Approximate Analysis of Natural Frequencies of a Circular Cylindrical Tank*

By Hisao KONDO**
Yoshiyuki YAMAMOTO***

This report deals with a method of evaluating coupled natural frequencies of a circular cylindrical tank. Starting from a variational principle, the authors obtain a functional relating to coupled oscillations between a linear elastic body of infinitesimal deformations and a perfect fluid of small wave heights. By imposing as constraints several of stationary conditions of the functional above-mentioned, an alternate functional is derived from which a complementary Rayleigh quotient is obtained. The method approximation for the complementary Rayleigh quotient gives approximate formulas of coupled natural frequencies for a circular cylindrical tank oscillating in an axisymmetric manner. Numerical examples show satisfactory coincidence between exact natural frequencies and approximate ones in the case of sloshing modes, whereas tolerable error of the approximation in the case of lower bulging modes.

Key Word: Hydroelasticiy, Variational Principle, Complementary Rayleigh Quotient, Natural Frequency, Approximate Analysis, Oil Storage Tank

1. Introduction

Coupled hydro-elastic oscillations of oil storage tanks in earthquakes have been a great concern of recent decades. In early studies, only the motions of the liquid were treated analytically(12) (18), and deformations of the container were ignored. Recently, finite element method, boundary integral equation method and so on have been applied to this problem. Khajebaz(3) and Komatsu(4) expressed the behavior of the liquid by means of the boundary integral equation method and that of the cylindrical tank by ring shell elements. Kiefling and Peng(5) used tetrahedron elements for the liquid and triangular plate elements for the tank shell. Sakai et al.(6) used disk elements for the liquid and ring shell elements for the cylindrical tank. When the boundary shape is as simple as that of a circular cylindrical tank, analytical solutions are possibly feasible(7)(8). One of the authors(9)(10) applied the solution method(11) for the outer problem of a cylinder with respect to a free surface wave to the axisymmetric free vibration analysis of a circular cylindrical tank, which is one of the cases where analytical solutions are relatively easy to obtain.

In the present investigation, referring to works published hitherto(12) (18), the authors derive from the variational principle a functional describing coupled oscillations between a linear elastic body of infinitesimal deformations and a liquid of small wave heights, and then introduce a complementary Rayleigh quotient from which approximate formulas for natural frequencies of a circular cylindrical tank oscillating in an axisymmetric manner are obtained.

2. A functional

Let us construct a functional describing coupled oscillations of a solid and a liquid. We regard the solid as a linear elastic one of infinitesimal deformations and the liquid as a perfect one of small wave heights. As shown in Fig. 1, \( V_s \) denotes the region of the solid, \( V_l \) the region of the liquid, \( S_f \) the free surface, \( S_p \) the contact boundary between the liquid and the solid where liquid pressure \( p \) is exerted, \( S_0 \) the boundary of the solid where surface tractions other than \( p \) are specified and \( S_d \) denotes the boundary of the solid where the displacement is specified. The normal is taken outward from the region considered, and therefore the outer normal \( \mathbf{n} \) to the liquid and that \( -\mathbf{n} \) to the solid are reverse to each other on \( S_p \). Taking a cartesian coordinate system \( o-x_1,x_2,x_3 \), we express the governing equation of the linear elastic body as follows:

\[
\varepsilon = \frac{1}{2} (\sigma_{xx} + \sigma_{yy}) \tag{1}
\]

\[
\sigma_{xx} = E \varepsilon_{xx} \tag{2a}
\]

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** Research Engineer, Ishikawajima-Harima Heavy Industries Co., Ltd., \( \overline{\mathbb{T}}135-01, \) 3-1-15, Toyosu, Koto-Ku, Tokyo, Japan.
*** Professor, Faculty of Engineering of University of Tokyo, \( \overline{\mathbb{T}}113, \) 7-3-1, Hongo, Bunkyo-ku, Tokyo, Japan.
equations (1) through (6) conform to usual notations (12), e.g., repeated indices mean to sum the relevant quantities within the prescribed range of the index, that is, from one (1) to three (3). $\epsilon_{ij}$ is a strain component, and $\sigma_{ij}$ a stress component. The index preceded by comma (comma) means a partial derivative, for example, $u_{i,j} = \partial u_i / \partial x_j$, $\sigma_{ij}$ is a stress component, $E_{ijkl}$ an elastic coefficient, $C_{ijkl}$ a flexibility coefficient, $f_i$ the body force per unit volume and $p_0$ is the density of the solid. The symbol $(dot)$ denotes the temporal derivative $\partial / \partial t$. $T_1$ is the surface tractions prescribed on $S_T$, and $u_j$ the displacement prescribed on $S_U$. The Lagrangian $L_0$ of the solid having Eqs. (1), (2a) and (6) as its constraint conditions and Eqs. (3) and (4) as its natural conditions are well known (12) as follows:

$$
\mathcal{L}_n = \int \left[ \frac{1}{2} \epsilon_{ijkl} \dot{\sigma}_{ijkl} - \frac{1}{2} B_{ijkl} \epsilon_{ijkl} + f_i \dot{u}_i \right] dV + \int_{S_T} \tau_{ijkl} \dot{u}_j dS + \int_{S_U} \mathbf{p}_0 \dot{u}_i dS
$$

(7)

The integral on $S_T$ is not contained in Eq. (7), but will be treated in the Lagrangian of the liquid subsequently.

![Fig1 Coupled Liquid-Solid System](image)

Let us consider the Lagrangian of the liquid. The Lagrangian of essentially nonlinear character is read in Bateman's text (13), which involves the vorticity but ignores the free surface conditions. Luke (14) extended it to cover the free surface conditions but ignored the vorticity. Luke's Lagrangian is unclear as for the process of deduction and not well applicable to the linear theory of liquid motions of small wave height. After the variational principle of the linear elastic body, the Lagrangian of the perfect fluid is formulated (15)(16). The velocity vector $\mathbf{v}$ is expressed as the gradient of a velocity potential $\phi$, that is, $\mathbf{v} = \nabla \phi$, which gives a different form of the kinetic energy of the perfect fluid from those in Eq. (7). In order to express the kinetic energy of the perfect fluid by means of $\phi$, we may refer to Toupin-Crandall's principle (17)(18) complementary to Hamilton's principle. The former principle has originally been introduced for the discrete system such as a mass-spring system, and may be extended (19) to treat the linear elastic body of infinitesimal deformations by means of a symmetric tensor $p_{ij}$ which is expressed like an impulse, in case where body force $f_i$ per unit volume is not exerted, as follows:

$$
\sigma_{ij} = p_0 \delta_{ij} \quad \text{in } V, \quad \sigma_{ij} = p_0 \delta_{ij} \quad \text{on } S_T \quad \text{and } \quad \sigma_{ij} = -p_0 \delta_{ij} \quad \text{on } S_U
$$

(8a)

(8b)

By imposing equation of motions (3) as a constraint, the displacement $u_j$ is expressed by $p_{ij}$, namely,

$$
\rho \dot{u}_i = p_{ij}
$$

(9)

Imposing Eqs. (2b) and (4) as further constraints, we have a complementary Lagrangian $L_C$ as follows:

$$
\mathcal{L}_C = \int \left[ \frac{1}{2} \epsilon_{ijkl} \dot{\sigma}_{ijkl} - \frac{1}{2} B_{ijkl} \epsilon_{ijkl} + f_i \dot{u}_i \right] dV + \int_{S_T} \tau_{ijkl} \dot{u}_j dS + \int_{S_U} \mathbf{p}_0 \dot{u}_i dS
$$

(10)

Equations (1) and (6) will be shown to be obtainable from Eq. (10) as its natural conditions. Noting

$$
\dot{\mathbf{p}}_{ij} = \rho \dot{u}_i \delta_{ij} = \dot{u}_i \delta_{ij} = (\dot{u}_i \delta_{ij}) - \frac{1}{2} (\dot{u}_i \delta_{ij}) - \frac{1}{2} (\dot{u}_i \delta_{ij})
$$

(11)

we integrate the first variation $\delta \mathcal{L}_C$ on an interval $t_1$ to $t_2$.

$$
\int_{t_1}^{t_2} \delta \mathcal{L}_C = \int_{t_1}^{t_2} \left[ \frac{1}{2} \epsilon_{ijkl} \delta \dot{\sigma}_{ijkl} - \frac{1}{2} B_{ijkl} \delta \epsilon_{ijkl} + \delta f_i \dot{u}_i \right] dV + \int_{S_T} \delta \tau_{ijkl} \dot{u}_j dS + \int_{S_U} \delta (\mathbf{p}_0 \dot{u}_i) dS
$$

(12)

By imposing the condition that $\delta \mathbf{p}_{ij}$ must be zero at $t = t_1, t_2$, the first term on the right side of Eq. (12) vanishes, and the remaining terms give temporal derivatives of Eqs. (1) and (6). Suppose the solid is in oscillation with natural frequency $\omega$. Putting $u_i = 0$ in Eq. (10), we obtain a relation complementary to Rayleigh quotient, which may be called complementary Rayleigh quotient hereafter, that is,

$$
\omega^2 = \int_{V} \frac{1}{2} \epsilon_{ijkl} \rho_0 \mathbf{p}_{ij} \mathbf{p}_{ij} dV
$$

(13)

where

$$
\rho = \rho_0
$$

(14)

Equation (14) is known to be a linearized...
Bernoulli's pressure equation which corresponds to Eq. (8a). Linearized equation of motions of the liquid without body force is given by
\[ \frac{\partial \rho}{\partial t} = -\nabla p \]  \hspace{1cm} \text{(15)}
where \( \rho \) is the density of the liquid. Substituting Eq. (14) into Eq. (15) and differentiating with respect to \( t \) yield
\[ \nabla \phi \]  \hspace{1cm} \text{(16)}
which is a well-known relation. Noting that the flexibility coefficient like \( C_{ijkl} \) is zero on account of the incompressibility of the liquid considered as a perfect fluid, we express \( L_0 \) as follows:
\[ L_0 = \int \nabla \phi \nabla \phi dV + \int \phi \nabla \phi \nabla \phi dS \]  \hspace{1cm} \text{(17)}
where \( V_L \) denotes the prescribed velocity component and \( S_L \) the boundary of \( V_L \) on which \( \phi = 0 \) is specified. When the boundary \( V_L, \) \( S_L, \) consists of \( \partial V \) and the boundary \( S_p \) on which the liquid pressure is prescribed, the natural conditions of Eq. (17) is given by
\[ \phi = 0 \] on \( V_L \) \hspace{1cm} \text{(18)}
\[ \frac{\partial \phi}{\partial n} = 0 \] on \( S_L \) \hspace{1cm} \text{(19)}
In Eq. (17) the free surface condition is ignored. When \( \partial V \) consists only of \( S_p, \) it can be shown by Green's theorem that \( L_0 \) of Eq. (17) expresses the kinetic energy of the liquid if Eqs. (18) and (19) are imposed as constraints, namely,
\[ L_0 = \int \nabla \phi \nabla \phi dV + \int \phi \nabla \phi \nabla \phi dS = -\int \frac{\partial \phi}{\partial t} \nabla \phi dV + \int \phi \nabla \phi \nabla \phi dS = \int \frac{\partial \phi}{\partial t} \nabla \phi dV \] \hspace{1cm} \text{(20)}
\( L_0 \) of Eq. (17) may be called a complementary kinetic energy of the liquid. We extend the functional \( L_0 \) so as to have Eq. (18) and the following as natural conditions:
\[ \phi = \zeta \] on \( S_L \) \hspace{1cm} \text{(20-a)}
\[ \frac{\partial \phi}{\partial n} = 0 \] on \( S_L \) \hspace{1cm} \text{(20-b)}
\[ \alpha \phi = \zeta \] on \( S_L \) \hspace{1cm} \text{(20-c)}
\[ \alpha \frac{\partial \phi}{\partial n} = 0 \] on \( S_L \) \hspace{1cm} \text{(20-d)}
where \( \zeta \) is the displacement of the free surface and \( g \) is the gravitational acceleration. Adding to Eq. (17) the potential energy due to the gravity and taking \( S_p = S_p + S_p, \) we consider the functional \( L_0, \)
\[ L_0 = -\int \frac{\partial \phi}{\partial t} \nabla \phi dV - \int \frac{\partial \phi}{\partial t} \zeta \nabla \phi dS + \int \frac{\partial \phi}{\partial \phi} \nabla \phi dS + \int \frac{\partial \phi}{\partial \phi} \nabla \phi dS \] \hspace{1cm} \text{(21)}
Adding Eq. of (7) to Eq. of (22) yields a functional \( L \) having \( \phi, \zeta \) and \( \zeta \) as its variational functions, which is our goal, namely,
\[ L = \int \frac{\partial \phi}{\partial t} \nabla \phi dV - \int \frac{\partial \phi}{\partial t} \zeta \nabla \phi dS + \int \alpha \phi \nabla \phi dS + \int \frac{\partial \phi}{\partial \phi} \nabla \phi dS + \int \frac{\partial \phi}{\partial \phi} \nabla \phi dS \] \hspace{1cm} \text{(22)}
To see this, we integrate the first variation \( \delta \phi \) on an interval \( t_1 \leq t \leq t_2, \) under the constraints to the variation \( \delta \phi \) etc. as follows:
\[ \text{at } t = t_1, t_2 \]
\[ \frac{\partial \phi}{\partial t} = 0, \quad \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 0 \] \hspace{1cm} \text{(23)}
Noting that the boundaries \( \partial V, \) \( \delta V \), of \( V_L \) and \( V_L \), respectively, are written as \( \partial V = S_L + S_p + S_p \) and \( \partial V = S_L + S_p + S_p, \) we integrate \( \delta \phi \) by parts with respect to \( t \) yield
\[ \int_0^t dt \int_{V_L} \left( \nabla \phi \nabla \phi - \frac{\partial \phi}{\partial t} \zeta \nabla \phi \right) dV + \int_0^t dt \int_{S_p} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dS \]
\[ = \left[ \int_0^t dt \int_{V_L} \left( \frac{\partial \phi}{\partial t} \zeta \nabla \phi \right) dV + \int_0^t dt \int_{S_p} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dS \right] \]
\[ = \left[ \int_0^t dt \int_{V_L} \left( \frac{\partial \phi}{\partial t} \zeta \nabla \phi \right) dV + \int_0^t dt \int_{S_p} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dS \right] \]
\[ = \left[ \int_0^t dt \int_{V_L} \left( \frac{\partial \phi}{\partial t} \zeta \nabla \phi \right) dV + \int_0^t dt \int_{S_p} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dS \right] \]
\[ = \left[ \int_0^t dt \int_{V_L} \left( \frac{\partial \phi}{\partial t} \zeta \nabla \phi \right) dV + \int_0^t dt \int_{S_p} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) dS \right] \]
As known from the right hand side of Eq. (25), the first term vanishes according to Eq. (24), the second term gives Eq. (3), the third term Eq. (4), the fourth term Eq. (18), the fifth and sixth terms give Eqs. (20.a) and (20.b), respectively, and the seventh and eighth terms give Eqs. (21.a) and (21.b), respectively. Hence \( L \) of Eq. (23) is a functional which we have sought for. In Eq. (23) the first line points to the Lagrangian of an linear elastic solid, while the second line points to the Lagrangian of the liquid.

3. A complementary Rayleigh quotient

We shall introduce a sort of a complementary Rayleigh quotient in order to calculate natural frequency \( \omega \) for coupled oscillations of a solid and a liquid. For this purpose we constrain several of natural conditions of \( L \) of Eq. (23). For a linear elastic solid Eqs. (1) through (6) where we set \( f_1 = 0 \) and \( T_0 = 0 \) and also regard \( p = -\nabla \phi \) as a known external force the displacement \( \phi \) is integrated as follows:
\[ w_i(X) = \int_{S_p} G_i(X, Y; \omega \phi)(Y) dS_Y \] \hspace{1cm} \text{(26)}
\[ G_i(X, Y; \omega \phi)(Y) = G_i(X, Y; \omega \phi) \] \hspace{1cm} \text{(27)}
where \( G_i(X, Y; \omega \phi) \) is Green's function for the linear elastic solid oscillating with circular frequency \( \omega \), which is symmetric with respect to spatial points \( X \) and \( Y \) because Eq. (3) is self-adjoint. The following equation will be referred to:
\[ \int_0^t dt \int_{S_p} \alpha \phi \nabla \phi dS \]
\[ = \int_0^t dt \int_{S_p} \alpha \phi \nabla \phi dS \int \frac{\partial \phi}{\partial \phi} \nabla \phi dV \]
\[ = \int_0^t dt \int_{S_p} \alpha \phi \nabla \phi dS \int \frac{\partial \phi}{\partial \phi} \nabla \phi dV \]
\[ = \int_0^t dt \int_{S_p} \alpha \phi \nabla \phi dS \int \frac{\partial \phi}{\partial \phi} \nabla \phi dV \]
Here Eqs. (24), (26) and (27) have been utilized.
After the above preparation, we shall introduce a functional $J(\phi)$ having $\phi$ only as a variational function. For the liquid of small wave height, Eqs. (18) and (20a) are imposed as constraints. Recalling that $L_0$ of Eq. (17) represents the kinetic energy of the liquid under the constraint of Eqs. (16) and (19), we replace the third, fifth and sixth terms of Eq. (23) for the kinetic energy of the liquid. The above procedure leads to the following functional $J(\phi)$:

$$J(\phi) = \int_{t_1}^{t_2} \left( \frac{1}{2} \rho \dot{\phi}, \dot{\phi} \right) + \int_{t_1}^{t_2} \rho \phi \, dt$$

$$= \int_{t_1}^{t_2} \rho \phi \, dt$$

In Eq. (29) $u_1$, $c_{11}$ and $c_{12}$ are related to $\phi$ by Eqs. (1) and (26). Equation (29) is known to consist of the Lagrangian of the solid and that of the liquid. Integrating the first variation $\delta J(\phi)$ on an interval $t_1 \leq t \leq t_2$ and noting that the right hand side of Eq. (28) is produced by the first term of Eq. (29), similarly to the derivation of Eq. (29), we obtain

$$\delta J(\phi) = \int_{t_1}^{t_2} \left( \int_{t_1}^{t_2} \rho \phi \, dt \right)$$

$$= \int_{t_1}^{t_2} \rho \phi \, dt$$

As is known from Eq. (30), the first term satisfies Eq. (21.b), the second term does Eq. (20.b) with the help of Eq. (20.a) and the third term vanishes according to Eq. (24). In other words, the natural condition of $J(\phi)$ is a kinematical condition between normal component of velocity of the solid and that of the liquid, in addition to free surface condition. In order to derive a complementary Rayleigh quotient from the right hand side Eq. (30), we put variations $\delta \phi$ and $\delta u_1$ as follows:

$$\delta \phi = \epsilon \delta \phi_0$$

$$\delta u_1 = \epsilon \delta u_1$$

where $\epsilon$ is an arbitrary small parameter, $\phi$ and $u_1$ are possible exact solutions and also $\delta \phi$ and $\delta u_1$ are independent of time $t$.

In Eq. (32) the relation (26) is accounted for, which connects $u_1$ and $\phi$. Omitting the first term of the right hand side of Eq. (30), substituting Eqs. (31) and (32) into it and taking $t_2 = t_1 + 2\pi/\omega$ yield

$$0 = \frac{2\pi}{\omega} \left( \int_{t_1}^{t_2} \rho \phi_0 \, dt + \int_{t_1}^{t_2} \rho \phi_0 \, dt - \int_{t_1}^{t_2} \rho \phi \, dt \right)$$

Solving Eq. (33) with respect to $\omega^2$ yields

$$\omega^2 = \frac{\int_{t_1}^{t_2} \rho \phi_0 \, dt + \int_{t_1}^{t_2} \rho \phi_0 \, dt}{\int_{t_1}^{t_2} \rho \phi_0 \, dt - \int_{t_1}^{t_2} \rho \phi \, dt}$$

where Eq. (26) is used for $u_1$. Equation (34) is a complementary Rayleigh quotient sought for, and is known to be similar in form to that of Eq. (13).

4. Applications to a circular cylindrical tank

To apply the complementary Rayleigh quotient to free vibration analysis of a circular cylindrical tank, we utilize the solution method (11) for the external problem of the cylinder with respect to a free surface wave. In axisymmetric case $\phi$ is given by

$$\phi = A_0 \cos \left( \frac{2\pi z}{a} \right) + \sum_{n=1}^{\infty} \left( A_n \cos \left( \frac{n \pi z}{a} \right) + B_n \sin \left( \frac{n \pi z}{a} \right) \right)$$

As shown in Fig. 2, $a$ is the radius of the tank, $d$ the undisturbed liquid depth, $r$ and $z$ are cylindrical coordinates. $J_0$ and $J_0$ are the cylinder functions in addition to $J_1$ and $J_1$ which will appear later. $\phi$ of Eq. (35) satisfies the condition that the liquid is still at the bottom $z = 0$, and also the free surface condition at $z = d$ by Eq. (36). Given $\omega$-value of Eq. (36) determines a set of eigenvalues $\lambda_0$ and $\lambda_n (n = 1, 2, 3, ...)$, and thus in case $n = 0$ $\lambda_0$ is implied instead of $\lambda_0$. The side wall of the tank is regarded as a circular cylindrical shell, which is a special case of a three-dimensional linear elastic solid and is treated as a two-dimensional continuum under Kirchhoff-Love hypothesis. Its fundamental unknowns are three components of the displacement belonging to the middle surface of the shell. $\phi_{0i}$ in Eq. (36) is approximated by radial component of the displacement which is integrated according to the membrane theory of shell as in the following:

$$\phi_{0i} = \frac{p a^2}{E h} \left( 1 - \frac{v^2}{E h} \right)$$

where $p$ is dynamic liquid pressure, $E$ Young's modulus and $h$ is the thickness of the shell. $\phi_{0i}$ of Eq. (38) does not satisfy the boundary condition at $z = 0$ and continuity condition at $z = d$, nevertheless on the whole it is a good approximation for most of oil storage tanks. If an equation of motions of the circular cylindrical shell which will be shown later is solved exactly, $\phi_{0i}$ is able to be corrected, but because of simplicity of the expression it is easier to deal with $\phi_{0i}$ of Eq. (38). Substituting Eq. (38) into (34) and expressing it in the cylindrical coordinate yield.

$$\omega^2 = \frac{\int_{0}^{a} \phi \phi_{0i} \, dr \, dz}{\int_{0}^{a} \phi \phi_{0i} \, dr \, dz}$$

Fig. 2 Profile of a Circular Cylindrical Tank
Natural modes in which the coefficient $A_0$ in Eq. (35) predominates are called sloshing modes. We approximate $\Phi$ in Eq. (40) by that obtained on the assumption of rigid side wall, namely:

$$\dot{\Phi} = A_0 \cos \frac{k_0}{\varepsilon} f_1 \frac{k_0}{a}$$ \hspace{1cm} \text{(41)}$$

where $f_1(k_0) = 0.04, k_0 = 3.6317069, k_0 = 0.0153567$. \hspace{1cm} \text{(42)}$

There exists an enumerable set of $k_0$ of Eq. (42), so does $\Phi$ of Eq. (41). Substituting Eqs. (41) and (42) into Eq. (40) yields

$$\frac{\omega^2}{\rho g} = \frac{1}{a \varepsilon} \left( \tan \frac{h_0}{a} + \frac{h_0}{a} \right) \text{pol} \frac{a}{a}$$ \hspace{1cm} \text{(43)}$$

Equation (43) is an approximate formula of the natural circular frequency in sloshing mode. If the first term of the denominator, which is due to deflection of the tank side wall, is ignored, Eq. (43) gives the natural circular frequency for a tank of rigid side wall. Because the above-mentioned term is positive, the deflection of the tank shell reduces natural frequencies in sloshing modes as compared with those for rigid side wall.

Natural modes in which a coefficient $A_m$ (m = 1) in Eq. (35) predominates are called bulging modes. We approximate $\Phi$ in this case by retaining only a term which involves $A_m$, namely,

$$\dot{\Phi} = A_m \cos \frac{m\pi x}{a} f_1 \frac{m\pi x}{a} \text{pol} \frac{m\pi x}{a} \text{pol} \frac{a}{a}$$ \hspace{1cm} \text{(44)}$$

$$m = 1, 2, 3, \ldots$$ \hspace{1cm} \text{(45)}$$

$\lambda_m$ of Eq. (45) is an approximation of $\lambda_m$ of Eq. (36), based on the anticipation that bulging modes occur when $\omega$-value is large. Because $\Phi$ in Eq. (44) vanishes at $x = a$ according to Eq. (45), the second term vanishes from both denominator and numerator of Eq. (40), so that potential energy and kinetic one of the free surface are ignored. Substituting Eq. (44) into Eq. (40) and integrating yield

$$\omega^2 = \frac{m\pi}{a} \left( \frac{1}{\rho g} \right) L_1 \left( \frac{m\pi x}{a} \right) \text{pol} \frac{a}{a}$$ \hspace{1cm} \text{(46)}$$

$$m = 1, 2, 3, \ldots$$

Equation (46) is an approximate formula of the natural circular frequency in bulging mode.

Regarding Eq. (39) as an eigen-value problem and solving it by an iteration process, we can use Eq. (43) or (46) as the first approximation for $\omega^2$. The iteration process is elaborated as follows. In the first place, the case of sloshing modes is dealt with. We use Eqs. (41) and 43 as the first approximations for $\phi$ and $\omega^2$, respectively. We obtain $\omega$ by integrating the equation of motions of the circular cylindrical shell for the axisymmetric case (10) as follows, with Poisson's ratio denoted by $\nu$:

$$\dot{\phi} \left[ 1 - \frac{1}{\varepsilon^2} \left( \frac{k_0}{a} \right)^2 \frac{b_0^2}{\varepsilon^2} \right] = \frac{\partial g}{\partial \varepsilon} \text{pol} \frac{0}{(d-x)} \text{pol} \frac{0}{(d-x)}$$ \hspace{1cm} \text{(47)}$$

where $\phi$ and $\omega^2$ are given by Eqs. (41) and (43), respectively. The set $\{\lambda_n\}$ is evaluated from Eq. (36), into the left hand side of which we substitute the $\omega$-value gained from Eq. (43). To determine the set $\{A_m\}$ in Eq. (35), we expand $\omega$ into a series of cosh $(\lambda_0 \rho a)$ and $\cos (\lambda_0 \eta a)$ and impose the kinematic condition such that $\omega = \omega / \eta^2$. For this purpose Eqs. (56) and (57) in Reference (10) are utilized. If we substitute $\phi$ and $\omega$ obtained above into Eq. (39), we have the second approximation of $\omega$. Hereafter, this iteration process will be continued until it converges. Depending on the starting value of Eq. (42) and the corresponding function of Eq. (41), there converges an arbitrary set of $\phi$, $\omega$ and $\omega^2$. As for bulging modes the iteration process is similar. But there are two different aspects. Namely, we must use Schmidt's orthogonalization process (22) to get higher bulging modes in sequence, and also utilize Eqs. (39) and (60) in Reference (10) to determine the set $\{A_m\}$. It should be noted that membrane solution $\omega$ of Eq. (38) is obtained from Eq. (47) by retaining only the term underlined (--.--), and therefore it is a very rough approximation of solution of Eq. (47).

5. Numerical examples

Based on the results of the foregoing chapter, a computer code is written to give numerical examples. The circular cylindrical tank is made of steel and the liquid is water. Details are as follows: radius of the tank $a$ is 25m, length of the tank $l$ is 30m, thickness of the side wall $a$ is 30mm, and the undisturbed depth of the liquid $d$ is 21.6m.

Fig. 3 shows a radial component of the displacement $w$ of the side wall. $w_0$ is the initial value of $w$ which is obtained from Eqs. (38) and (41) for sloshing modes, or from Eqs. (38) and (44) for bulging modes.

![Fig. 3 Starting Value $w_0$ and Converged Value $w$ of Radial Component of the Displacement of the Side Wall](image-url)
$w_i$ is a converged value of $w$ which is obtained by the iteration process described in the foregoing chapter. Two sets of figures on the left side of Fig. 3 indicate the first sloshing mode and the second one, which show a fairly good coincidence between $w_i$ and $w_k$ except near the free surface at $z=0$. Two sets on the right side of Fig. 3 indicate the first bulging mode and the second one, which show a fairly good agreement except near the bottom at $z=0$.

Table 1 shows natural circular frequencies of the first five sloshing modes and as many bulging modes. Values in the first column are obtained from the eigen-matrix of Reference (10). Values in the second column are approximate ones obtained from Eq. (43) for sloshing modes and Eq. (46) for bulging modes. Values in the third column are converged ones of complementary Rayleigh quotient (39) solved by the iteration process. Approximate values in the second column are very accurate for sloshing modes, but the relative error for bulging modes grows gradually to the extent that the relative error is 5% for the fifth bulging mode. Values in the first column agree with those in the third column to the extent that the relative error is $10^{-3}$ at the most.

Table 1  Accuracy of the Natural Circular Frequencies (rad/s) 

<table>
<thead>
<tr>
<th>mode</th>
<th>ref. 10</th>
<th>1st approx.</th>
<th>converged</th>
</tr>
</thead>
<tbody>
<tr>
<td>mainly</td>
<td>1</td>
<td>1.223789</td>
<td>1.223789</td>
</tr>
<tr>
<td>in</td>
<td>2</td>
<td>1.658230</td>
<td>1.658230</td>
</tr>
<tr>
<td>sloshing</td>
<td>3</td>
<td>1.996913</td>
<td>1.996900</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2.285293</td>
<td>2.285279</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2.540901</td>
<td>2.540885</td>
</tr>
<tr>
<td>mainly</td>
<td>1</td>
<td>22.09557</td>
<td>22.09557</td>
</tr>
<tr>
<td>in</td>
<td>2</td>
<td>43.76193</td>
<td>44.12022</td>
</tr>
<tr>
<td>bulging</td>
<td>3</td>
<td>56.82922</td>
<td>58.21969</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>66.88753</td>
<td>69.68940</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>75.34688</td>
<td>79.16511</td>
</tr>
</tbody>
</table>

Table 2 gives results on a long oil storage tank which has length (l) and liquid depth (d) of 100m and 50m, respectively, and the same size as those for Table 1 other than l and d. In this case the values in the second column are known to be good approximations for both sloshing modes and bulging ones. By comparing Table 1 with Table 2 we observe that natural circular frequencies of sloshing modes are nearly coincident between them, and that natural circular frequencies of bulging modes are almost inversely proportional to the square root of the liquid depth.

Table 2  Accuracy of the Natural Circular Frequencies (rad/s) 

<table>
<thead>
<tr>
<th>mode</th>
<th>ref. 10</th>
<th>1st approx.</th>
<th>converged</th>
</tr>
</thead>
<tbody>
<tr>
<td>mainly</td>
<td>1</td>
<td>1.225422</td>
<td>1.225414</td>
</tr>
<tr>
<td>in</td>
<td>2</td>
<td>1.658239</td>
<td>1.658228</td>
</tr>
<tr>
<td>sloshing</td>
<td>3</td>
<td>1.996913</td>
<td>1.996900</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2.285293</td>
<td>2.285279</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2.540901</td>
<td>2.540886</td>
</tr>
<tr>
<td>mainly</td>
<td>1</td>
<td>10.72299</td>
<td>10.64856</td>
</tr>
<tr>
<td>in</td>
<td>2</td>
<td>26.36285</td>
<td>26.40186</td>
</tr>
<tr>
<td>bulging</td>
<td>3</td>
<td>36.22089</td>
<td>36.55147</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>43.60393</td>
<td>44.31845</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>49.71226</td>
<td>50.87748</td>
</tr>
</tbody>
</table>

6. Conclusions

We conclude from the present analysis as follows:

(1) A functional describing the coupled oscillations between a linear elastic solid of infinitesimal deformations and a liquid of small wave heights is obtained from a variational principle.

(2) By imposing several of natural conditions of the functional as constraints, an alternate functional is introduced from which a complementary Rayleigh quotient is obtained.

(3) The first approximations for the complementary Rayleigh quotient give approximate formulas of coupled natural frequencies for a circular cylindrical tank oscillating in an axisymmetric manner. Numerical examples show satisfactory coincidence between exact natural frequencies and approximate ones in the case of sloshing modes, with a tolerable error of the approximation in the case of lower bulging modes.

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