comparisons of them with experiments and results by the large matrix method are made to examine the validity and effectiveness of the proposed method.

1. Introduction

In view of the degrees of freedom, two approaches are available for a static and a dynamic analysis of a structural system which have already been discretized by a certain method, for example, the finite element method.

(1) One is the method using a matrix obtained by assembling mass, damping and stiffness matrices of elements from the consideration of the equilibrium and the compatible conditions at nodal points,

(2) The other is the method using matrices which give the relations between adjacent state vectors of elements. This method is generally known as the transfer matrix method.

We will hereafter term the former method "large matrix method" for convenience.

Both of these methods have been frequently used for a natural frequency analysis and a steady state response analysis under a harmonic excitation. However, only large matrix method has been used for a general dynamic response analysis of structures subjected to such a complicated load as impulsive one because of the difficulty in the calculation by the transfer matrix method. Various procedures called the step-by-step integration technique are widely used there. Newmark's β method and Wilson's θ method are two of typical methods among them.

Consider the case of a large system. As the degrees of freedom are very large, the

application of the step-by-step integration techniques involves great difficulties resulting from the lack of required memory and the increase in computation time using the computer system.

In this study, a new method of dynamic response analysis termed "step-by-step transfer matrix method" is presented to overcome above mentioned difficulties.

In the method, any algorithm of the existing step-by-step integration procedures can be used without any consideration and a modified transfer matrix method is utilized for the dynamic response analysis at each short time increment. Numerical examples are obtained in order to examine the validity and the effectiveness of the proposed method.

2. Algorithms of step-by-step integration techniques

In the step-by-step integration procedures, the response of the system is evaluated for a series of short time increments. The motion of the system is calculated successively by the direct integration of the equation of motion at every time increment. Existing various algorithms may be classified into the following two groups :

- (i) conditionally stable methods :
 - (a) linear acceleration method, and
 - (b) Newmark's β method,
 - (c) others.
- (ii) unconditionally stable methods :
 - (a) Wilson's θ method,
 - (b) Houbolt's method and
 - (c) others.

Accordingly we will take Newmark's β method and Wilson's θ method as typical methods in the following discussion and give some brief explanations necessary for the derivation of the step-by-step transfer matrix method.

2.1 Newmark's β method

Assume that all of displacements,

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velocities and accelerations at an instant t are known, then the following three equations become basic ones :

(a) approximate equation of displacements at the time $t+\Delta t$

$$q(t+\Delta t) = q(t) + \Delta t \dot{q}(t) + \left(\frac{1}{2} - \beta\right) (\Delta t)^2 \ddot{q}(t) + \beta (\Delta t)^2 \ddot{q}(t+\Delta t) \quad \dots\dots\dots (1)$$

(b) approximate equation of velocities at the time $t+\Delta t$

$$\dot{q}(t+\Delta t) = \dot{q}(t) + \frac{\Delta t}{2} \{\ddot{q}(t+\Delta t) + \ddot{q}(t)\} \quad \dots\dots\dots (2)$$

(c) equation of motion at the time $t+\Delta t$

$$Q(t+\Delta t) = M(t+\Delta t) \ddot{q}(t+\Delta t) + C(t+\Delta t) \dot{q}(t+\Delta t) + K(t+\Delta t) q(t+\Delta t) \quad \dots\dots\dots (3)$$

where Q represents a force vector and M, C, K symbolize the mass, damping, stiffness matrices respectively. When the parameter β equals to $1/6$, the procedure becomes identical to the linear acceleration method.

From Eqs. (1) ~ (3), the displacements, velocities and accelerations at the time $t+\Delta t$ can be calculated.

2.2 Wilson's θ method

Wilson's θ method has almost the same algorithm as the standard linear acceleration method. But the following interpolation is used to have a stable solution ; Quantities at the time $t+\Delta t$ are evaluated indirectly by use of a parameter θ over an extended time interval $t \sim t+\theta\Delta t$ as in Eqs. (4) ~ (6).

$$q(t+\theta\Delta t) = q(t) + \theta\Delta t \dot{q}(t) + \frac{(\theta\Delta t)^2}{3} \ddot{q}(t) + \frac{1}{6} (\theta\Delta t)^2 \ddot{q}(t+\theta\Delta t) \quad \dots\dots\dots (4)$$

$$\dot{q}(t+\theta\Delta t) = \dot{q}(t) + \frac{\theta\Delta t}{2} \{\ddot{q}(t+\theta\Delta t) + \ddot{q}(t)\} \quad \dots\dots\dots (5)$$

$$Q(t+\theta\Delta t) = M(t+\theta\Delta t) \ddot{q}(t+\theta\Delta t) + C(t+\theta\Delta t) \dot{q}(t+\theta\Delta t) + K(t+\theta\Delta t) q(t+\theta\Delta t) \quad \dots\dots\dots (6)$$

$\ddot{q}(t+\Delta t)$ is computed from the linear interpolation with respect to the accelerations :

$$\ddot{q}(t+\Delta t) = \frac{1}{\theta} \{(\theta-1)\ddot{q}(t) + \ddot{q}(t+\theta\Delta t)\} \quad \dots\dots\dots (7)$$

The next procedure is identical to the linear acceleration method. Thus we have the displacements and velocities

$$q(t+\Delta t) = q(t) + \Delta t \dot{q}(t) + \frac{(\Delta t)^2}{3} \ddot{q}(t) + \frac{(\Delta t)^2}{6} \ddot{q}(t+\Delta t) \quad \dots\dots\dots (8)$$

$$\dot{q}(t+\Delta t) = \dot{q}(t) + \frac{\Delta t}{2} \{\ddot{q}(t) + \ddot{q}(t+\Delta t)\} \quad \dots\dots\dots (9)$$

When either Newmark's β method or Wilson's θ method as mentioned above is used along with the large matrix method, the property matrices and vectors appearing in those equations need the whole degrees of freedom of the system. An increase

in the degrees of freedom causes serious difficulties on account of the lack of required memory cores and the increase in the computation time in the case of solving these equations simultaneously.

However, if a modified transfer matrix method at every time step described later is used along with one of the step-by-step integration procedures, the necessary degrees of freedom in Eqs. (1) ~ (9) equal nearly to those of each element and thus it is possible to overcome the computational difficulties.

3. Proposal of dynamic response analysis by step-by-step transfer matrix method

Only natural frequency analysis and steady state response analysis under a harmonic excitation are able to be treated by the standard transfer matrix method due to the characteristic of the procedures. It is difficult to apply the standard transfer matrix method along with a step-by-step integration technique to a dynamic problem with a complicated loading history. Consequently in this study, a new approach termed "step-by-step transfer matrix method" will be presented so that it could be applied to a general dynamic response problem. Although this method can be used along with any of step-by-step integration techniques, we will restrict our discussion to the case using Newmark's β method Wilson's θ method.

3.1 Formulation of step-by-step transfer matrix of element

The standard transfer matrix defines the relation between adjacent state vectors (generalized force vector involving moments and generalized displacement vector involving slope etc.) of elements.

It is possible to take any of the following three quantities as a state vector in the method to be proposed ;

- (i) displacements and forces,
- (ii) velocities and forces,
- (iii) accelerations and forces.

Of course, the concept of a force vector contains here a moment vector and a displacement vector a slope vector etc.

3.1.1 Step-by-step transfer matrix method using Newmark's method

(a) Transfer matrix for a state vector expressed in terms of forces and displacements

Solving $q(t+\Delta t)$ in Eq. (1) through Eq. (3) simultaneously and rearranging it with respect to the displacements lead to

$$Q(t+\Delta t) = \left[\frac{1}{\beta(\Delta t)^2} M(t+\Delta t) + \frac{1}{2\beta\Delta t} C(t+\Delta t) + K(t+\Delta t) \right] q(t+\Delta t) - \left[\frac{1}{\beta(\Delta t)^2} M(t+\Delta t) \left\{ q(t) + \Delta t \dot{q}(t) + \left(\frac{1}{2} - \beta\right) (\Delta t)^2 \ddot{q}(t) \right\} + \frac{1}{2\beta\Delta t} C(t+\Delta t) \left\{ \dot{q}(t) + (1-2\beta)\Delta t \ddot{q}(t) + \left(\frac{1}{2} - \beta\right) (\Delta t)^2 \ddot{q}(t) \right\} \right] \quad \dots\dots\dots (10)$$

or simply

$$Q(t+\Delta t) = UQ(t) + G \quad \dots\dots\dots(11)$$

where U represents the relevant term to $q(t+\Delta t)$ and the components in G consist of $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$

Rewritten with respect to the left and the right quantities, the equation becomes

$$\begin{Bmatrix} -Q^L(t+\Delta t) \\ Q^R(t+\Delta t) \end{Bmatrix} = \begin{bmatrix} u_A & u_B \\ u_B^T & u_C \end{bmatrix} \begin{Bmatrix} q^L(t+\Delta t) \\ q^R(t+\Delta t) \end{Bmatrix} + \begin{Bmatrix} g^L \\ g^R \end{Bmatrix} \quad \dots\dots\dots(12)$$

where u_A , u_B , u_C are submatrices of U and superscript T denotes a transposed quantity and g^L , g^R are the components of G . If we take these displacements and forces as a state vector, it is rather difficult to express Eq.(12) directly in the form of the transfer matrix because an additional term remains on the right hand of Eq.(12).

This term may, however, be brought together as a single term by adding unity to the state vector and then we can derive the relation of the transmission as follows :

$$\begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^R = \begin{bmatrix} -u_B^{-1}u_A & -u_B^{-1} & -u_B^{-1}g \\ u_B^T - u_C u_B^{-1}u_A & -u_C u_B^{-1} & g^R - u_C u_B^{-1}g^L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(13)$$

where the bracket gives the extended transfer matrix which transmits the extended state vector and thus it is called hereafter "step-by-step transfer matrix". Eq.(13) is written simply as

$$\begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^R = \begin{bmatrix} t_A & t_B & t_C \\ t_D & t_E & t_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(14)$$

where 0 designates a row vector with zero components.

For reference, the complete forms of $t_A \sim t_F$ are attached as an appendix at the end.

(b) Transfer matrix for a state vector expressed in terms of accelerations and forces

Substitution of Eqs.(1),(2) into Eq.(3) gives a relation with respect to the acceleration as follows :

$$\begin{aligned} Q(t+\Delta t) = & \left\{ M(t+\Delta t) + \frac{\Delta t}{2} C(t+\Delta t) + \beta(\Delta t)^2 K(t+\Delta t) \right\} \ddot{q}(t+\Delta t) \\ & + C(t+\Delta t) \left\{ \dot{q}(t) + \frac{\Delta t}{2} \ddot{q}(t) \right\} \\ & + K(t+\Delta t) \left\{ q(t) + \Delta t \dot{q}(t) + \left(\frac{1}{2} - \beta \right) (\Delta t)^2 \ddot{q}(t) \right\} \end{aligned} \quad \dots\dots\dots(15)$$

Noting the left and the right quantities of the element and expressing simply, we have

$$\begin{Bmatrix} -Q^L(t+\Delta t) \\ Q^R(t+\Delta t) \end{Bmatrix} = \begin{bmatrix} u_A & u_B \\ u_B^T & u_C \end{bmatrix} \begin{Bmatrix} \ddot{q}^L(t+\Delta t) \\ \ddot{q}^R(t+\Delta t) \end{Bmatrix} + \begin{Bmatrix} g^L \\ g^R \end{Bmatrix} \quad \dots\dots\dots(16)$$

Hence we have a similar relation between the extended state vectors to the case of (a).

$$\begin{Bmatrix} \ddot{q}(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^R = \begin{bmatrix} -u_B^{-1}u_A & -u_B^{-1} & -u_B^{-1}g \\ u_B^T - u_C u_B^{-1}u_A & -u_C u_B^{-1} & g^R - u_C u_B^{-1}g^L \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(17)$$

If we eliminate the term of accelerations at the time $t+\Delta t$ from Eq.(1) through Eq.(3) and retain only the associated term with velocities, we can derive the step-by-step transfer matrix in the same way as in the former cases of (a) and (b).

3.1.2 Step-by-step transfer matrix method using Wilson's θ method

In this case, it is also possible to take any of the state vectors shown in (i) ~ (iii) to be transmitted. For example, when the accelerations and forces are taken as a state vector (that is the case (ii)), substituting Eqs.(4),(5) into Eq.(6) and arranging with respect to the accelerations lead to

$$\begin{aligned} Q(t+\theta\Delta t) = & \left\{ M + \frac{\theta\Delta t}{2} C + \frac{(\theta\Delta t)^2}{6} K \right\} \ddot{q}(t+\theta\Delta t) \\ & + C \left\{ \dot{q}(t) + \frac{\theta\Delta t}{2} \ddot{q}(t) \right\} + K \left\{ q(t) + \theta\Delta t \dot{q}(t) + \frac{(\theta\Delta t)^2}{3} \ddot{q}(t) \right\} \end{aligned} \quad \dots\dots\dots(18)$$

Rewriting this in view of the left and the right quantities and applying the linear interpolation technique to the accelerations, we can formulate the step-by-step transfer matrix in the same way as in the case of Newmark's β method. Since choosing other state vectors as in the cases of (i) and (iii) yields almost the same procedure, the explanation will be omitted here.

3.2. Step-by-step transfer matrix for external forces

We will discuss only the case of taking forces as a state vector for simplicity. The same discussions can be applied to the cases of taking other state vectors. The forces in a state vector to be transmitted by the step-by-step transfer matrix are stress resultants. Some consideration is needed to deal with the applied external forces at i 'th nodal point. This can be treated by formulating the following step-by-step transfer matrix :

$$\begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^R = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q(t+\Delta t) \\ Q(t+\Delta t) \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(19)$$

or directly adding the relation in Eq.(19) to Eq.(14) leads to

$$\begin{Bmatrix} \mathbf{q}(t+\Delta t) \\ \mathbf{Q}(t+\Delta t) \\ 1 \end{Bmatrix}^R = \begin{bmatrix} \mathbf{t}_A & \mathbf{t}_B & \mathbf{t}_C \\ \mathbf{t}_D & \mathbf{t}_E & \mathbf{t}_F - \mathbf{F} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} \mathbf{q}(t+\Delta t) \\ \mathbf{Q}(t+\Delta t) \\ 1 \end{Bmatrix}^L \quad (20)$$

where $\mathbf{I}$ represents a unit matrix and $\mathbf{0}$ a zero matrix or a zero vector.

3.3 Dynamic response analysis of whole system

A formulation of the step-by-step transfer matrix of the whole system is necessary for the dynamic response analysis of the system. We hereafter consider mainly the most adaptable structural system for the analysis by the transfer matrix method, that is, a chain type structural system of which elements are connected in series as shown in Fig.1.

Fig.1 Chain type structural system

All other structural systems can be reduced to this type of structural system as described later. Suppose the step-by-step transfer matrix of each element is known, then the relation between the left and the right state vectors is as follows:

$$\mathbf{Z}_i^R = \mathbf{T}_i \mathbf{Z}_i^L \quad (i=1, 2, \dots, n) \quad (21)$$

Using other relations

$$\mathbf{Z}_i^R = \mathbf{Z}_{i-1}^L, \quad \mathbf{Z}_{i+1}^L = \mathbf{Z}_i^R \quad (i=1, 2, \dots, n) \quad (22)$$

we obtain the expression between the two state vectors at both ends of the system as follows

$$\mathbf{Z}_n^R = \mathbf{T}_n \mathbf{T}_{n-1} \dots \mathbf{T}_2 \mathbf{T}_1 \mathbf{Z}_1^L = \mathbf{T} \mathbf{Z}_1^L \quad (23)$$

The matrix $\mathbf{T}$ in Eq.(22) gives the step-by-step transfer matrix of the system and to be obtained as the successive products of each transfer matrix.

Giving boundary conditions at the both ends of the system and the initial conditions at the time t_0 to Eq.(23), we can calculate the dynamic response of each nodal point at every time increment. We often see a case where only initial accelerations are unknown, while other initial displacements and velocities are known. In such a case, a procedure of solving the equation of motion at the initial time simultaneously is often adopted for the analysis based on the large matrix method. However, we can calculate the initial accelerations by the same step-by-step transfer matrix method as used in the analysis at every time increment;

The equation of motion at the initial time is written as

$$\mathbf{Q}(t_0) = \mathbf{M}(t_0) \ddot{\mathbf{q}}(t_0) + \mathbf{C}(t_0) \dot{\mathbf{q}}(t_0) + \mathbf{K}(t_0) \mathbf{q}(t_0) \quad (24)$$

Rewriting this with respect to the unknown accelerations, we have

$$\mathbf{Q}(t_0) = \mathbf{M}(t_0) \ddot{\mathbf{q}}(t_0) + \mathbf{H} \quad (25)$$

where $\mathbf{H}$ contains the initial displacements and velocities. Equation(25) has a similar form to Eq.(11). Accordingly the formulation of the step-by-step transfer matrix for the accelerations and forces gives an evaluation of the unknown accelerations using the approximate same degrees of freedom of elements. A simple flow chart is given in Fig.2 to show the procedure of the step-by-step transfer matrix method.

Fig.2 Procedure by step-by-step transfer matrix method

3.4 Some additional properties of step-by-step transfer matrix

3.4.1 Driving point stiffness matrix

Assume that state vectors at both ends of the system are related to each other by a step-by-step transfer matrix of the system as shown by

$$\begin{Bmatrix} \mathbf{q} \\ \mathbf{Q} \\ 1 \end{Bmatrix}^R = \begin{bmatrix} \mathbf{t}_A & \mathbf{t}_B & \mathbf{t}_C \\ \mathbf{t}_D & \mathbf{t}_E & \mathbf{t}_F - \mathbf{F} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{Q} \\ 1 \end{Bmatrix}^L \quad (26)$$

Then if we give such a prescribed boundary condition as "clamped" or "free" etc. to the state vector at either end of the system, the relation between a displacement vector and a force vector at the other end is of the form:

$$\begin{Bmatrix} \mathbf{Q} \\ 1 \end{Bmatrix} = \begin{bmatrix} \mathbf{K}^* \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ 1 \end{Bmatrix} \quad (27)$$

$\mathbf{K}^*$ relates the extended displacement vector to the extended force vector and is considered to be a kind of stiffness matrix. We will call it "stiffness matrix at the driving point" or simply "driving point stiffness matrix" (Table 1). We used the terminology "driving point stiffness matrix" above and will use it later. However, the property of the matrix will be close to that of a mass matrix in the case taking other types of vectors.

3.4.2. Absorbing matrix

It is comparatively simple and easy to deal with a chain-type structural system by the step-by-step transfer matrix method. On the other hand, a structural system having a branch as shown in Fig.3 is often seen in the practical problems. We can also treat this type of structural system by the step-by-step transfer matrix as follows; First the step-by-step transfer matrix in the branch system is derived. To this end, the relation between a starting point state vector and an ending point one in the branch system is obtained by successive multiplication of step-by-step transfer matrices of elements. Giving a boundary condition to the ending point state vector and deriving a driving point stiffness matrix at the starting point lead to the following relation between Q^L and Q^R .

$$\begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L = \begin{bmatrix} t_A & t_B & t_C \\ t_D & t_E & t_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^R \dots\dots\dots(28)$$

In addition, if we note the main system primary, we can regard the driving point stiffness matrix as a special element inserted at the starting point of the branch system. The property of this special element is given by the driving point stiffness matrix as follows:

$$q^L = q^R = q^b \dots\dots\dots(29)$$

$$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^L - \begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{Bmatrix} Q \\ 1 \end{Bmatrix}^b = [K^*] \begin{Bmatrix} q \\ 1 \end{Bmatrix}^b \dots\dots\dots(30)$$

$$\times \begin{Bmatrix} q \\ 1 \end{Bmatrix}^b = \begin{bmatrix} k_A^* & k_B^* \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^b$$

From Eq.(30) we can derive the step-by-step transfer matrix of the form:

$$\begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} I & 0 & 0 \\ -k_A^* & I & -k_B^* \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L \dots\dots\dots(31)$$

This step-by-step transfer matrix is called "absorbing transfer matrix" or simply "absorbing matrix". Although this matrix is originally formulated to deal with elastic constraints, we can treat the branch type structural system by using it along with a driving point stiffness matrix.

3.4.3 Recurrence formula for driving point stiffness matrix

Consider a structural system whose end being constrained or bounded by a driving point stiffness matrix as shown in Fig.4.

Then the relation between the left and the right state vectors is written by

$$\begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} t_A & t_B & t_C \\ t_D & t_E & t_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L \dots\dots\dots(32)$$

Table 1 Examples of driving point stiffness matrices

boundary condition	driving point stiffness matrix K^*
free	$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} t_D t_A^{-1} - t_E t_A^{-1} t_C + t_F & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^L$
clamped	$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} t_E t_B^{-1} - t_F t_B^{-1} t_C + t_F & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^L$
free	$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} -t_E^{-1} t_D & -t_E^{-1} t_F \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^L$
clamped	$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} -t_E^{-1} t_A & -t_E^{-1} t_C \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^L$

Fig.3 Branch type structural system

Fig.4 Structural system of which one end is constrained by driving point stiffness matrix

If we invert the direction of transfer, we have the following reciprocal transfer matrix.

$$\begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L = \begin{bmatrix} i_A & i_B & i_C \\ i_D & i_E & i_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^R \dots\dots\dots(33)$$

The state vector at the right end is constrained by the driving point stiffness

matrix as

$$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} K^* \\ 0 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} k_A^* & k_B^* \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^R \quad \dots\dots\dots(34)$$

From Eqs. (33), (34),

$$\begin{Bmatrix} Q \\ 1 \end{Bmatrix}^L = \begin{bmatrix} (\bar{i}_D + \bar{i}_E k_A^*)(\bar{i}_A + \bar{i}_B k_A^*)^{-1} - (\bar{i}_D + \bar{i}_E k_A^*)(\bar{i}_A + \bar{i}_B k_A^*)^{-1}(\bar{i}_B k_B^* + \bar{i}_C) + (\bar{i}_E k_B^* + \bar{i}_F) \\ 0 \end{bmatrix} \begin{Bmatrix} q \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(35)$$

This yields a new driving point stiffness matrix at the left end of the system and the property of each element is able to be absorbed into such a driving point stiffness matrix in turn. Hence Eq. (35) is regarded as a recurrence formula for the driving point stiffness matrix.

3.4.4. Bundle type structural system

Finally we are to deal with a bundle type structural system as shown in Fig.5. We can easily treat it by deriving each stiffness matrix from the transfer matrices T_1, T_2 of each element and assembling them and then transforming the assembled matrix into a transfer matrix. The formulation of a stiffness matrix from a transfer matrix is done on by

$$\begin{Bmatrix} -Q^L \\ Q^R \\ 1 \end{Bmatrix} = \begin{bmatrix} t_B^{-1}t_A & -t_B^{-1} & t_B^{-1}t_C \\ t_D - t_E t_B^{-1}t_A & t_E t_B^{-1} & t_F - t_E t_B^{-1}t_C \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q^L \\ q^R \\ 1 \end{Bmatrix}$$

$$= \begin{bmatrix} k_A & k_B & k_C \\ k_D & k_E & k_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q^L \\ q^R \\ 1 \end{Bmatrix} \quad \dots\dots\dots(36)$$

Inversely the formulation of a transfer matrix from a stiffness matrix is done by

$$\begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^R = \begin{bmatrix} -k_B^{-1}k_A & -k_B^{-1} & -k_B^{-1}k_C \\ k_D - k_E k_B^{-1}k_A & k_E k_B^{-1} & k_F - k_E k_B^{-1}k_C \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L$$

$$= \begin{bmatrix} t_A & t_B & t_C \\ t_D & t_E & t_F \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} q \\ Q \\ 1 \end{Bmatrix}^L \quad \dots\dots\dots(37)$$

Equations (36), (37) have similar transformations but the signs of elements in these matrices are partly different.

4. Examinations by numerical examples

To examine the proposed method we developed a computer program for dynamic response analysis of planar framed structures by the step-by-step transfer matrix method and obtained several numerical examples. In this program, we used Newmark's β method and took displacements and forces as a state vector to be transmitted. Additionally we aimed at developing a general purpose program so as to deal with a branch type and a bundle type structural systems. The computer program consists of a main program and nineteen subroutines. If we input the program by punched cards, about 1,000 cards will be necessary. As compared with use of the large matrix method, the number of subroutines rather increases but required memory cores for

Fig.5 Bundle type structural system

variable area become smaller, about 5,200 words with double precision. The dynamic response of a structure composed of about 50 elements at the most can be calculated by this.

The following examples were obtained all assuming that a parameter β equals to 1/4 and a time increment Δt equals to 0.2 ms. Next we will show the comparison with experiments using model structures and with the results by the large matrix method.

4.1 Comparison with experiments

In the experiments to be compared with numerical results, impulsive loads by hammering were applied to various types of model structures made of mild steel (SS41) and response displacement histories were recorded (Fig.6).

Fig.6 Various types of models

The comparisons between numerical results by the proposed method and experiments are shown in Fig.7 through in Fig.12. For reference, configurations of the models and the points at which a load applied and response data were recorded are also shown in these figures. Dots on the figures of each model indicate the nodal points. The numerical results agree fairly well with experiments and the accuracy of the proposed method is confirmed.

Fig.7 Dynamic response of model A

Fig.8 Dynamic response of model B (No.1)

Fig.9 Fig.9 Dynamic response of model C (No.1)

Fig.10 Dynamic response of model C (No.1)

Fig.11 Dynamic response of model C (No.2)

Fig.12 Dynamic response of model D

4.3. Comparison with results by large matrix method

A simple comparison was made with results by the large matrix method to confirm the effectiveness of the proposed method. Both methods were used for the dynamic response analysis of a cantilever beam subjected to a half-sine impulsive load at its free end and computation times corresponding to various numbers of elements were compared as shown in Fig.13.

In the large matrix method, the computation time is in proportion to the approximate square of the number of element. In the step-by-step transfer matrix method, it varies proportionally as the number of elements increases.

Thus the step-by-step transfer matrix

method becomes more effective as the number of elements increases.

Fig.13 Comparison with large matrix method
($t=2$ ms)

5. Conclusions

A new method, termed "step-by-step transfer matrix method" was presented to solve the difficulties with the shortage of memory cores and the increase in computation time in using the computer. In this method, we can utilize the various concepts of the standard transfer matrix method and use any algorithm of existing step-by-step integration techniques. Any of the following quantities might be chosen as a state vector in this method :

- (i) displacements and forces,
- (ii) velocities and forces,
- (iii) accelerations and forces

In order to deal widely with various structures, some important relations of a driving point stiffness matrix, an absorbing matrix and a recurrence formula for a driving point matrix were introduced which enabled us to apply this method not only to a chain type structure most adaptable for the transfer matrix method but also to a branch type structure and a bundle type structure. Displacement responses of planar framed structures subjected to impulsive loads were given by this method and the comparisons with experiments confirmed the accuracy of it. In addition effectiveness was also confirmed by the comparison with the results by the large matrix method. It will be further necessary to have more various numerical results and examine them in detail.

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Appendix

Complete forms of the elements $t_A \sim t_F$ in the step-by-step transfer matrix of element :

$$t_A = - \left[\frac{1}{\beta(\Delta t)^2} m_B + \frac{1}{2\beta\Delta t} c_B + k_B \right]^{-1} \\ \times \left[\frac{1}{\beta(\Delta t)^2} m_A + \frac{1}{2\beta\Delta t} c_A + k_A \right]$$

$$t_B = - \left[\frac{1}{\beta(\Delta t)^2} m_B + \frac{1}{2\beta\Delta t} c_B + k_B \right]^{-1}$$

$$t_D = - \left[\frac{1}{\beta(\Delta t)^2} m_C + \frac{1}{2\beta\Delta t} c_C + k_C \right] \left[\frac{1}{\beta(\Delta t)^2} m_B + \frac{1}{2\beta\Delta t} c_B + k_B \right] \\ \times \left[\frac{1}{\beta(\Delta t)^2} m_A + \frac{1}{2\beta\Delta t} c_A + k_A \right] + \left[\frac{1}{\beta(\Delta t)^2} m_B + \frac{1}{2\beta\Delta t} c_B + k_B \right]$$

$$t_E = - \left[\frac{1}{\beta(\Delta t)^2} m_C + \frac{1}{2\beta\Delta t} c_C + k_C \right] \left[\frac{1}{\beta(\Delta t)^2} m_B + \frac{1}{2\beta\Delta t} c_B + k_B \right]^{-1}$$

$$g^L = -m_A \left\{ \frac{1}{\beta(\Delta t)^2} q^L(t) + \frac{1}{\beta\Delta t} \dot{q}^L(t) + \left(\frac{1}{2\beta} - 1 \right) \ddot{q}^L(t) \right\} \\ + m_B \left\{ \frac{1}{\beta(\Delta t)^2} q^R(t) + \frac{1}{\beta\Delta t} \dot{q}^R(t) - \left(\frac{1}{2\beta} - 1 \right) \ddot{q}^R(t) \right\} \\ + c_A \left\{ \frac{1}{2\beta\Delta t} q^L(t) + \left(\frac{1}{2\beta} - 1 \right) \dot{q}^L(t) + \left(\frac{1}{4\beta} - 1 \right) \Delta t \ddot{q}^L(t) \right\} \\ + c_B \left\{ \frac{1}{2\beta\Delta t} q^R(t) + \left(\frac{1}{2\beta} - 1 \right) \dot{q}^R(t) + \left(\frac{1}{4\beta} - 1 \right) \Delta t \ddot{q}^R(t) \right\}$$

g^R : replace m_A, m_B, c_A, c_B in g^L
by $m_A^T, m_B^T, c_A^T, c_B^T$ respectively
 $t_C = t_B g^L, \quad t_F = g^R + t_D g^L$

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