Out-of-plane Vibrations of Curved Bars with Varying Cross-section*

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In this paper, the out-of-plane free vibrations of a curved bar of which the cross-section varies are analyzed by using the classical theory. The equations of vibration and the boundary conditions are determined from the stationary conditions of the Lagrangian of the bar in a period of vibration. The equations of vibration are solved exactly in a series solution. As numerical examples, the natural frequencies and the mode shapes of symmetric elliptic arc bars with both clamped ends and with both simply supported ends are obtained. And the variations of the natural frequencies and the mode shapes for various cross-sections are investigated.

Key Words; Vibration, Out-Plane Vibration, Curved Bar, Natural Frequency, Mode Shape.

1. Introduction

Plane curved bars with varying cross-section have been much used in the piping systems of chemical plants, atomic plants and as members of structures. Therefore, to clarify the vibrations of the curved bars is important for an earthquake-proof design of piping system or structures. Some papers have been published on the vibration problems of circular arc bars with varying cross-section. Hashimoto and others⁽¹⁾ have analyzed the in-plane vibrations of circular arcs with uniformly changing cross section and Irie and others^{(2),(3)} have done the in-plane vibration of a free-clamped slender arc with varying cross-section and the out-of-plane vibration of an arc bar of variable cross-section. But it seems that much research has not been done on the in-plane and out-of-plane vibrations of general plane curved bars of which the cross-sections vary. The authors^{(4),(11)} have reported researches on the vibrations in a plane comprising the center line and those perpendicular to the plane of a uniform plane curved bar with varying curvature. And also, they⁽¹²⁾ have analyzed recently the in-plane vibrations of a plane curved bar with varying cross-section. Meanwhile, Irie and others⁽¹³⁾ have reported a research on the out-of-plane vibrations of plane curved bars with varying cross-section. They have obtained the steady state out-of-plane response of a Timoshenko's curved beam with internal damping (free-clamped) by use of the transfer matrix approach. However, the free vibrations have not been solved exactly yet and the characteristics of vibration have not been

clarified sufficiently yet.

From this point of view, we analyze, in this paper, the out-of-plane free vibrations of a plane curved bar with an arbitrary varying cross-section by the classical theory. The equations of vibration and the boundary conditions are determined from the stationary conditions of the Lagrangian of the curved bar with a varying cross-section in a period of vibration. The equations of vibration are solved exactly in a series solution. As numerical examples, the natural frequencies and the mode shapes of symmetric elliptic arc bars with both clamped ends and with both simply supported ends are obtained. And the vibrations of the natural frequencies and the mode shapes for the various cross-sections are investigated.

2. Theory

2.1 Lagrangian of vibration in a period and equations of vibration

Let us consider a vibration perpendicular to the plane comprising the center line of a curved bar, of which the center line is a plane curve and the cross-section varies. Let l denote the representative length; s , the length along center line; $1/\rho$, the curvature at any point of center line; p , the circular frequency, the time; $v \sin pt$, $\phi \sin pt$ the bending and the torsional displacements of bar, respectively. Let also γ/g , E , S , I , I_p and C denote the density, the modulus of longitudinal elasticity, the cross-sectional area, the moment of inertia of area, the polar moment of inertia of area and the torsional rigidity of bar, respectively. We put as follows.

$$S=S_0H_1(s), \quad I_p=I_{p0}H_2(s), \quad I=I_0H_3(s), \quad C=C_0H_4(s) \dots \dots \dots (1)$$

where S_0 , I_{p0} , I_0 and C_0 are the cross-sectional area, the polar moment of inertia of area, the moment of inertia of area and the torsional rigidity at the origin of s on the center line and $H_1(s) \sim H_4(s)$ are functions of s . And we put the nondimensional frequencies α_0 , β_0 , the slenderness ratio ν and the ratio of bending rigidity to torsional rigidity ν as follows.

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$$\alpha_0^4 = \gamma S_0 p^2 l^4 / EI_0 g, \quad \beta_0^2 = \gamma I_{p0} p^2 l^2 / C_0 g, \quad \bar{v}^2 = S_0 l^2 / I_{p0}, \quad \nu = C_0 / EI_0 \quad (2)$$

We neglect the effects of the shear deformation and the rotatory inertia of the bar. Considering the kinetic energies due to v and ϕ and the bending and the torsional strain energies, we obtain the Lagrangian of the curved bar in a period of vibration as follows⁵;

$$L = \frac{1}{2} \int_{s_1}^{s_2} \left[\frac{\gamma S p^2 v^2}{g} + \frac{\gamma I_p p^2 \phi^2}{g} - EI U^2 - CW^2 \right] ds \quad (3)$$

where s_1 and s_2 are the values of s at both ends and U and W are

$$U = \frac{d^2 v}{ds^2} - \frac{\phi}{\rho}, \quad W = \frac{d\phi}{ds} + \frac{1}{\rho} \frac{dv}{ds} \quad (4)$$

Assuming γ/g and E are constants and rewriting Eq.(3) by using Eqs.(1) and (2), we have

$$L = \frac{EI_0}{2} \int_{s_1}^{s_2} \left[\left(\frac{\alpha_0}{l} \right)^4 H_1 v^2 + \nu \left(\frac{\beta_0}{l} \right)^2 H_2 \phi^2 - H_3 U^2 - \nu H_4 W^2 \right] ds \quad (5)$$

Taking the first variation of L to obtain the stationary condition of the Lagrangian $\delta L=0$, we have

$$\begin{aligned} \delta L / EI_0 = & \int_{s_1}^{s_2} \left[\left(\frac{\alpha_0}{l} \right)^4 H_1 v - \frac{d^2}{ds^2} (H_3 U) + \nu \frac{d}{ds} \left(\frac{H_4 W}{\rho} \right) \right] \delta v ds + \int_{s_1}^{s_2} \left[\nu \left(\frac{\beta_0}{l} \right)^2 H_2 \phi + \frac{H_3 U}{\rho} \right. \\ & \left. + \nu \frac{d}{ds} (H_4 W) \right] \delta \phi ds - \left[\frac{d}{ds} (H_3 U) - \nu \frac{H_4 W}{\rho} \right] \delta v + \nu H_4 W \delta \phi + H_3 U \delta \left(\frac{dv}{ds} \right) \Big|_{s_1}^{s_2} \quad (6) \end{aligned}$$

Euler's equations (equations of vibration) are

$$\left(\frac{\alpha_0}{l} \right)^4 H_1 v - \frac{d^2}{ds^2} (H_3 U) + \nu \frac{d}{ds} \left(\frac{H_4 W}{\rho} \right) = 0, \quad \nu \left(\frac{\beta_0}{l} \right)^2 H_2 \phi + \frac{H_3 U}{\rho} + \nu \frac{d}{ds} (H_4 W) = 0 \quad (7)$$

The third term of the right side of Eq.(6) denotes the boundary condition, in which the quantities $EI_0 [d(H_3 U)/ds - \nu H_4 W/\rho]$, $\nu EI_0 H_4 W$ and $EI_0 H_3 U$ denote the shearing force, the twisting moment and the bending moment, respectively.

Let us consider the following transformation of the variable since it is very difficult to solve Eq.(7) leaving s as a variable.

$$\frac{dx}{ds} = \frac{1}{\rho} = G \Psi'(x) \quad (8)$$

Where G is a constant of which the dimension is a reciprocal of length and x is a variable that denotes an angle between the tangent at the origin of s and the one at any point on the center line. And $\Psi(x)$ is a function of x , which can be obtained in a simple expression for all the curved bars whose curvatures are obtained by some expressions⁶⁾. As an example, let us consider the curvature is expressed by an ellipse. Then the equation is expressed by the rectangular coordinates (ξ_1, ξ_2) given by $\xi_1 = a \cos \theta$, $\xi_2 = b \sin \theta$, in which $2a$, $2b$ and θ are the major and the minor axes and a parameter, respectively. Putting the representative length l and the flatness of bar μ as

$$l = \sqrt{(a^2 + b^2)/2}, \quad \mu = (a^2 - b^2)/(a^2 + b^2) \quad (9)$$

we obtain from Eq.(8)

$$\tan x = \sqrt{\frac{1+\mu}{1-\mu}} \tan \theta, \quad \frac{1}{Gl} = 1 - \mu^2, \quad \Psi(x) = (1 + \mu \cos 2x)^{3/2} \quad (10)$$

Rewriting Eq.(5) by using Eq.(8), we have

$$L = \frac{EI_0 G^3}{2} \int_{x_1}^{x_2} \left[\left(\frac{\alpha_0}{Gl} \right)^4 H_1 v^2 + \nu \left(\frac{\beta_0}{Gl} \right)^2 H_2 \Psi'^2 \Phi^2 - H_3 X^2 - \nu H_4 Y^2 \right] \frac{dx}{\Psi} \quad (11)$$

where x_1 and x_2 are the values of x corresponding to s_1 and s_2 and

$$\phi = G \Psi \Phi, \quad X = \Psi \frac{d}{dx} \left(\Psi \frac{dv}{dx} \right) - \Psi^2 \Phi, \quad Y = \Psi \frac{d}{dx} (\Psi \Phi) + \Psi^2 \frac{dv}{dx} \quad (12)$$

The first variation of L , i.e., δL is

$$\begin{aligned} \delta L / EI_0 G^3 = & \int_{x_1}^{x_2} \left[\left(\frac{\alpha_0}{Gl} \right)^4 H_1 v - \Psi \frac{d}{dx} \left\{ \Psi \frac{d(H_3 X)}{dx} \right\} + \nu \Psi^2 \frac{d}{dx} (\Psi H_4 Y) \right] \frac{\delta v}{\Psi} dx + \int_{x_1}^{x_2} \left[\nu \left(\frac{\beta_0}{Gl} \right)^2 H_2 \Phi \right. \\ & \left. + H_3 X + \nu \frac{d(H_4 Y)}{dx} \right] \Psi \delta \Phi dx - \left[\Psi \left[H_3 X \delta \left(\frac{dv}{dx} \right) - \left(\frac{dH_3 X}{dx} - \nu H_4 Y \right) \delta v + \nu H_4 Y \delta \Phi \right] \right]_{x_1}^{x_2} \quad (13) \end{aligned}$$

Euler's equations (equations of vibration) are from $\delta L=0$

$$\left(\frac{\alpha_0}{Gl} \right)^4 H_1 v - \Psi \frac{d}{dx} \left\{ \Psi \frac{d(H_3 X)}{dx} \right\} + \nu \Psi^2 \frac{d}{dx} (\Psi H_4 Y) = 0, \quad \nu \left(\frac{\beta_0}{Gl} \right)^2 H_2 \Phi + H_3 X + \nu \frac{d(H_4 Y)}{dx} = 0 \quad (14)$$

2.2 Solutions of equations of vibration

We put Ψ^2 as follows.

$$\Psi^2 = \Psi_0 \sum_{n=0}^{\infty} \zeta_n x^{2n} \quad (15)$$

This is valid for curved bars of which the curvatures are expressed as even functions of x . Especially for the ellipse mentioned above, we have

$$\psi_0 = (1 + \mu)^3, \quad \zeta_0 = 1, \quad \zeta_n = \frac{3\mu^3(-1)^n 2^{2n}}{(1 + \mu)^3 (2n)!} \left[\frac{1 + 3^{2n-1}}{4} + \frac{1}{\mu} + \frac{1}{\mu^2} \right] \dots (16)$$

Now let us consider such a case that $H_1(x)$, $H_2(x)$, $H_3(x)$ and $H_4(x)$ are expressed as follows provided that A_n , B_n , D_n and E_n are known coefficients.

$$H_1(x) = \sum_{n=0}^{\infty} A_n x^{2n}, \quad H_2(x) = \sum_{n=0}^{\infty} B_n x^{2n}, \quad H_3(x) = \sum_{n=0}^{\infty} D_n x^{2n}, \quad H_4(x) = \sum_{n=0}^{\infty} E_n x^{2n} \dots (17)$$

Then, two kinds of solutions are obtained for Eq.(14), that is, one solution with v and ϕ as even functions of x and the other with them as odd functions of x . The former is a solution for symmetric vibration and the latter for antisymmetric vibration. Let us consider the two cases separately.

(i) In the case that v and ϕ are even functions of x , we put as follows;

$$v = \sum_{n=0}^{\infty} P_n x^{2n}, \quad \phi = \sum_{n=0}^{\infty} Q_n x^{2n} \dots (18)$$

where P_n and Q_n are undetermined coefficients which are obtained in turn as follows. Substituting Eq.(18) into Eq.(14), we have

$$\left. \begin{aligned} \sum_{n=0}^{\infty} \psi_0^2 [-(2n+1)(2n+2)(2n+3)(2n+4)D_0 \zeta_0^2 P_{n+2} + f_{na}] x^{2n} &= 0 \\ \sum_{n=0}^{\infty} \psi_0 [(2n+1)(2n+2)v E_0 \zeta_0 Q_{n+1} + f_{nb}] x^{2n} &= 0 \end{aligned} \right\} \dots (19)$$

where

$$\left. \begin{aligned} f_{na} &= K_1 \sum_{p=0}^n A_{n-p} P_p - \sum_{r=1}^n 2r(n+r) \zeta_{n+1-r} \left[\sum_{q=0}^r D_{r-q} \left\{ \sum_{p=1}^{q+1} 2p(q+p) \zeta_{q-p+1} P_p \right. \right. \\ &\quad \left. \left. - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} \right] - (2n+1)(2n+2) \zeta_0 \left[\sum_{q=0}^n D_{n+1-q} \left\{ \sum_{p=1}^{q+1} 2p(q+p) \zeta_{q-p+1} P_p \right. \right. \\ &\quad \left. \left. - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} + D_0 \left\{ \sum_{p=1}^{n+1} 2p(n+1+p) \zeta_{n+2-p} P_p - \sum_{p=0}^{n+1} \zeta_{n+1-p} Q_p \right\} \right] \\ &\quad + \nu \sum_{r=0}^n (n+r+1) \zeta_{n-r} \left[\sum_{q=0}^r E_{r-q} \sum_{p=0}^{q+1} [2pP_p + (q+p+1)Q_p] \zeta_{q+1-p} \right] \\ f_{nb} &= K_2 \sum_{p=0}^n B_{n-p} Q_p + \sum_{q=0}^n D_{n-q} \left\{ \sum_{p=1}^{q+1} 2p(q+p) \zeta_{q-p+1} P_p - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} \\ &\quad + (2n+1) \nu \left[\sum_{q=0}^{n-1} E_{n-q} \left\{ \sum_{p=0}^{q+1} [2pP_p + (q+p+1)Q_p] \zeta_{q+1-p} \right\} \right. \\ &\quad \left. + E_0 \left\{ (2n+2) \zeta_0 P_{n+1} + \sum_{p=0}^n \zeta_{n+1-p} [2pP_p + (n+p+1)Q_p] \right\} \right] \\ K_1 &= \left(\frac{\alpha_0}{Gl} \right)^4 / \psi_0^2, \quad K_2 = \nu \left(\frac{\beta_0}{Gl} \right) / \psi_0 \end{aligned} \right\} \dots (20)$$

From Eq.(19), the coefficients Q_{n+1} and P_{n+2} ($n \geq 0$) are obtained in turn as follows with P_0 , E_1 and Q_0 left undetermined. Provided that $\sum_{i=0}^n (i \geq j)$, we have

$$Q_{n+1} = -f_{nb} / [(2n+1)(2n+2)v E_0 \zeta_0], \quad P_{n+2} = f_{na} / [(2n+1)(2n+2)(2n+3)(2n+4)D_0 \zeta_0^2] \dots (21)$$

Thus we obtain three pairs of independent solutions.

(ii) In the case that v and ϕ are odd functions of x , we put as follows.

$$v = \sum_{n=0}^{\infty} P_n x^{2n+1}, \quad \phi = \sum_{n=0}^{\infty} Q_n x^{2n+1} \dots (22)$$

where P_n and Q_n are undetermined coefficients. Substituting Eq.(22) into Eq.(14), we have

$$\left. \begin{aligned} \sum_{n=0}^{\infty} \psi_0^2 [-(2n+2)(2n+3)(2n+4)(2n+5)D_0 \zeta_0^2 P_{n+2} + f_{na}] x^{2n+1} &= 0 \\ \sum_{n=0}^{\infty} \psi_0 [(2n+2)(2n+3)v E_0 \zeta_0 Q_{n+1} + f_{nb}] x^{2n+1} &= 0 \end{aligned} \right\} \dots (23)$$

where the definitions of K_1 and K_2 are the same as those of Eq.(20) and

$$\left. \begin{aligned} f_{na} &= K_1 \sum_{p=0}^n A_{n-p} P_p - \sum_{r=1}^n (2r+1)(n+r+1) \zeta_{n+1-r} \left[\sum_{q=0}^r D_{r-q} \left\{ \sum_{p=0}^{n+1} (2p+1)(p+q+1) \zeta_{q+1-p} P_p \right. \right. \\ &\quad \left. \left. - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} \right] - (2n+2)(2n+3) \zeta_0 \left[\sum_{q=0}^n D_{n+1-q} \left\{ \sum_{p=0}^{q+1} (2p+1)(p+q+1) \zeta_{q+1-p} P_p - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} \right. \\ &\quad \left. + D_0 \sum_{p=0}^{n+1} \{ (2p+1)(p+n+2) \zeta_{n+2-p} P_p - \zeta_{n+1-p} Q_p \} \right] + (n+1) \nu E_0 \zeta_0 (P_0 + Q_0) \zeta_{n+1} \\ &\quad + \nu \sum_{r=0}^n (n+r+2) \zeta_{n-r} \left[\zeta_0 (P_0 + Q_0) E_{r+1} + \sum_{q=0}^r E_{r-q} \left\{ \sum_{p=0}^{q+1} \zeta_{q+1-p} [(q+p+2)Q_p + (2p+1)P_p] \right\} \right] \end{aligned} \right\}$$

$$\begin{aligned}
f_{n0} = & K_2 \sum_{p=0}^n B_{n-p} Q_p + \sum_{q=0}^n D_{n-q} \left\{ \sum_{p=0}^{q+1} (2p+1)(q+p+1) \zeta_{q+1-p} P_p - \sum_{p=0}^q \zeta_{q-p} Q_p \right\} \\
& + \nu(2n+2) \left\{ \zeta_0 (P_0 + Q_0) E_{n+1} + \sum_{q=0}^{n-1} E_{n-q} \left[\sum_{p=0}^{q+1} \zeta_{q+1-p} ((q+p+2) Q_p \right. \right. \\
& \left. \left. + (2p+1) P_p) \right] + E_0 \left[\sum_{p=0}^n \zeta_{n+1-p} ((n+p+2) Q_p + (2p+1) P_p) + (2n+3) \zeta_0 P_{n+1} \right] \right\} \\
& \dots \dots \dots (24)
\end{aligned}$$

From Eq.(23), Q_{n+1} and P_{n+2} ($n \geq 0$) are obtained in turn as follows with P_0 , P_1 and Q_0 left undetermined.

$$\left. \begin{aligned}
Q_{n+1} &= -f_{n0} / [(2n+2)(2n+3)\nu E_0 \zeta_0] \\
P_{n+2} &= f_{n0} / [(2n+2)(2n+3)(2n+4)(2n+5)D_0 \zeta_0^2]
\end{aligned} \right\} \dots (25)$$

Hence, we have three pairs of independent solutions. Thus, we obtain six pairs of independent solutions. The general solutions of Eq.(14) are expressed by combining linearly these six pairs of independent solutions. However, when the boundary conditions of both ends are the same, the vibrations are divided into a symmetric vibration and an antisymmetric one about $x=0$. The expression obtained by combining linearly the three pairs of independent solutions from Eq.(18) gives the general solutions for symmetric vibration and one from Eq.(22) for antisymmetric vibration, respectively.

3. Numerical calculations

We have made numerical calculations for symmetric elliptic arc bars with clamped ends and also with simply supported ends in the case where $\mu=0.2$, $\bar{\nu}=25$, 100 and obtained the natural frequencies and the corresponding mode shapes. Assuming that the elliptic arc bar has a circular cross-section where the diameter $\bar{d}$ varies with $\bar{d}=\bar{d}_0(1+kx^2)$, we have taken $H_1(x) \sim H_4(x)$ in Eq.(17) as

$$\left. \begin{aligned}
H_1(x) &= (1+kx^2)^2 \\
H_2(x) &= H_3(x) = H_4(x) = (1+kx^2)^4
\end{aligned} \right\} \dots (26)$$

where k is an arbitrary constant. In the numerical calculations, we have taken Poisson's ratio 0.3 and $k=0, \pm 0.2$. By the way, the authors, in Reference (5), had already obtained the frequencies in the case of $k=0$ (uniform cross-section) for various values of $\bar{\nu}$ and μ . We compared some results of this paper with those of Reference (5) for

Fig.1 Elliptic arc bar with varying diameter ($\bar{\nu}=25$, $\mu=0.2$, $k=-0.2$)

frequencies in the case where $\bar{\nu}=25$, $\mu=0.2$ and we confirmed that both results agree precisely. For reference, we show the shape and the variation of diameter $\bar{d}$ of the elliptic arc bar ($\xi_1 = a \cos \theta$, $\xi_2 = b \sin \theta$) in the case that $\bar{\nu}=25$, $\mu=0.2$, $k=-0.2$, $\bar{d}_0=10 \text{ mm}$ in Fig.1.

The boundary conditions at both ends ($x=\pm x_0$) are $\nu=d\nu/dx=\Phi=0$ when both ends are clamped and $\nu=\Phi=X=0$ when both ends are simply supported. We have calculated ν and Φ of Eqs.(18) and (22) by retaining 70 terms of the coefficients P_n and Q_n , respectively. The range of convergence of the solutions is different according to α_0 , μ , $\bar{\nu}$ and so on. Within a range of our calculation, the accuracies of ν , Φ , $d\nu/dx$ and X are up to 7~8 digits where $x \leq 1.6$. But the convergence of the solutions is poor where x is over 1.6 and it is impossible to obtain the frequency where $x=1.8$, because the accuracies of the solutions are a few digits.

The general solutions of Eq.(14) for symmetric vibration, as mentioned before, are expressed by combining linearly three pairs of independent solutions (ν_1, Φ_1), (ν_2, Φ_2), (ν_3, Φ_3) obtained from Eq.(18) as follows;

$$\begin{aligned}
\nu &= \lambda_1 \nu_1 + \lambda_2 \nu_2 + \lambda_3 \nu_3, & \Phi &= \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \lambda_3 \Phi_3 \\
& \dots \dots \dots (27)
\end{aligned}$$

where $\lambda_1 \sim \lambda_3$ are arbitrary constants.

Considering the boundary conditions mentioned above, we obtain a frequency equation in the form of a third order determinant. We can obtain similarly a frequency equation for antisymmetric vibration with use of the solutions of Eq.(22). We have determined the three pairs of independent solutions by assigning ($P_0=1, P_1=Q_0=0$), ($P_1=1, Q_0=P_0=0$) and ($Q_0=1, P_0=P_1=0$). And we have carried out the calculations by the following procedure. Namely, we have determined, at first, the coefficients P_n and Q_n for given values of α_0 , μ , ν , $\bar{\nu}$, k and ζ_n and then, we have obtained the relation between α_0 and x_0 by searching for the values of x_0 which make the

Fig.2 Frequency curve (both ends clamped, symmetric vibration, $\bar{\nu}=25$)

frequency equation zero.

Figure 2 shows the relation between α_0 and x_0 of a symmetric elliptic arc bar for symmetric vibration in the case that both ends are clamped and $\mu=0.2$, $\bar{\nu}=25$, and Fig.3 does that for antisymmetric vibration. In these figures, we show the curves for the first through the fourth modes of vibration in the case where $k=0, \pm 0.2$ in the order from below. As seen from the figures, with $k=0$ (uniform cross-section) as a turning point, the frequencies at $k>0$ (S, I_p, I and C become large) become higher than those at $k=0$ and on the contrary, the frequencies at $k<0$ become lower. Where x_0 is small (the bar is short), much difference of frequency due to k is not recognized, but the difference becomes large as x_0 becomes large or as the vibration becomes stronger. And even in the case of $k \neq 0$, a transfer phenomenon can be seen in the frequency curves, to which we refer again in Figs.6 and 7. In Figs.2 and 3, the values of α_0 when $x_0=\pi/2$ denote the frequencies of a symmetric semi-elliptic arc bar. For reference, the values of frequencies when $k=0$ are $\alpha_0=1.317, 3.290, 5.356, 6.028$ for the first through the fourth modes of symmetric vibration and $\alpha_0=2.256, 4.328, 6.376, 7.106$ for those of antisymmetric vibration.

Figures 4 and 5 show the relation between α_0 and x_0 of a symmetric elliptic arc bar with clamped ends in the case where $\bar{\nu}=100$. Figure 4 denotes the frequency curves for symmetric vibration and Fig.5 do the ones for antisymmetric vibration, in which the curves are for the first through the fourth modes of vibration in the order from below. The variations of the frequency curves due to k are almost similar to those of $\bar{\nu}=25$. But the curves show a monotonous change unlike the case of $\bar{\nu}=25$ and a transfer phenomenon can not be seen in the range of our graphical representation. The values of α_0 when $x_0=\pi/2$ in the case where $k=0$ are 1.318 (first mode), 3.296 (second mode), 5.367 (third mode), 7.408 (fourth mode) for symmetric vibration and 2.259 (first mode), 4.337 (second mode), 6.390 (third mode), 8.429 (fourth mode) for antisymmetric vibration.

Figures 6 and 7 show the comparison between the frequency curves of $\bar{\nu}=25$ and those of $\bar{\nu}=100$ in the case that both ends are clamped and $k=0.2$. The frequency curves are given for the first through the fourth modes of vibration. Figure 6 shows the comparison

Fig.3 Frequency curve (both ends clamped, antisymmetric vibration, $\bar{\nu}=25$)

for symmetric vibration and Fig.7 does that for antisymmetric vibration. When we compare the frequency curves for $\bar{\nu}=25$ with those for $\bar{\nu}=100$, such a phenomenon can be seen that both the curves for the first mode of vibration almost coincide (the curves for $\bar{\nu}=100$ lie slightly to the right side of those for $\bar{\nu}=25$) and those for higher mode of vibration not less than the second are fairly apart from each other where x_0 is small and α_0 is large. But the curves for $\bar{\nu}=25$ gradually approach the ones for $\bar{\nu}=100$ from the left side as x_0 becomes large and α_0 becomes small. The authors call the phenomenon 'a transfer phenomenon in the frequency curves'. In Fig.6 and 7, the numbers (1)~(14) on the frequency curves denote the points at which the modes (ν, Φ) are calculated. Just as in the frequency curves, a transfer phenomenon can be seen in the mode shapes, which are shown in Fig.14 and 15.

Figures 8 to 11 show the relation between α_0 and x_0 of a symmetric elliptic arc bar with simply supported ends. They are the frequency curves for the first through the fourth modes of vibration. The tendency of the curves is almost the same as in the case that both ends are clamped. In the figures for symmetric vibration the values of x_0 when $x_0=\pi/2$ denote the frequencies of a perfect elliptic arc bar (an elliptic ring) vibrating symmetrically about the ξ_1 axis and antisymmetrically about the ξ_2 axis, and the values for the first through the fourth modes of vibration are 0, 2.727, 4.831, 6.028 in the case of $k=0, \bar{\nu}=25$ and 0, 2.731, 4.840, 6.839 in the case of $k=0, \bar{\nu}=100$. And in the

Fig.4 Frequency curve (both ends clamped, symmetric vibration, $\bar{\nu}=100$)

Fig.5 Frequency curve (both ends clamped, antisymmetric vibration, $\bar{\nu}=100$)

figures for antisymmetric vibration, the values of α_0 when $x_0 = \pi/2$ denote the frequencies of an elliptic ring vibrating antisymmetrically about the ξ_1 and the ξ_2 axes, and the values for the first through the fourth modes of vibration are 1.622, 3.790, 5.858, 7.106 in the case of $k=0$, $\bar{\nu}=25$ and 1.624, 3.797, 5.871, 7.910 in the case of $k=0$, $\bar{\nu}=100$.

Figures 12 and 13 show the comparison between the frequency curves of $\bar{\nu}=25$ and those of $\bar{\nu}=100$ in the case that both ends are simply supported and $k=0.2$. The tendency is similar to the case that both ends are clamped. Both the curves for the first mode of vibration almost coincide in the range of our graphical representation (Strictly speaking, the curves for $\bar{\nu}=100$ lie slightly to the right side of those for $\bar{\nu}=25$) and a transfer phenomenon can not be seen. But in the higher mode of vibration not less than the second, a transfer phenomenon can also be seen. In Figs. 12 and 13, the numbers (1)~(14) on the frequency curves denote the points at which the modes are calculated and the modes are shown in Figs. 16 and 17.

Figure 14 shows the mode shapes for symmetric vibration of the symmetric elliptic arc bar with clamped ends. They are the mode shapes corresponding to the points on the frequency curves in Fig. 6, in which the mode shapes of (1)~(4) are the ones for $\bar{\nu}=100$ and those of (5)~(14) are for $\bar{\nu}=25$. In the calculations of mode shapes, λ_1 of Eq. (24) is taken to be unity and the scales

Fig. 6 Frequency curve (both ends clamped, symmetric vibration, $k=0.2$)

The numbers (1)~(14) denote the points at which the modes are calculated.

Fig. 7 Frequency curve (both ends clamped, antisymmetric vibration, $k=0.2$)

The numbers (1)~(14) denote the points at which the modes are calculated.

of ν and Φ are magnified or reduced appropriately to represent them clearly. This is the same in Figs. 15 to 17. When we pay attention to the mode shapes for $\bar{\nu}=100$, we can observe that the number of nodes of ν and Φ is zero in the first mode, two in the second mode, four in the third mode, six in the fourth mode, and that the variation is regular. When we compare the mode shapes for $\bar{\nu}=25$ with those for $\bar{\nu}=100$, such a transfer phenomenon can be seen that the mode shapes for $\bar{\nu}=25$ and $\bar{\nu}=100$ are considerably different even in the same mode of vibration where the frequency curves for $\bar{\nu}=25$ and $\bar{\nu}=100$ deviate from each other, but the mode shapes for $\bar{\nu}=25$, varying complicatedly on the way, gradually approach those for $\bar{\nu}=100$ as the frequency curves for $\bar{\nu}=25$ approach the ones for $\bar{\nu}=100$. When we look at the mode shapes for the third mode of vibration in the case of $\bar{\nu}=25$, for example, we can observe that the mode shapes of (9) are very similar to those of (2) (the mode shapes for the second mode of vibration in the case of $\bar{\nu}=100$) and the mode shapes thus vary as complicatedly as (10), (11) with the transfer of the frequency curves. And the mode shapes of (12) where the curves of $\bar{\nu}=25$ and $\bar{\nu}=100$ coincide are almost the same as those of (3) (the mode shapes for the third mode of vibration in the case of $\bar{\nu}=100$).

Figure 15 shows the mode shapes for antisymmetric vibration of the symmetric elliptic arc bar with clamped ends. They are the mode shapes corresponding to the points on the frequency curves in Fig. 7. The mode shapes of (1)~(4) are the ones for $\bar{\nu}=100$ and

Fig. 8 Frequency curve (both ends simply supported, symmetric vibration, $\bar{\nu}=25$)

Fig. 9 Frequency curve (both ends simply supported, antisymmetric vibration, $\bar{\nu}=25$)

those of (5)~(14) are for $\bar{v}=25$. In the mode shapes for $\bar{v}=100$, the number of nodes of v and ϕ varies comparatively regularly such that the number of nodes of v and ϕ is one in the first mode, three in the second mode, five in the third mode, seven in the fourth mode. The mode shapes for $\bar{v}=25$ are considerably different from those for $\bar{v}=100$ even in the same mode of vibration where the frequency curves for $\bar{v}=25$ and $\bar{v}=100$ deviate from each other. However, as in the case of symmetric vibration, the mode shapes for $\bar{v}=25$, varying complicatedly on the way, gradually approach those for $\bar{v}=100$ with the transformation of the frequency curves.

Figures 16 and 17 show the mode shapes of the symmetric elliptic arc bar with simply supported ends. They vary almost similarly to the case that both ends are clamped, except that $dv/dx \neq 0$ at both ends.

4. Conclusions

In this paper, by using the classical theory, we have analyzed the free out-of-plane vibrations of curved bars with varying cross-section which have been much used in the piping systems of various plants and as the members of structures. As numerical examples, we have obtained the natural frequencies and the mode shapes of symmetric elliptic arc bars with both clamped ends and with both simply supported ends, and clarified the variations of the natural frequencies and the mode shapes due to the variations of the cross-section. The results can be summarized as follows:

Fig.10 Frequency curve (both ends simply supported, symmetric vibration, $\bar{v}=100$)

Fig.11 Frequency curve (both ends simply supported, antisymmetric vibration, $\bar{v}=100$)

(1) When the cross-section becomes large (S , I_p , I and C becomes large), the natural frequency becomes high compared with the case of uniform cross-section but on the contrary, the natural frequency becomes low when the cross-section becomes small.

(2) The variation of the natural frequency corresponding to the change of the cross-section becomes large as the curved bar becomes long or as the vibration becomes higher.

(3) Transfer phenomena appear in the frequency curves and the mode shapes even for a curved bar with varying cross-section when the slenderness ratio is as small as $\bar{v}=25$.

The numerical calculations in this paper were carried out by using ACOS Series 77 NEAC System 900 Computer of Tohoku University.

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Fig.13 Frequency curve (both ends simply supported, antisymmetric vibration, $k=0.2$)

The numbers (1)~(14) denote the points at which the modes are calculated.

Fig.12 Frequency curve (both ends simply supported, symmetric vibration, $k=0.2$)

The numbers (1)~(14) denote the points at which the modes are calculated.

Fig.14 Mode (both ends clamped, symmetric vibration, $k=0.2$)

The mode shapes corresponding to the points on the frequency curves in Fig.6 are shown. The modes ①~④ are the ones for $\bar{v}=100$ and ⑤~⑭ are for $\bar{v}=25$.

Fig.15 Mode (both ends clamped, antisymmetric vibration, $k=0.2$)

The mode shapes corresponding to the points on the frequency curves in Fig.7 are shown. The modes ①~④ are the ones for $\bar{v}=100$ and ⑤~⑭ are for $\bar{v}=25$.

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Fig.16 Mode (both ends simply supported, symmetric vibration, $k=0.2$)

The mode shapes corresponding to the points on the frequency curves in Fig.12 are shown. The modes ①~④ are the ones for $\bar{v}=100$ and ⑤~⑭ are for $\bar{v}=25$.

Fig.17 Mode (both ends simply supported, antisymmetric vibration, $k=0.2$)

The mode shapes corresponding to the points on the frequency curves in Fig.13 are shown. The modes ①~④ are the ones for $\bar{v}=100$ and ⑤~⑭ are for $\bar{v}=25$.

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