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On Integral Equation Methods
for Elastic Boundary Value Problems*
(1st. Report, Direct Methods for Static Problems)

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The applications of the boundary integral equation methods (direct methods) to static elasticity problems are investigated. It is shown that considerable errors may occur in numerical solutions if integral equations are used uncritically. Several suggestions and modifications for the integral equations are presented.

1. Introduction

Since the appearance of the well-known Sherman-Lauricella's integral equations⁽¹⁾, a number of investigations have been published on the integral equation formulations of the elastic boundary value problems. The equations called boundary integral equations (direct methods) are derived by Kupradze⁽²⁾, and shown to be effective for numerical analyses of the actual elasticity problems by Rizzo⁽³⁾. Although many investigators have been interested in these methods, few of them have studied the question whether the integral equations give the solutions of the elasticity problems or not.

In this paper, the applications of the boundary integral equation methods (direct methods) to the static, isotropic elasticity problems are investigated. The equivalency between the integral equation formulations and the original elasticity problems is examined by the use of a fictitious displacement field defined in the domain exterior to the elastic body. An example of the numerical solution is shown, in which there appear considerable errors due to the non-equivalency of the integral equation to the elasticity problem. Several suggestions and appropriately modified integral equations are presented.

2. Finite elastic bodies

We use Cartesian coordinate system x_i and the standard tensor notations. Subscripts take the values 1, 2, 3 in three-dimensional cases, and 1, 2 in two-dimensional cases. Summation convention is assumed. The subscripts following commas indicate the differentiations with respect to the coordinate variables at the field points where the displacements are defined.

The fundamental solution $U_i^{(j)}(P, Q)$ is the x_i component of the displacement vector at a point P in an infinite elastic body when the unit concentrate body force is

applied at a point Q in the x_j direction, and is given by

$$U_i^{(j)}(P, Q) = \frac{1}{16\pi\mu} \frac{3-4\nu}{1-\nu} \frac{1}{R} \delta_{ij} + \frac{1}{16\pi\mu} \frac{1}{1-\nu} \frac{1}{R} R_{,i} R_{,j} \dots\dots\dots (1)$$

in three-dimensional cases⁽⁴⁾, and by

$$U_i^{(j)}(P, Q) = \frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \ln \frac{l}{R} \delta_{ij} + \frac{1}{8\pi\mu} \frac{1}{1-\nu} R_{,i} R_{,j} \dots\dots\dots (1)^{*1}$$

in two-dimensional cases (plane strain). Here μ is the modulus of rigidity, ν is Poisson's ratio, δ_{ij} is Kronecker's delta and R denotes the distance between points P and Q.

The second fundamental solution $T_i^{(j)}(P, Q)$ is the x_i component of the displacement vector at a point P in an infinite elastic body when an infinitesimal dislocation loop, which is normal to a unit vector $\mathbf{n}$ and has unit dislocation in the x_j direction, exists at a point Q. If we make use of the elastic constants tensor C_{ijkl} defined as

$$C_{ijkl} = \frac{2\nu}{1-2\nu} \mu \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk}$$

the solution $T_i^{(j)}(P, Q)$ is written as $C_{mjkl} n_m(Q) U_k^{(j)},_l(Q, P)$ and we have

$$T_i^{(j)}(P, Q) = \frac{1}{8\pi} \frac{1-2\nu}{1-\nu} \frac{1}{R^2} [R_{,i} n_j - R_{,j} n_i + \left\{ \delta_{ij} + \frac{3}{1-2\nu} R_{,i} R_{,j} \right\} R_{,k} n_k] \dots\dots\dots (2)$$

*1 Here l denotes an arbitrary positive constant with dimension of length. Although many investigators put $l=1$, we use this expression to clarify that the argument of the logarithm is a non-dimensional number.

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in three-dimensional cases ⁽⁴⁾, and

$$T_{ij}^{(j)}(P, Q) = \frac{1}{4\pi} \frac{1-2\nu}{1-\nu} \frac{1}{R} \left[R, n_i - R, n_j + \left\{ \delta_{ij} + \frac{2}{1-2\nu} R, i R, j \right\} R, k n_k \right] \dots\dots\dots (2)'$$

in two-dimensional cases. Here n_i denotes the x_i component of the vector $\mathbf{n}$.

Suppose a finite, simply-connected isotropic elastic body D whose elastic constants are μ and ν , denote the boundary of D as S and the infinite domain exterior to D as D^c . In the elastic boundary value problem, we shall determine the displacement $u_i(P)$ ($P \in D$) which satisfies the displacement equation

$$\mu u_{i,jj}(P) + \frac{\mu}{1-2\nu} u_{j,i}(P) + f_i(P) = 0 \quad (P \in D) \dots\dots\dots (3)$$

together with the boundary conditions of the form

$$\left. \begin{aligned} [u_i(P)]^- &= g_i(P) & (P \in S_d) \\ [t_i^{(n)}(P)]^- &= [C_{ijkl} n_j(P) u_{k,i}(P)]^- = h_i(P) & (P \in S_t) \end{aligned} \right\} (S = S_d \cup S_t) \dots\dots\dots (4)$$

in which $[]^-$ indicates the limit values from the interior domain D , the body force is represented by $f_i(P)$, the surface displacement by $g_i(P)$, the surface traction by $h_i(P)$.

Now we introduce the function $v_i(P)$ expressed in the form

$$v_i(P) = \int_S \phi_j(Q) U_i^{(j)}(P, Q) ds(Q) - \int_S \varphi_j(Q) T_i^{(j)}(P, Q) ds(Q) + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) \dots\dots\dots (5)$$

where

$$\phi_i(P) = g_i(P) \quad (P \in S_d), \quad \phi_i(P) = h_i(P) \quad (P \in S_t) \dots\dots\dots (6)$$

and consider the condition for the unknown $\varphi_i(P)$ ($P \in S_t$) and $\phi_i(P)$ ($P \in S_d$):

$$v_i(P) = 0 \quad (P \in D^c) \dots\dots\dots (7)$$

The functional equation (7), which must be satisfied over the domain D^c , is equivalent to the preceding boundary value problem and the following relations hold.

$$\left. \begin{aligned} u_i(P) &= v_i(P) & (P \in D) \\ [u_i(P)]^- &= \varphi_i(P), \quad [t_i^{(n)}(P)]^- = \phi_i(P) & (P \in S) \end{aligned} \right\} \dots\dots\dots (8)$$

This equivalency may be deduced as follows:

(i) If we know the solution $u_i(P)$ of the elasticity problem eqs.(3) and (4), and set $\varphi_i(P)$ and $\phi_i(P)$ as eq.(8), then eq.(6) is established automatically, and eq.(7) is satisfied since it becomes one of Somigliana's identities ⁽⁵⁾

$$\begin{aligned} & \int_S [t_j^{(n)}(Q)]^- U_i^{(j)}(P, Q) ds(Q) - \int_S [u_j(Q)]^- T_i^{(j)}(P, Q) ds(Q) \\ & + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) = \begin{cases} u_i(P) & (P \in D) \\ u_i(P)/2 & (P \in S) \\ 0 & (P \in D^c) \end{cases} \dots\dots\dots (9) \end{aligned}$$

(ii) From the properties of the elastic potentials ^(5,6), the function $v_i(P)$ defined by eq.(5) satisfies the displacement equation (3) in the domain D and the following relations, for arbitrary $\varphi_i(P)$ and $\phi_i(P)$,

$$\left. \begin{aligned} [v_i(P)]^+ - [v_i(P)]^- &= -\varphi_i(P) \\ [C_{ijkl} n_j(P) v_{k,i}(P)]^+ - [C_{ijkl} n_j(P) v_{k,i}(P)]^- &= -\phi_i(P) \end{aligned} \right\} (P \in S) \dots\dots\dots (10)$$

where $[]^+$ indicates the limit values from the exterior domain D^c . Hence the functional equation (7) with eq.(6) implies the boundary conditions for the displacement $v_i(P)$ in the forms

$$[v_i(P)]^- = g_i(P) \quad (P \in S_d), \quad [C_{ijkl} n_j(P) v_{k,i}(P)]^- = h_i(P) \quad (P \in S_t) \dots\dots\dots (11)$$

Thus we see that $v_i(P)$ derived from the solution of the functional equation (7) is the solution of the elasticity problem eqs.(3) and (4).

Equation (7) is known as the functional equation of Kupradze ⁽²⁾. The boundary integral

equation (direct method) can be derived from this equation by letting the point P in eq.(7) approach the boundary S from the exterior domain D^e , and the result is

$$[v_i(P)]^+ = 0 \quad (P \in S) \dots\dots\dots (12)^{*2}$$

in which the limit value $[v_i(P)]^+$ is given by the use of the behaviour of the elastic potentials⁽⁶⁾ as

$$[v_i(P)]^+ = \int_S \phi_j(Q) U_{i,j}(P, Q) ds(Q) - \frac{1}{2} \varphi_i(P) - \int_S \varphi_j(Q) T_{i,j}(P, Q) ds(Q) + \int_D f_j(Q) U_{i,j}(P, Q) dv(Q) \dots\dots\dots (13)$$

Let us examine the equivalency between the integral equation (12) and the functional equation (7).

(i) If we notice that the function $v_i(P)$ given by eq.(5) satisfies the displacement equation with no body force in the domain D^e , we can regard $v_i(P)$ as a displacement induced in the fictitious elastic body D^e with elastic constants μ and ν . Then the integral equation (12) can be interpreted as the boundary condition $[v_i(P)]^+ = 0$ ($P \in S$). Hence if the function $v_i(P)$ satisfies eq.(12) and the conditions at infinity of the forms

$$v_i(P) = O(R_0^{-1}), \quad v_{i,j}(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \dots\dots\dots (14)$$

in three-dimensional cases, and

$$v_i(P) = O(1), \quad v_{i,j}(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \dots\dots\dots (14)'$$

in two-dimensional cases where R_0 denotes the distance between the point P and the origin, then we see that the uniqueness theorem of elasticity^{*3} guarantess that $v_i(P) = 0$ ($P \in D^e$), i.e., the functional equation (7) holds.

(ii) On the other hand, it is obvious that the function $v_i(P)$ derived from the solution of eq.(7) satisfies eqs.(12) and (14).

Thus we have the integral equation and the complementary conditions which are equivalent to the original elasticity problem. Using this equivalency, we may investigate the integral equation for the respective elasticity problems according to the natures of the given boundary conditions.

2.1 Displacement problems

In three-dimensional cases, we see from eqs.(1), (2) and (5) that the conditions at infinity (14) are satisfied by arbitrary $\varphi_i(P)$ and $\phi_i(P)$. Then the integral equation (12) turns out to be equivalent to the elasticity problem, wherein displacement $u_i(P)$ is determined uniquely because the boundary displacement is specified. Hence, by virtue of the relations (3), the solution $\phi_i(P) = [t_i^{(m)}(P)]^-$ of the integral equation (12) is unique.

In two-dimensional cases, eqs.(1)', (2)' and (5) give

$$\left. \begin{aligned} v_i(P) &= \left[\int_S \phi_j(Q) ds(Q) + \int_D f_j(Q) dv(Q) \right] \left[\frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \ln \frac{l}{R_0} + \text{const.} \right] + O(R_0^{-1}) \\ v_{i,j}(P) &= - \left[\int_S \phi_j(Q) ds(Q) + \int_D f_j(Q) dv(Q) \right] \frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \frac{1}{R_0} R_{0,j} + O(R_0^{-2}) \end{aligned} \right\} \quad (R_0 \rightarrow \infty) \dots\dots\dots (15)$$

Hence, the integral equation (12) is equivalent to the elasticity problem only if the following condition is satisfied.

*2 When the point P approaches the boundary S from the interior domain D in eq.(9), we have⁽⁸⁾

$$[u_i(P)]^- = \int_S [t_j^{(m)}(Q)]^- U_{i,j}(P, Q) ds(Q) + \frac{1}{2} [u_i(P)]^- - \int_S [u_j(Q)]^- T_{i,j}(P, Q) ds(Q) + \int_D f_j(Q) U_{i,j}(P, Q) dv(Q)$$

If the relations (8) are established, the above equation is identical to eq.(12).

*3 The uniqueness theorem of elasticity is stated as follows⁽⁷⁾. In the cases of finite elastic bodies:

(i) For displacement problems or mixed problems, the displacement is determined uniquely. For traction problems, the displacement is unique if we do not take into account the rigid body displacements of translation and rotation.

Similarly, in the cases of three-dimensional infinite elastic bodies:

(ii) If the displacement satisfies such conditions at infinity that $u_i(P) = O(R_0^{-1})$ and $u_{i,j}(P) = O(R_0^{-2})$ as $R_0 \rightarrow \infty$, the displacement is always determined uniquely.

And in the cases of two-dimensional infinite elastic bodies:

(iii) If the displacement satisfies such conditions at infinity that $u_i(P) = O(1)$ and $u_{i,j}(P) = O(R_0^{-2})$ as $R_0 \rightarrow \infty$, the displacement is determined uniquely for displacement and mixed problems, and is determined uniquely excluding the rigid body displacement of translation for traction problems.

$$\int_S \phi_i(Q) ds(Q) + \int_D f_i(Q) dv(Q) = 0 \quad (i=1,2) \quad (16)$$

According to the relations (8), $[t_i^{(n)}(P)]$ is a solution $\phi_i(P)$ of eq.(12) and satisfies eq.(16). This suggests the existence of a homogeneous solution of eq.(12) which does not satisfy eq.(16).

As an example, let us consider a problem on a circular plate of radius r_0 (plane strain) as shown in Fig.1. The homogeneous equation of eq.(12) is written as

$$\int_S \phi_{0i}(Q) \left[\frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \ln \frac{l}{R} \delta_{ij} + \frac{1}{8\pi\mu} \frac{1}{1-\nu} R_{,i} R_{,j} \right] ds(Q) = 0 \quad (P \in S) \quad (17)$$

Referring to Fig.1, we have

$$R = 2r_0 \sin \frac{|\theta_P - \theta_Q|}{2}, \quad R_{,1} = -\operatorname{sgn}(\theta_P - \theta_Q) \sin \frac{\theta_P + \theta_Q}{2}, \quad R_{,2} = \operatorname{sgn}(\theta_P - \theta_Q) \cos \frac{\theta_P + \theta_Q}{2}$$

If we represent $\phi_{0i}(P)$ in the form of Fourier's series

$$\phi_{0i}(P) = \frac{1}{2} b_{0i} + \sum_{n=1}^{\infty} [b_{ni} \cos n\theta_P + b_{ni}' \sin n\theta_P] \quad (18)$$

and substitute it into eq.(17), then after integrations we obtain

$$\begin{aligned} & \frac{1}{8\pi\mu} \frac{3-4\nu}{1-\nu} \pi r_0 \left[b_{0i} \ln \frac{l}{r_0} + \sum_{n=1}^{\infty} \frac{1}{n} [b_{ni} \cos n\theta_P + b_{ni}' \sin n\theta_P] \right] \\ & + \frac{1}{8\pi\mu} \frac{1}{1-\nu} \frac{\pi r_0}{2} [b_{0i} + (-1)^i \{b_{1i} \cos \theta_P - b_{1i}' \sin \theta_P\} \\ & - \delta_{1i} \{b_{12} \sin \theta_P + b_{12}' \cos \theta_P\} - \delta_{2i} \{b_{11} \sin \theta_P + b_{11}' \cos \theta_P\}] = 0 \quad (0 \leq \theta_P < 2\pi) \end{aligned}$$

Hence, we deduce the conditions for Fourier's coefficients as:

$$(1+2(3-4\nu) \ln l/r_0) b_{0i} = 0, \quad b_{ni} = b_{ni}' = 0 \quad (n=1,2,3,\dots) \quad (19)$$

Consequently, we conclude that the integral equation (12) has a non-trivial homogeneous solution $\phi_{0i}(P) (= \text{const.})$ if the radius of the circle r_0 and the definite constant l in the expression (1) satisfy the relation

$$1+2(3-4\nu) \ln l/r_0 = 0 \quad (20)$$

Next we examine the effects of the existence of the homogeneous solution on numerical solutions of eq.(12). We discretize eq.(12) and obtain numerical solutions, by the use of the boundary elements of constant densities. That is, we take N nodes P_n ($r_0 \cos(n-1)\pi/N$, $r_0 \sin(n-1)\pi/N$, $n=1,2,\dots,N$) on the circumference, approximate the boundary by N line elements n ($=P_n P_{n+1}$, with $P_{N+1}=P_1$), perform the integrations assuming that the densities of $\varphi_i(P)$ and $\phi_i(P)$ are constant in each element and determine the unknown $\phi_i(P)$ so that eq.(12) is satisfied at the middle points of the elements. In the cases that $\nu=0.3$, the body force is absent and the boundary displacement is one derived from the complex functions⁽⁸⁾ $\varphi(z) = 1/(z-2r_0)$ and $\psi(z) = 0$ ($z=x_1+ix_2$), we take 24 elements and carry out some numerical calculations. Fig.2 shows the relation of the resultant force f in x_1 direction ($= |\int_S \phi_i(Q) ds(Q)|$) versus l/r_0 . Here the units of length and stress are r_0 and μ respectively. The resultant force f takes its maximum 9617 at $l=0.75220$ which slightly deviates from the theoretical value $l=0.75747$ predicted by eq.(20). The solutions $[t_i^{(n)}(P)] (= \phi_i(P))$ obtained are tabulated in the fourth to the seventh columns of Table 1, in the cases of $l=0.74$ and $l=0.75220$. The error is considerably large if l is close to 0.75220.

Fig.1 Circular Plate

Fig.2 Variation of the resultant force f with parameter l

The phenomena similar to the above are known in boundary integral equations applied for the plane potential theory, wherein several techniques to overcome them have been presented^(9,10). In this paper, following the strategy adopted in Sherman-Lauricella's equations⁽¹⁾, we propose a modified integral equation which is equivalent to the pair of eqs.(12) and (16), and hence to the elasticity problem.

Let P_0 be a point in D , R_i be unknown constants expressed by $\phi_i(P)$ as

$$R_i = \int_S \phi_i(Q) ds(Q) + \int_D f_i(Q) dv(Q) \quad \dots\dots\dots (21)$$

and F_i be a non-zero constant with appropriate dimension. If we add two terms in expression (5) and define the function $v_i^*(P)$ in the form

$$v_i^*(P) = v_i(P) - R_j U_i^{(j)}(P, P_0) + F_i R_j \delta_{ij} \quad \dots\dots\dots (22)$$

and consider the following integral equation for $\phi_i(P)$.

$$[v_i^*(P)]^+ = 0 \quad (P \in S) \quad \dots\dots\dots (23)$$

Then the following may be stated:

- (i) If a function $\phi_i(P)$ satisfies eqs.(12) and (16), then it also satisfies eq.(23).
- (ii) The function $v_i^*(P)$ satisfies the displacement equation with no body force in the domain D^c . From eqs.(1)', (2)' and (22), it follows

$$v_i^*(P) = F_i R_i + O(R_0^{-1}), \quad v_{i,j}^*(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \quad \dots\dots\dots (24)$$

and the displacement $v_i^*(P)$ satisfies the conditions at infinity (14)'. Therefore, we can regard $v_i^*(P)$ as a displacement induced in the fictitious elastic body D^c , and interpret eq.(23) as the boundary condition for the displacement $v_i^*(P)$. Hence if eq.(23) is satisfied, the uniqueness theorem of elasticity shows that $v_i^*(P) = 0 (P \in D^c)$. From eq.(24) it follows that $R_i = 0$ and $v_i^*(P) = v_i(P)$. Then we see eqs.(12) and (16) are satisfied.

Thus we obtain the integral equation (23) which is equivalent to the elasticity problem. The numerical solutions of this equation for the preceding cases are shown in the eighth and ninth columns of Table 1. Here we put P_0 at the origin and $F_i = 1$. It is seen that satisfactory results are obtained.

2.2 Traction problems

In the cases that the surface traction is prescribed as the boundary condition, the resultant force and the moment of the surface traction and the body force are equal to zero by assumption. Hence, eq.(16) is satisfied and the integral equation (12) is equivalent to the elasticity problem in two-dimensional cases as well as in three-dimensional cases. By using the uniqueness theorem of elasticity and the relations (8), we conclude that the unknown function $\phi_i(P) = [u_i(P)]^-$ can be determined uniquely if we do not take into account the rigid body

Table 1 Numerical Solutions

Element	Boundary condition		Numerical solution of eq.(12)				Numerical solution of eq.(23)		Exact solution	
			$\lambda=0.74$		$\lambda=0.75220$		$\lambda=0.74, 0.75220$			
	u_1	u_2	$t_1^{(n)}$	$t_2^{(n)}$	$t_1^{(n)}$	$t_2^{(n)}$	$t_1^{(n)}$	$t_2^{(n)}$	$t_1^{(n)}$	$t_2^{(n)}$
1	-0.398	-0.175	-0.301	-1.036	1539	15.05	-0.216	-1.036	-0.093	-0.960
2	-0.375	-0.374	-0.464	-1.718	1538	16.45	-0.380	-1.718	-0.402	-1.602
3	-0.331	-0.376	-0.423	-1.235	1536	18.14	-0.339	-1.235	-0.380	-1.195
4	-0.295	-0.302	-0.292	-0.808	1534	18.59	-0.208	-0.807	-0.234	-0.802
5	-0.277	-0.220	-0.178	-0.579	1532	17.64	-0.094	-0.579	-0.108	-0.580
6	-0.276	-0.149	-0.086	-0.456	1530	15.71	-0.003	-0.456	-0.010	-0.458
7	-0.286	-0.095	-0.009	-0.379	1531	13.40	0.075	-0.379	0.071	-0.380
8	-0.302	-0.055	0.059	-0.315	1532	11.38	0.143	-0.316	0.140	-0.316
9	-0.320	-0.029	0.117	-0.253	1534	10.23	0.201	-0.254	0.198	-0.254
10	-0.336	-0.013	0.163	-0.187	1536	10.27	0.247	-0.187	0.245	-0.187
11	-0.348	-0.005	0.195	-0.115	1538	11.52	0.279	-0.115	0.277	-0.115
12	-0.355	-0.001	0.211	-0.039	1540	13.66	0.296	-0.039	0.294	-0.039

displacements of translation and rotation. This fact that the rigid body displacement u_{Ri} is a homogeneous solution of eq.(12) is evident from Somigliana's identities (9) applied for u_{Ri} , which is

$$-\int_S u_{Ri}(Q) T_i^{(j)}(P, Q) ds(Q) = \begin{cases} u_{Ri}(P) & (P \in D) \\ u_{Ri}(P)/2 & (P \in S) \\ 0 & (P \in D^c) \end{cases} \quad (25)$$

In order to determine the unknown function definitely, we must constrain the rigid body motion. For this purpose, we can fix some boundary points⁽³⁾. Yet we propose an alternative method in which the integral equation is modified to be equivalent to the elasticity problem with constraint conditions, and hence it has a unique solution.

First, we deal with three-dimensional cases. Let P_0 be a point in D , and v_{0i} and ω_{0ij} be unknown constants derived from eq.(5) as

$$v_{0i} = v_i(P_0), \quad \omega_{0ij} = \omega_{ij}(P_0) = [v_{j,i}(P_0) - v_{i,j}(P_0)]/2 \quad (26)$$

and assume F_1 and F_2 to be non-zero constants with appropriate dimensions. We introduce the function $v_i^*(P)$ derived from $v_i(P)$ of eq.(5) by adding two terms as

$$v_i^*(P) = v_i(P) + F_1 v_{0j} U_i^{(j)}(P, P_0) + F_2 \omega_{0jk} U_i^{(k),j}(P, P_0) \quad (27)$$

and consider the following integral equation

$$[v_i^*(P)]^+ = 0 \quad (P \in S) \quad (28)$$

Then the following equivalency is established:

(i) Equation (28) is satisfied if we substitute into $\varphi_i(P)$ the surface displacement $[u_i(P)]^-$ obtained from the elasticity problem with the constraint conditions for the displacement $u_i(P_0) = 0$ and the rotation $\omega_{ij}(P_0) = 0$.

(ii) The function $v_i^*(P)$ satisfies the displacement equation with no body force in the domain D^c , and the conditions at infinity. Hence if eq.(28) is satisfied, we have $v_i^*(P) = 0$ ($P \in D^c$). In this case, if we notice that the resultant force and moment of external forces are zero and the terms which have the resultant force or moment are the second and the third ones in the right hand side of eq.(27), we have $v_{0i} = 0$ and $\omega_{0ij} = 0$, and hence $v_i^*(P) = v_i(P)$. Thus the displacement $v_i^*(P)$ ($P \in D$) derived from the solution of eq.(28) is the solution of the elasticity problem with the constraint conditions.

Similarly, we set $v_i^*(P)$ in two-dimensional cases

$$v_i^*(P) = v_i(P) + F_1 v_{0i} + F_2 \omega_{0ij} U_i^{(j)}(P, P_0) \quad (29)$$

and consider the integral equation

$$[v_i^*(P)]^+ = 0 \quad (P \in S) \quad (30)$$

Then we observe the equivalency:

(i) Equation (30) is satisfied if we substitute into $\varphi_i(P)$ the surface displacement $[u_i(P)]^-$ obtained from the elasticity problem with the constraint conditions $u_i(P_0) = 0$ and $\omega_{ij}(P_0) = 0$.

(ii) The function $v_i^*(P)$ defined by eq.(29) satisfies the displacement equation with no body force in the domain D^c , and the conditions at infinity, since the resultant force of the external forces is zero by assumption. Hence if eq.(30) is satisfied, we have $v_i^*(P) = 0$ ($P \in D^c$). It follows that $v_{0i} = 0$, for the displacement at infinity is $F_1 v_{0i}$. And it follows also that $\omega_{0ij} = 0$, for the term which has the resultant moment is the third one only in the right hand side of eq.(29). Then we have $v_i^*(P) = v_i(P)$. Thus the displacement $v_i^*(P)$ ($P \in D$) derived from the solution of eq.(30) is the solution of the elasticity problem with the constraint conditions.

2.3 Mixed problems

In mixed problems, displacements are prescribed on some portion of S , and tractions on the rest. For the same reason as stated for displacement problems, we may use eq.(12) in three-dimensional cases and eq.(23) in two-dimensional cases.

3. Infinite elastic bodies

Now we treat problems on an infinite isotropic elastic body which occupies the whole space excluding the finite, simply connected domain. The same symbols are used as in the preceding sections, but it must be noticed that D denotes the infinite domain, D^c denotes the finite domain exterior to D and the normal vector $\mathbf{n}$ points to D^c .

The solution $u_i(P)$ of the elasticity problem is that which satisfies the displacement equation in D , the boundary conditions on S and the conditions at infinity.

In order to derive Somigliana's identities for infinite elastic bodies, we apply the identities for finite bodies (9) to the finite domain bounded by the surface S and the spherical surface S_0 of radius R_0 , and put $R_0 \rightarrow \infty$ assuming that the displacement satisfies the conditions at infinity. Then we have

$$\int_S [t_j^{(n)}(Q)]^- U_i^{(j)}(P, Q) ds(Q) - \int_S [u_j(Q)]^- T_i^{(j)}(P, Q) ds(Q) + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) + u_{\infty i} = \begin{cases} u_i(P) & (P \in D) \\ u_i(P)/2 & (P \in S) \\ 0 & (P \in D^c) \end{cases} \quad (31)$$

Here $u_{\infty i}$ is the term which represents the displacement at infinity, i.e., zero in three-dimensional cases and a constant in two-dimensional cases.

If we set the function $v_i(P)$ as

$$v_i(P) = \int_S \phi_j(Q) U_i^{(j)}(P, Q) ds(Q) - \int_S \varphi_j(Q) T_i^{(j)}(P, Q) ds(Q) + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) + v_{\infty i} \quad (32)$$

where $v_{\infty i}$ denotes zero (in three-dimensional cases) or a constant (in two-dimensional cases), then it is easily seen that the pair of the functional equation of Kupradze

$$v_i(P) = 0 \quad (P \in D^c) \quad (33)$$

and the conditions at infinity of the forms

$$v_i(P) = O(R_0^{-1}), \quad v_{i,j}(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \quad (34)$$

in three-dimensional cases and

$$v_i(P) = O(1), \quad v_{i,j}(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \quad (34)^{*4}$$

in two-dimensional cases, is equivalent to the elasticity problem, and the following relations are established.

$$\left. \begin{aligned} u_i(P) &= v_i(P) & (P \in D) \\ [u_i(P)]^- &= \varphi_i(P), & [t_i^{(m)}(P)]^- = \phi_i(P) & (P \in S) \end{aligned} \right\} \quad (35)$$

Since D^c is a finite domain, we conclude from the uniqueness theorem of elasticity applied for the domain D^c that the functional equation of Kupradze (33) is equivalent to the integral equation

$$[v_i(P)]^+ = 0 \quad (P \in S) \quad (36)$$

Next, we examine the integral equation for respective boundary problems according to the natures of the given boundary conditions.

3.1 Displacement problems

In three-dimensional cases, the conditions at infinity (34) are satisfied by arbitrary $\varphi_i(P)$ and $\phi_i(P)$. Hence the integral equation (36) is equivalent to the elasticity problem. Since the solution $u_i(P)$ of the elasticity problem is unique, the solution $\phi_i(P) = [t_i^{(m)}(P)]^-$ of the integral equation is also determined uniquely.

In two-dimensional displacement boundary value problems, it is well-known that the value of the resultant force of the external force is necessary to determine the displacement⁽¹⁾. Since the displacement diverges logarithmically at infinity if the resultant force is not zero, we consider the cases that the following condition is satisfied.

$$\int_S \phi_i(Q) ds(Q) + \int_D f_i(Q) dv(Q) = 0 \quad (37)$$

We have from (1)', (2)', (32) and (37), the estimation

$$v_i(P) = v_{\infty i} + O(R_0^{-1}), \quad v_{i,j}(P) = O(R_0^{-2}) \quad (R_0 \rightarrow \infty) \quad (38)$$

and see that the conditions at infinity (34)' are satisfied. Hence the simultaneous equations (36) and (37) for $\phi_i(P)$ and $v_{\infty i}$ are equivalent to the elasticity problem. Since the solution of the latter is determined uniquely, so are the solutions $\phi_i(P)$ and $v_{\infty i}$ of eqs. (36) and (37).

*4 In some literatures, the conditions at infinity are assumed in the forms of $u_i(P) = o(1)$ as $R_0 \rightarrow \infty$. Solutions satisfying those conditions, however, do not exist for displacement problems.

3.2 Traction problems

In three-dimensional cases, the conditions at infinity are satisfied by arbitrary $\varphi_i(P)$ and $\phi_i(P)$, and the integral equation (36) is equivalent to the elasticity problem. Since the solution of the elasticity problem is determined uniquely, the solution $\varphi_i(P)=[u_i(P)]^-$ of the integral equation (36) is also unique.

In two-dimensional cases, it is assumed that the resultant force of the external forces vanishes (if not, the displacement at infinity diverges), and the conditions at infinity (34)' are satisfied. Hence the integral equation (36) is equivalent to the elasticity problem. The solution $u_i(P)$ of the elasticity problem is determined uniquely if we do not take into account the rigid body displacements of translation. Consequently the solution $\varphi_i(P)=[u_i(P)]^-$ of the integral equation (36) is determined uniquely if we substitute arbitrary value (e.g., zero) into v_{out} of eq.(32).

Contrary to the problems on finite elastic bodies, the rigid body displacement u_{Ri} is not a homogeneous solution of the integral equation (36) for infinite elastic bodies. This fact is obvious from the identities

$$\int_S u_{Rj}(Q) T_i^{(j)}(P, Q) ds(Q) = \begin{cases} u_{Ri}(P) & (P \in D^c) \\ u_{Ri}(P)/2 & (P \in S) \\ 0 & (P \in D) \end{cases} \quad \dots\dots\dots (39)$$

which we derive from eq.(25) noting the direction of the outward normal n .

3.3 Mixed problems

As for the displacement problems, the mixed problems may be treated by eq.(36) in three-dimensional cases, and by eqs.(36) and (37) in two-dimensional cases.

4. Conclusions

We examine the integral equation meth-

ods for the static, isotropic elasticity problems. Then we find that if we use the following integral equations according to the prescribed boundary conditions, we can obtain the unique solutions, which give the solutions of the elasticity problems.

In the cases of finite elastic bodies: for displacement problems, use eq.(12) with eq.(5) [eq.(23) with eq.(22)], for traction problems, use eq.(28) with eq.(27) [eq.(30) with eq.(29)], for mixed problems, use eq.(12) with eq.(5) [eq.(23) with eq.(22)].

In the cases of the infinite elastic bodies:

for displacement problems, use eq.(36) with eq.(32) [eqs.(36) and (37) with eq.(32)], for traction problems, use eq.(36) with eq.(32) [eq. (36) with eq.(32) ($v_{out}=0$)], for mixed problems, use eq.(36) with eq.(32) [eqs.(36) and (37) with eq.(32)]. Here we indicate those for two-dimensional problems in bracket.

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