Bifurcation of an Obliquely Fiber-Reinforced Elastic Slab under Axial Loads*

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In the framework of the theory of continua with kinematical constraint of inextensibility, bifurcation from homogeneous finite deformation of an obliquely fiber-reinforced elastic slab under axial loads, both tensile and compressive, is studied. The matrix is assumed to be Blatz-Ko foam rubber. In spite of the uniform axial stress loading, the initial homogeneous deformation consists not only of axial extension or contraction but also of pure shear deformation owing to the oblique reinforcement. The bifurcation is analyzed on the basis of the incremental deformation theory. For various wavelength-to-thickness ratios, the critical loads are shown in the form of curves as a function of fiber inclination.

Key Words: Elasticity, Bifurcation, Oblique Fiber-Reinforcement, Slab, Rubber, Critical Load

1. Introduction

The mechanics of fiber-reinforced materials was studied extensively within the framework of the theory of continua with internal kinematical constraints of inextensibility along the directions of the fibers. The fibers were treated as being completely flexible and continuously distributed. This idealized theory appears in books written by Tugsdell and Noll⁽¹⁾, Pipkin⁽²⁾, and Spencer⁽³⁾.

To the author's knowledge, there are a few works dealing with the instability or bifurcation of a finitely deformed, ideal fiber-reinforced elastic body. Previously, the author took up the bifurcation problems of an ideal fiber-reinforced slab (in the plane-strain state)⁽⁴⁾ and strip (in the plane-stress state)⁽⁵⁾ under axial loads. Both the slab and strip were reinforced in the transverse direction, and their matrix was specified as two kinds of rubber having particular forms of the strain-energy function proposed by Blatz and Ko (solid and foam rubbers)⁽⁶⁾. In another previous paper⁽⁷⁾, a universal solution was obtained for the similar problem concerning an axially fiber-reinforced slab. This solution includes the stress singularities in the boundary fibers represented by the Dirac delta distribution.

The present paper is concerned with the bifurcation from the initial homogeneous finite deformation of an elastic slab reinforced obliquely by a single family of fibers and subjected to axial loads, both compressive and tensile. Because of this oblique reinforcement, the homogeneous deformation consists not only of axial extension or contraction but also of pure shear

deformation in spite of the uniform axial stress loading. The bifurcation is analyzed on the basis of Biot's mechanics of incremental deformations⁽⁸⁾. The matrix is assumed to be the Blatz-Ko polyurethane foam rubber⁽⁶⁾, and the slab to be in the plane-strain state prior to and after bifurcation.

Most of the basic equations associated with the initial finite deformations and the superposed infinitesimal deformations of the ideal fiber-reinforced material are omitted in the present paper for the sake of brevity. All of these equations are given in the previous paper⁽⁴⁾.

2. Initial Homogeneous Finite Deformation

Let (X, Y, Z) and (x, y, z) denote, respectively, the material and spatial coordinates of a material point in a body when referred to the same Cartesian system. Consider an infinite elastic slab, whose center line coincides with the Y -axis in its natural state and with the y -axis in its deformed state, as shown in Figs.1(a) and (b). The finite de-

Fig.1 An obliquely fiber-reinforced elastic slab: (a) Natural state, (b) Homogeneously deformed state, (c) Bifurcated state

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formation due to an axial load, T_0 (Cauchy stress), alters the thickness, $2H_0$, of the slab and the direction, Θ , of its fibers into $2h$ and θ , respectively.

The deformation gradient F corresponding to the homogeneous plane deformation shown in Fig.1(b) is given by

$$\tilde{F} = \begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1)$$

where all components are constant. It follows from the definition of the left Cauchy-Green deformation tensor, $\tilde{B} = \tilde{F}\tilde{F}^T$, that

$$\tilde{B} = \begin{bmatrix} F_{11}^2 & F_{11}F_{21} & 0 \\ F_{11}F_{21} & F_{22}^2 + F_{21}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

whose inverse tensor is

$$\tilde{B}^{-1} = III^{-1} \begin{bmatrix} F_{22}^2 + F_{21}^2 & -F_{11}F_{22} & 0 \\ -F_{11}F_{21} & F_{11}^2 & 0 \\ 0 & 0 & III \end{bmatrix}, \quad (3)$$

with

$$III = (F_{11}F_{22})^2. \quad (4)$$

If H denotes a unit vector in the fiber direction in the natural state, that is,

$$H = [\cos\Theta \sin\Theta \ 0]^T, \quad (5)$$

then the deformation carries this unit vector into

$$h = FH = \begin{bmatrix} F_{11}\cos\Theta \\ F_{21}\cos\Theta + F_{22}\sin\Theta \\ 0 \end{bmatrix}. \quad (6)$$

In the previous papers^(4,5), the author considered the reinforcement of Blatz-Ko polyurethane foam⁽⁶⁾ by a family of inextensible, completely flexible fibers along the preferred direction H . The stress-deformation relation for this idealized fiber-reinforced foam rubber is given by

$$T = \mu_0 \{ 1 - (1/\sqrt{III}) \tilde{B}^{-1} \} - Qh\tilde{h}, \quad (7)$$

where T denotes the Cauchy stress tensor, and 1 the unit tensor. The coefficient μ_0 is the shear modulus in the natural state. The last term on the right hand side represents the reaction stress due the kinematical constraint of inextensibility,

$$\text{tr}(h\tilde{h}) = 1, \quad (8)$$

with $\text{tr}(\)$ denoting the trace and $(\)\tilde{(\)}$ the tensor product.

Applying the boundary conditions $T_{11} = T_{12} = 0$ and the loading condition $T_{22} = T_0$ to the stress components obtained from Eq.(7) yields the following equations:

$$1 - (1/\sqrt{III}) (\tilde{B}^{-1})_{11} - (Q/\mu_0) h_1^2 = 0, \quad (9)$$

$$-(1/\sqrt{III}) (\tilde{B}^{-1})_{12} - (Q/\mu_0) h_1 h_2 = 0, \quad (10)$$

$$1 - (1/\sqrt{III}) (\tilde{B}^{-1})_{22} - (Q/\mu_0) h_2^2 = T_0/\mu_0. \quad (11)$$

Furthermore, the constraint condition (8) implies

$$h_1^2 + h_2^2 = 1. \quad (12)$$

For the given values of T_0/μ_0 and Θ , Eqs. (9)-(12) are a complete system of equations with respect to four unknowns, F_{11} , F_{21} , F_{22} , Q/μ_0 . Once the values of these unknowns are determined, the angle, θ , between the x -axis and the fiber direction in the deformed state is easily obtained by

$$\theta = \tan^{-1}(h_2/h_1). \quad (13)$$

3. Bifurcation Analysis

In order to examine whether the slab bifurcates at a certain value of the applied load from the homogeneous finite deformation obtained in the previous section, let us superpose a nonhomogeneous infinitesimal plane deformation on the finite one. The displacement vector associated with this infinitesimal deformation is denoted by $u(u, v, 0)$ where $u = u(x, y)$ and $v = v(x, y)$. The strain tensor is denoted by

$$e = \frac{1}{2} \{ (\partial u / \partial x) + (\partial v / \partial x) T \}, \quad e_k = 0, \quad (14)$$

where k is the unit vector along the z -axis. The second equation comes from the plane-strain assumption. The kinematical constraint of fiber-inextensibility for the superposed infinitesimal deformation is

$$\text{tr}(eh\tilde{h}) = 0, \quad (15)$$

which was obtained in Ref.(4). The incremental stress-incremental strain relation corresponding to the stress-deformation relation (7) is easily obtained from Eqs.(15) and (17) in Ref.(4) as follows:

$$s = (\mu_0 / \sqrt{III}) (e\tilde{B}^{-1} + \tilde{B}^{-1}e + \tilde{B}^{-1}\text{tr } e) - qh\tilde{h} - Q(eh\tilde{h} + h\tilde{e}h), \quad (16)$$

where s is the incremental stress tensor based on the Jaumann stress rate. The second term in the right hand side represents the indeterminate reaction stress corresponding to the condition (15).

Since the nonvanishing component of T is only $T_{22} = T_0$, the equilibrium equations for the incremental stress components are

$$\begin{aligned} \partial s_{11} / \partial x + \partial s_{12} / \partial y + T_0 \partial \omega_{12} / \partial y &= 0, \\ \partial s_{12} / \partial x + \partial s_{22} / \partial y + T_0 \partial \omega_{12} / \partial x &= 0, \end{aligned} \quad (17)$$

where ω_{12} is the local rotation about the z -axis associated with the incremental deformation. The boundary conditions for free tractions are

$$\begin{aligned} \Delta f_1 = s_{11} &= 0 \quad \text{at } x = \pm h, \\ \Delta f_2 = s_{12} - T_0 e_{12} &= 0 \quad \text{at } x = \pm h. \end{aligned} \quad (18)$$

Substituting Eqs.(14) into (16) and then the resulting equations into (17) yields the equilibrium equations in terms of q , u , and v :

$$-h_1^2 \partial q / \partial x - h_1 h_2 \partial q / \partial y$$

$$\begin{aligned}
& +\phi_{11}\partial^2 u/\partial x^2 + \frac{1}{2}(\phi_{33}+T_0)\partial^2 u/\partial y^2 \\
& + \frac{1}{2}(\phi_{13}+2\phi_{31})\partial^2 u/\partial x\partial y \\
& + \frac{1}{2}\phi_{13}\partial^2 v/\partial x^2 + \phi_{32}\partial^2 v/\partial y^2 \\
& + \frac{1}{2}(2\phi_{12}+\phi_{33}-T_0)\partial^2 v/\partial x\partial y=0, \\
& -h_1h_2\partial q/\partial x - h_2^2\partial q/\partial y \\
& + \frac{1}{2}(\phi_{33}-T_0)\partial^2 v/\partial x^2 + \phi_{22}\partial^2 v/\partial y^2 \\
& + \phi_{31}\partial^2 u/\partial x^2 + \frac{1}{2}\phi_{23}\partial^2 u/\partial y^2 \\
& + \frac{1}{2}(2\phi_{32}+\phi_{23})\partial^2 v/\partial x\partial y \\
& + \frac{1}{2}(2\phi_{21}+\phi_{33}+T_0)\partial^2 u/\partial x\partial y=0,
\end{aligned} \quad (19)$$

where ϕ_{11} , ϕ_{12} , etc. are given in the Appendix. Elimination of q from these two equations gives

$$\begin{aligned}
& \psi_{11}\partial^2 u/\partial x^2 + \psi_{12}\partial^2 u/\partial x\partial y + \psi_{13}\partial^2 u/\partial y^2 \\
& + \psi_{21}\partial^2 v/\partial x^2 + \psi_{22}\partial^2 v/\partial x\partial y + \psi_{23}\partial^2 v/\partial y^2=0,
\end{aligned} \quad (21)$$

where ψ_{11} , ψ_{12} , etc. are also given in the Appendix. By using Eqs.(14) in (15), the inextensibility condition is rewritten in terms of u and v as

$$\begin{aligned}
& \partial u/\partial x + (h_2/h_1)\partial u/\partial y \\
& + (h_2/h_1)\{\partial v/\partial x + (h_2/h_1)\partial v/\partial y\}=0,
\end{aligned} \quad (22)$$

Equations (21) and (22) are a complete system of differential equations with respect to u and v . Let us assume solutions for this system in the following form:

$$\begin{aligned}
u &= A \exp\{i\pi(\zeta x+y)/\ell\}, \\
v &= B \exp\{i\pi(\zeta x+y)/\ell\},
\end{aligned} \quad (23)$$

where A and B are constants, and ℓ , the half wavelength in the y -direction. ζ is a constant to be determined so as to satisfy the above system. Substituting these assumed solutions into Eqs.(21) and (22) yields

$$A(\psi_{11}\zeta^2 + \psi_{12}\zeta + \psi_{13}) + B(\psi_{21}\zeta^2 + \psi_{22}\zeta + \psi_{23})=0, \quad (24)$$

$$(\zeta + h_2/h_1)(A + B h_2/h_1)=0. \quad (25)$$

From the first factor in the left hand side of (25), it follows that

$$\zeta = -(h_2/h_1) = \zeta_1. \quad (26)$$

Eliminating A and B from Eqs.(24) and (25), we obtain a quadratic equation in ζ , whose coefficients a , b , c are given in the Appendix. The solutions for this equation are

$$\zeta = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}) = (\zeta_2, \zeta_3). \quad (27)$$

Thus, if the integral constants are denoted by A_m ($m=1,2,3$), the solutions for Eqs.(21) and (22) are given by

$$\begin{aligned}
u &= \sum_{m=1}^3 A_m \exp\{i\pi(\zeta_m x+y)/\ell\}, \\
v &= \sum_{m=1}^3 A_m \kappa_m \exp\{i\pi(\zeta_m x+y)/\ell\},
\end{aligned} \quad (28)$$

where κ_m has been obtained from Eq.(24) as follows:

$$\begin{aligned}
\kappa_m &= -(\psi_{11}\zeta_m^2 + \psi_{12}\zeta_m + \psi_{13}) \\
& \quad / (\psi_{21}\zeta_m^2 + \psi_{22}\zeta_m + \psi_{23}).
\end{aligned} \quad (29)$$

Next, let us turn to Eq.(19), which is rewritten by using the solutions for u and v , (28), as

$$\begin{aligned}
& h_1^2\{\partial q/\partial x + (h_2/h_1)\partial q/\partial y\} \\
& = \sum_{m=1}^3 A_m (i\pi/\ell)^2 [(\phi_{11} + \frac{1}{2}\phi_{13}\kappa_m)\zeta_m^2 \\
& + \{(\phi_{13}+2\phi_{31}) + (2\phi_{12} + \phi_{33}-T_0)\kappa_m\}\zeta_m \\
& + \frac{1}{2}(\phi_{33}+T_0) + 2\phi_{32}\kappa_m] \exp\{i\pi(\zeta_m x+y)/\ell\}.
\end{aligned} \quad (30)$$

The general solution for this nonhomogeneous equation is easily obtained, if A_0 is an additional integral constant, as

$$\begin{aligned}
q &= (i\pi/\ell) [A_0 + A_1 v_1 (i\pi/\ell)x] \exp\{i\pi(\zeta_1 x+y)/\ell\} \\
& + \sum_{m=1}^3 A_m v_m \exp\{i\pi(\zeta_m x+y)/\ell\},
\end{aligned} \quad (31)$$

where

$$\begin{aligned}
v_m &= v'_m [(\phi_{11} + \frac{1}{2}\phi_{13}\kappa_m)\zeta_m^2 + \{(\phi_{13}+2\phi_{31}) + (2\phi_{12} + \\
& + \phi_{33}-T_0)\kappa_m\}\zeta_m + \frac{1}{2}(\phi_{33}+T_0) + 2\phi_{32}\kappa_m}],
\end{aligned} \quad (32)$$

with

$$v'_1 = 1/h_1^2, \quad v'_m = 1/h_1^2 (\zeta_m + h_2/h_1) \quad (m=2,3). \quad (33)$$

Finally, let us consider the boundary conditions at $x=\pm h$ for the superposed infinitesimal deformations, (18). Substituting Eqs.(14) into (16) and using the resulting equations in (18), we obtain

$$\begin{aligned}
& \phi_{11}\partial u/\partial x + \phi_{12}\partial v/\partial y + \frac{1}{2}\phi_{13}(\partial u/\partial y + \partial v/\partial x) \\
& - h_1^2 q = 0 \quad \text{at } x=\pm h, \\
& \phi_{31}\partial u/\partial x + \phi_{32}\partial v/\partial y + \frac{1}{2}(\phi_{33}-T_0)(\partial u/\partial y + \partial v/\partial x) \\
& - h_1 h_2 q = 0 \quad \text{at } x=\pm h.
\end{aligned} \quad (34)$$

Referring to these boundary conditions, the solutions given by Eqs.(28) and (31) yield homogeneous linear algebraic equations with respect to the unknowns A_m ($m=0,1,2,3$). For a nontrivial set of solutions of these equations, the determinant of their coefficients, D_{mn} ($m,n=0,1,2,3$), must vanish, resulting in the following equation:

$$|D_{mn}| = 0. \quad (35)$$

This gives the critical load, T_{0c} , for the given Θ and ℓ/h .

4. Numerical Computations and Results

For the given values of T_0/μ_0 and Θ , the system of Eqs.(9)-(12) was solved with respect to F_{11} , F_{22} , F_{21} , Q/μ_0 by the Newton-Raphson method. Then, based on the foregoing bifurcation analysis, the critical loads, T_{0c}/μ_0 , at which the homogeneously deformed slab bifurcates, were determined by solving Eq.(35) by the trial and error method. The results are shown in Figs.2 and 3, where the critical loads are given as functions of Θ for various values of the wavelength-to-thickness ratio ℓ/h .

Fig.2 represents the case of tensile load. The critical load depends upon the wavelength-to-thickness ratio. This presents a contrast to the case in which $\Theta=0$, where the critical load is independent of ℓ/h ⁽⁴⁾. For all wavelength-to-thickness ratios, the critical load increases as the fibers rise from $\Theta=0$. For $\Theta=\pi/2$, the critical load may be unbounded. Furthermore, the critical loads for $\Theta \rightarrow 0$ (indicated by the hollow circles on

Fig.2 Critical loads for the case of tensile load

Fig.3 Critical loads for the case of compressive load

the vertical axis) do not coincide with the critical load for the shear mode for $\Theta=0$ (shown by the solid circle) (1). This may be due to the fact that the deformation prior to bifurcation includes not only extension but also shear deformation from its beginning.

The results for the case of compression are depicted in Fig.3. The critical load depends on the wavelength-to-thickness ratio for all $0 < \Theta < \pi/2$. The critical loads for $\Theta \rightarrow 0$ coincide with those at which the slab with $\Theta=0$ bifurcates with antisymmetric modes (4). From these values, the critical loads for the smaller ℓ/h go up gradually, reach their peaks, and then come down, as Θ varies from 0 to $\pi/2$. For all ℓ/h , the critical loads tend to $T_{0c}/\mu_0 = -1.0$, as $\Theta \rightarrow \pi/2$. This value is the same as that for an axially fiber-reinforced elastic slab, in whose boundary fibers the stress singularities represented by the Dirac delta distribution appear (7). The critical loads for the larger ℓ/h , e.g. $\ell/h=10$, remain nearly constant, and then rapidly go up to $T_{0c}/\mu_0 = -1.0$. It is expected from the tendency of those curves that, for very large ℓ/h , the critical load remains almost zero over the whole range of Θ except near $\Theta = \pi/2$, and abruptly rises to $T_{0c}/\mu_0 = -1.0$ near $\Theta = \pi/2$. From this fact, it may be conjectured that the bifurcation of the axially fiber-reinforced elastic slab under compression is

sensitive to the imperfection in which the fibers incline a little from the axis of the slab. That is, however slight the inclination of the fibers may be, the critical load for very large ℓ/h is far smaller than that of a perfect slab.

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Appendix

The coefficients appearing in Eqs.(19)-(21), and (27) are given as follows:

$$\begin{aligned}\phi_{11} &= 3\mu_0 (B^{-1})_{11} / \sqrt{III} - 2Qh_1^2, \\ \phi_{12} &= \mu_0 (B^{-1})_{11} / \sqrt{III}, \\ \phi_{13} &= 2\mu_0 (B^{-1})_{12} / \sqrt{III} - 2Qh_1h_2, \\ \phi_{21} &= \mu_0 (B^{-1})_{22} / \sqrt{III}, \\ \phi_{22} &= 3\mu_0 (B^{-1})_{22} / \sqrt{III} - 2Qh_2^2, \\ \phi_{23} &= \phi_{13}, \\ \phi_{31} &= 2\mu_0 (B^{-1})_{12} / \sqrt{III} - Qh_1h_2, \\ \phi_{32} &= \phi_{31}, \\ \phi_{33} &= \mu_0 \{ (B^{-1})_{11} + (B^{-1})_{22} \} / \sqrt{III} - Q\end{aligned}$$

$$\begin{aligned}\psi_{11} &= (h_2/h_1) \phi_{11} - \phi_{31}, \\ \psi_{12} &= \frac{1}{2} \{ (h_2/h_1) (\phi_{13} + 2\phi_{31}) - (2\phi_{21} + \phi_{33} + T_0) \}, \\ \psi_{13} &= \frac{1}{2} \{ (h_2/h_1) (\phi_{33} + T_0) - \phi_{23} \}, \\ \psi_{21} &= \frac{1}{2} \{ (h_2/h_1) \phi_{13} - (\phi_{33} - T_0) \}, \\ \psi_{22} &= \frac{1}{2} \{ (h_2/h_1) (2\phi_{12} + \phi_{33} - T_0) - (2\phi_{32} + \phi_{23}) \}, \\ \psi_{23} &= (h_2/h_1) \phi_{32} - \phi_{22}.\end{aligned}$$

$$a = (h_2/h_1) \psi_{11} - \psi_{21},$$

$$b = (h_2/h_1) \psi_{12} - \psi_{22},$$

$$c = (h_2/h_1) \psi_{13} - \psi_{23}.$$