On Steady Flow through a Channel

Consisting of an Uneven Wall and a Plane Wall

Part 1. Case of No Relative Motion in Two Walls*

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A steady viscous flow through a two-dimensional, infinite channel consisting of an uneven wall and a plane wall is theoretically investigated under a given pressure gradient. The profile of the uneven wall is assumed to be periodic in the direction of the mean flow. A systematic expansion procedure is developed for the case where the ratio of the mean channel width to the period of the uneven wall, k , is small. An approximate solution is obtained up to the order of k^2 . As a result, the flow rate is explicitly found as a function of the profile of the uneven wall, k and the Reynolds number. The unevenness of the wall always decreases the flow rate in proportion to k^2 . The inertia effect on the flow rate first arises from the second order of k . The flow rate and the streamline are concretely obtained for the case of a sinusoidal wall. The approximate solutions show in good agreement with the solutions obtained by a numerical calculation.

1. Introduction

The study of laminar flow in long pipes with a wavy wall has received interest in the recent years due to the importance of such flows in hydrodynamic lubrication⁽¹⁾⁽²⁾, sealing⁽³⁾ and biophysical problems⁽⁴⁾⁽⁵⁾. The characteristics from a hydrodynamical view point in these cases are, firstly, flows of a comparatively slow velocity in a very long pipe and, secondly, the presence of the wall waviness with an amplitude comparable with the mean radius of pipes. It seems from the first reason that the fluid is highly viscous, that is, the Reynolds number is small and flows are governed by an equation neglecting the inertia terms such as the Reynolds equation in the lubrication theory. However, the second reason, that is, the amplitude of the wall waviness being comparable with the mean radius of the pipe, means that the spatial variation in the velocity is rather large. Therefore, the inertia terms due to the spatial variation in the equation of motion are not always small. To study flows in such states, the fluid flow is assumed to be incompressible and viscous, and is analyzed by taking the inertia terms in the equation of motion into account. However, it is very difficult to solve the equation of motion for viscous fluid analytically. Then, the problem should be analyzed by using some approximating method. If a small parameter is contained in the differential system considered, a solution can be obtained by per-

turbing the governing equations with respect to its small parameter. In case of a small amplitude of the wavy wall in the two-dimensional channel flow, an approximate solution is found by authors⁽⁶⁾ assuming the amplitude to be small compared with the mean width of the channel. Meanwhile, it is one of the characteristics of such flows that the amplitude of the wall waviness is comparable with the mean width of the channel. Thus, taking this fact into account, the problem should be solved. If the spatial variation in the amplitude of the wall waviness is small and periodic, the ratio of the mean width of the channel to the period of the wall waviness is considered small. Then, perturbing the governing equations with respect to this small ratio, the solutions, in the case that the amplitude of the waviness is comparable with the mean width of the channel, are obtained in the present paper. The profiles of the wavy wall are not at all restricted except the condition mentioned above. In the engineering practice, the pressure difference between the entrance and the exit of a channel is usually prescribed, and the flow rate is unknown. Therefore, in this paper, the flow rate is mainly examined under such a condition that the pressure gradient over one period of the wall waviness is a given constant value. In addition, the approximate solutions are compared with those obtained by the numerical procedure and the degree of the accuracy for the approximate solutions is discussed.

2. Governing Equations

Consider a steady flow in a long channel consisting of a wavy wall and a plane wall. The channel is two-dimensional and its mean width is h . The wavy wall varies spatially with a period λ as shown in Fig. 1. The incompressible viscous fluid is assumed to flow under an externally given pressure gradient

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in this channel. The x-axis is taken along the plane wall and the y-axis is taken in the direction from the plane wall to the wavy wall. Denoting the x and y velocity components with u and v, and the pressure with p, the equation of continuity and the equation of motion of the fluid with the density ρ and the kinematic viscosity ν are

$$u_x + v_y = 0 \quad (1)$$

$$u u_x + v u_y = -\frac{1}{\rho} p_x + \nu (u_{xx} + u_{yy}) \quad (2)$$

$$u v_x + v v_y = -\frac{1}{\rho} p_y + \nu (v_{xx} + v_{yy}) \quad (3)$$

Fig.1 Schematic diagram of a channel

Here and after, the subscripts x and y represent the differentiation with respect to x and y. Next consider the boundary conditions for these equations. If the local width of the channel is denoted with $h(x)$, the following condition holds from the no slip condition on the wavy and plane walls:

$$y=0, h: u=v=0 \quad (4)$$

Since the fluid is assumed to flow steadily under an externally given pressure difference as mentioned in the previous section, the pressure gradient averaged over the cross section of the channel and over one period of the wavy wall must be equal to a given constant pressure gradient $(\partial p / \partial x)_0$. That is, the following condition for the pressure holds:

$$\frac{1}{\lambda} \int_x^{x+\lambda} \left(\frac{1}{h} \int_0^h p_x dy \right) dx = \left(\frac{\partial p}{\partial x} \right)_0 \quad (5)$$

Now introduce a stream function as

$$u = \psi_y, \quad v = -\psi_x \quad (6)$$

and the non-dimensional variables in the following. Here, x and y are nondimensionalized with respect to λ , the period of the waviness and $\bar{h}$, the mean width of the channel, respectively. The stream function ψ is nondimensionalized with respect to the product of $\bar{h}$ and the mean velocity $U = -(\partial p / \partial x)_0 \bar{h}^2 / 12 \rho \nu$, with which the fluid flows under the constant pressure gradient $(\partial p / \partial x)_0$ in a straight channel of the mean width $\bar{h}$. If letters used in the equations of the dimensional form are also used in the equations of the dimensionless

from as they are, the non-dimensional governing equations are obtained as follows:

$$\psi_{yyyy} = kR(\psi_y \psi_{xy} - \psi_x \psi_{yy}) - 2k^2 \psi_{xxyy} - k^3 R(\psi_x \psi_{xy} - \psi_y \psi_{xx}) - k^4 \psi_{xxxx} \quad (7)$$

$$y=0, h: \psi_x = \psi_y = 0 \quad (8)$$

$$\int_x^{x+\lambda} \left[\frac{1}{h} \int_0^h \{-\psi_{yyy} + kR(\psi_y \psi_{xy} - \psi_x \psi_{yy}) - k^2 \psi_{xxyy}\} dy \right] dx = 12 \quad (9)$$

where, R is the Reynolds number $U \bar{h} / \nu$ and $k = \bar{h} / \lambda$ is the ratio of the mean width to the period. Since the profile of the wavy wall is periodic with respect to x, the following condition holds:

$$h(x+1) = h(x) \quad (10)$$

Equations (7)~(9) are the equations governing the problem considered.

3. Solving Method and Approximate Equations

As mentioned in section 1, it is assumed that the spatial variation in the width of the channel is gradual and the period of the wall waviness is much longer than the mean width $\bar{h}$ of the channel, that is, k is very small. It should be noticed that the amplitude of the waviness is totally independent of the width of the channel.

If the stream function is expanded with respect to the small parameter k as

$$\psi(x, y) = \psi_0(x, y) + k\psi_1(x, y) + k^2\psi_2(x, y) + \dots \quad (11)$$

the approximate equations are obtained as follows:

$$O(k^0): \psi_{0yyyy} = 0 \quad (12)$$

$$y=0, h: \psi_{0x} = \psi_{0y} = 0 \quad (13)$$

$$\int_x^{x+\lambda} \left\{ \frac{1}{h} \int_0^h (-\psi_{0yyy}) dy \right\} dx = 12 \quad (14)$$

$$O(k^1): \psi_{1yyyy} = R(\psi_{0y} \psi_{0xy} - \psi_{0x} \psi_{0yy}) \quad (15)$$

$$y=0, h: \psi_{1x} = \psi_{1y} = 0 \quad (16)$$

$$\int_x^{x+\lambda} \left[\frac{1}{h} \int_0^h \{-\psi_{1yyy} + R(\psi_{0y} \psi_{0xy} - \psi_{0x} \psi_{0yy})\} dy \right] dx = 0 \quad (17)$$

$$O(k^2): \psi_{2yyyy} = R(\psi_{0y} \psi_{1xy} + \psi_{1y} \psi_{0xy} - \psi_{0x} \psi_{1yy} - \psi_{1x} \psi_{0yy}) - 2\psi_{0xxyy} \quad (18)$$

$$y=0, h: \psi_{2x} = \psi_{2y} = 0 \quad (19)$$

$$\int_x^{x+\lambda} \left[\frac{1}{h} \int_0^h \{-\psi_{2yyy} + R(\psi_{0y} \psi_{1xy} + \psi_{1y} \psi_{0xy} - \psi_{0x} \psi_{1yy} - \psi_{1x} \psi_{0yy}) - \psi_{0xxyy}\} dy \right] dx = 0 \quad (20)$$

These equations are solved up to the third order of k , since the effect of the inertia terms in the equation of motion on the flow rate first arises in the third order equation. Sobey found the numerical solutions in the channel with sinusoidal wavy walls⁽⁴⁾. He

compared his results with those obtained from the equation using the boundary layer type approximation are proposed by Smith⁽⁷⁾. These equations correspond to the approximate equations up to the second order obtained above. It is noticed that the flow rate is externally given in Sobey's analysis, but the pressure gradient is externally given in this paper.

4. Approximate Solutions

Since all the differential operators of the left-hand sides of the approximate equations (12), (15) and (18) are the same, the approximate stream function of the i th order, ϕ_i , is obtained as follows:

$$\phi_i = A_i(x)y^2 + B_i(x)y^3 + \phi_{ii} \quad (i=0, 1, 2)$$

where, A_i and B_i are arbitrary functions with respect to x . The stream function is assumed to become zero on the plane wall and ϕ_{ii} is a special solution of the stream function of the i th order. If these approximate stream functions are integrated once and are substituted into the condition for the pressure, the integration constants are determined. Thus, the approximate stream functions up to the third order are obtained as

$$\phi_0 = C_0(3\xi^2 - 2\xi^3) \dots\dots\dots (21)$$

$$\phi_1 = C_0^2 R h x \left(-\frac{3}{35}\xi^2 + \frac{9}{35}\xi^3 - \frac{3}{5}\xi^4 + \frac{3}{5}\xi^5 - \frac{6}{35}\xi^7 \right) \dots\dots\dots (22)$$

$$\begin{aligned} \phi_2 = & C_0^3 R^2 \left\{ h x^2 \left(-\frac{79}{53900}\xi^2 + \frac{26}{13475}\xi^3 + \frac{6}{175}\xi^4 - \frac{33}{350}\xi^6 \right. \right. \\ & + \frac{54}{1225}\xi^7 + \frac{3}{28}\xi^8 - \frac{11}{70}\xi^9 + \frac{2}{25}\xi^{10} - \frac{4}{275}\xi^{11} \Big) \\ & + h h_{xx} \left(\frac{2}{2695}\xi^2 - \frac{39}{26950}\xi^3 - \frac{3}{350}\xi^4 + \frac{9}{350}\xi^6 \right. \\ & \left. \left. - \frac{9}{1225}\xi^7 - \frac{3}{70}\xi^8 + \frac{2}{35}\xi^9 - \frac{1}{35}\xi^{10} + \frac{2}{385}\xi^{11} \right) \right\} \\ & + C_0 \left\{ h x^2 \left(\frac{9}{5}\xi^2 - \frac{6}{5}\xi^3 - 3\xi^4 + \frac{12}{5}\xi^5 \right) \right. \\ & \left. + h h_{xx} \left(-\frac{1}{5}\xi^2 - \frac{1}{5}\xi^3 + \xi^4 - \frac{3}{5}\xi^5 \right) \right\} \\ & + C_2(3\xi^2 - 2\xi^3) \dots\dots\dots (23) \end{aligned}$$

where, $\xi = y/h$, and C_0 and C_2 are the constants determined by the profile of the wavy wall, that is,

$$C_0 = 1 / \int_0^1 \frac{dx}{h^3} \dots\dots\dots (24)$$

$$C_2 = - \left(\frac{13}{26950} R^2 C_0^4 + \frac{3}{10} C_0^2 \right) \int_0^1 \frac{h x^2}{h^3} dx \dots\dots\dots (25)$$

If h is set equal to unity in the first approximate solution, the velocity of the plane Poiseuille flow in a straight channel, that is,

$$u = - \frac{1}{2\rho\nu} \left(\frac{\partial p}{\partial x} \right)_0 y(\bar{h} - y) \dots\dots\dots (26)$$

in the dimensional form, is derived from that approximate solution. The first approximate

solution corresponds to the solution of the Reynolds equation. Its solution is found to depend on the width of the channel in form of the third power of h ⁽⁸⁾. The approximate solutions derived above are obtained in the form of the general formula as a function of arbitrary profile $h(x)$ which is not at all restricted except that the period of the wavy wall is very long compared with the mean width of the channel. Then, the flow behaviors in a channel with any wavy wall, for example, velocity, pressure and viscous stress distributions, are easily found from Eq. (11) into which ϕ_0 , ϕ_1 and ϕ_2 are substituted. The flow rate, which is one of the flow properties averaged spatially over the channel, is mainly found in the following sections.

5. Flow Rate

5.1 General case

Consider the flow rates of this flow which are important in practical problems. Since the stream function is chosen to be equal to zero on the plane wall as mentioned in the previous section, the value of the stream function on the wavy wall becomes the flow rate of a steady flow under an externally given pressure gradient $(\partial p / \partial x)_0$. Then, the flow rate Q is found by substituting the values on the wavy wall, $y=h$, of ϕ_0 , ϕ_1 and ϕ_2 obtained in the previous section into Eq. (11). Since ϕ is a function of x and y , Q depends seemingly on x . However, the wavy wall is regarded as one of the stream lines. Then, $\phi(x, h)$ becomes a constant. In fact, the values of the first, second and third approximate solutions on the wavy wall are equal to the constants, C_0 , 0 and C_2 , respectively. They are determined by the Reynolds number and the profile of the wall. Thus, the flow rate Q is found as

$$Q = C_0 + k^2 C_2 + O(k^3) \dots\dots\dots (27)$$

The first term C_0 on the right-hand side of this equation corresponds to the flow rate of a highly viscous flow. It depends on the local width h in form of the third power of it as a case of the plane Poiseuille flow in the straight channel. It is found from Eq. (27) that the flow rate depends on the period of the wavy wall in the form proportional to the second power of k . Its proportional constant C_2 is as clear from Eq. (25) always negative. Then the flow rate always decreases as the period of the waviness decreases. Its decreasing rate is proportional to the second power of the slope h_x of the wall profile. It consists of two parts, that is, one being dependent on the square of Reynolds number and the other independent of it. The term depending and the one not depending on the Reynolds number on the right-hand side of Eq. (25) are based on the inertia terms and the viscous ones of the equations of motion (2) and (3), respectively. Then, the waviness of the wall is found to decrease the flow rate through the inertia force and the viscous force. It

should be noticed that the flow is blocked not only by the viscous force, but also by the inertia force.

5.2 Case of sinusoidal wall

Consider a sinusoidal wall with the amplitude δ , that is,

$$h=1-\delta \cos 2\pi x \quad \dots\dots\dots(28)$$

as one of the representative examples of the wavy wall. Since the integrated values contained in C_0 and C_2 are found as

$$\int_0^1 \frac{dx}{h^3} = \left(1 + \frac{\delta^2}{2}\right) / (1 - \delta^2)^{5/2} \quad \dots\dots\dots(29)$$

$$\int_0^1 \frac{h x^2}{h^3} dx = 2\pi^2 \delta^2 / (1 - \delta^2)^{5/2} \quad \dots\dots\dots(30)$$

the flow rate is obtained as a function of the Reynolds number R , the wave number k and the amplitude δ of the wavy wall. The coefficients C_0 and C_2 of the flow rate expanded with respect to k are dependent on the amplitude δ as shown in Figs. 2 and 3. The flow rate C_0 in case of small wave number is found from Fig. 2 to decrease as the amplitude δ increases. When δ is unity, that is, two walls contact each other, the flow rate becomes zero. The third approximation C_2 of the flow rate is found from Fig. 3 to take a negative value contrary to the case of C_0 and reach a minimum value at an amplitude in the range from zero to unity. The values of the amplitude corresponding to the minimum values hardly depend on the Reynolds number and are equal to about $1/2$. The flow rate is influenced by the profile of the wavy wall through the parameters k and δ and, of course, decreases as the amplitude δ increases. But the flow rate is strongly influenced by the period of the waviness when the amplitude is nearly equal to $1/2$. Then, in case of δ being equal to $1/2$, the flow rate Q is shown in Figs. 4 and 5 to decrease with

Fig.2 Relation between flow rate and amplitude due to first approximate solution.

Fig.3 Relation between flow rate and amplitude due to third approximate solution.

Fig.4 Relation between flow rate and wave number.

Fig.5 Relation between flow rate and Reynolds number.

increasing k and R . Since, in case of small amplitude, Eqs. (29) and (30) are expanded in a series form of the second power of δ , the coefficients C_0 and C_2 of power series expanded with respect to k are expressed as

$$C_0 = 1 - 3\delta^2 + O(\delta^4) \quad \dots\dots\dots(31)$$

$$C_2 = -2\pi^2 \delta^2 \left(\frac{13}{26950} R^2 + \frac{3}{10} \right) + O(\delta^4) \quad \dots\dots\dots(32)$$

Therefore, in case of small amplitude, the

flow rate is found to decrease linearly with the second power of $\delta^{(6)}$.

6. Stream Lines

Consider the relation between the flow rate and the profile of the wavy wall mentioned in the preceding section on the basis of the stream line for the sinusoidal wavy wall given by Eq. (28). The stream lines corresponding to the first, second and third approximate solutions, ψ_0 , ψ_1 and ψ_2 , which are given by Eqs. (21), (22) and (23) by the aid of Eqs. (29) and (30), are drawn in Figs. 6 (a), (b) and (c), respectively. It is noted in these figures that only the regions from $x=0$ to $1/2$ are treated, because these stream lines are symmetrical with respect to the line $x=1/2$. The stream lines for the first approximate solution shown in Fig. 6 (a) depend on only the amplitude δ of the sinusoidal wavy wall. These stream lines are bent by the waviness of the wall, but all the directions of the flow on each stream line are the same, that is, the direction adverse to the given pressure gradient. The stream lines for the second approximate solution, shown in Fig. 6 (b), depend on the amplitude δ and the Reynolds number R . Since ψ_1 is merely proportional to R , the behavior of the stream lines in this figure does not fundamentally change even if the Reynolds number changes. This flow is expressed with closed stream lines, because the flow rate is equal to zero. In the diverging region of the channel, the flow is decelerated near the wavy and plane walls and accelerated in the central part. Meanwhile in the converging region the trend is reversed. Thus, the effects of the wavy wall and the plane wall on the flow are basically the same except the difference in the degree of these effects. This is also understood from the fact that the two layers consisting of closed stream lines with the clockwise and anti-clockwise directions arise near the wavy wall and the plane wall, respectively, as

shown in Fig. 6 (b). The stream lines for the third approximate solution, shown in Fig. 6 (c), depend on the amplitude δ and the Reynolds number R . Decrease in the flow rate is found from this figure to be directly caused by the third approximate solution. A flow adverse to the main flow arises near the wavy wall and bends toward the plane wall at the throat. A steady flow with a flow rate is induced by ψ_2 in the direction adverse to the main stream. It is bent much more compared with the degree of the waviness of the wavy wall. The remaining region is occupied by a flow with a closed stream line. Thus, the wavy wall is found to have a blocking effect larger than the plane wall. The true stream lines, which are obtained by substituting those approximate solutions into Eq. (11), are shown in Fig. 7 for a small velocity and in Fig. 8 for a large velocity, respectively.

Fig.6 (b) Stream lines by second approximate solution.

Fig.6 (a) Stream lines by first approximate solution.

Fig.6 (c) Stream lines by third approximate solution.

Fig.7 Stream lines comparing with those obtained by numerical method (in case of small Reynolds number)

Fig.8 Stream lines by approximate solution (in case of large Reynolds number)

7. Separation

As the flow velocity increases and becomes large, separation will arise in the diverging region of the channel. Consider the velocity gradient along the walls to find the separation point. If the velocity component in the direction along the wall is denoted by v_{11} and the coordinate in the direction normal to the wall is denoted by n , the velocity gradient on the wall, $\partial v_{11}/\partial n$ is expressed in the form of the second derivative with respect to y . Thus, the separation point on the wavy wall is given by

$$\phi_{yy}|_{y=h}=0 \quad (33)$$

Since from Eqs. (21)~(23)

$$\phi_{0yy}|_{y=h} = -6C_0 \frac{1}{h^2} \quad (34)$$

$$\phi_{1yy}|_{y=h} = \frac{6}{35} RC_0^2 \frac{h_x}{h^2} \quad (35)$$

$$\phi_{2yy}|_{y=h} = R^2 C_0^3 \left(\frac{79}{26950} \frac{h_x^2}{h^2} - \frac{4}{2695} \frac{h_{xx}}{h} \right) + C_0 \left(\frac{42}{5} \frac{h_x^2}{h^2} - \frac{8}{5} \frac{h_{xx}}{h} \right) - 6C_2 \frac{1}{h^2} \quad (36)$$

the separation point is found to depend on the wall profile itself h , its slope h_x and its curvature h_{xx} . The critical state in which separation just occurs is shown in the $k - R$ plane with the parameter δ as Fig. 9. In the region of the upper side of the solid curves, separation arises. It is found from Fig. 9 that separation can be easily induced as the Reynolds number R and the wave number k increase. Moreover, separation becomes easy to occur with increasing amplitude of the wavy wall.

Fig.9 The critical curve when separation just occurs

8. Comparison between Approximate and Numerical Solutions

Numerical solutions are obtained to examine the degree of accuracy in the approximate solutions obtained in the preceding sections. Transforming a flow region in the channel with the wavy and plane walls into one with the plane walls, Eqs. (7)~(9) are solved by a finite difference method as referred to the appendix. The stream lines due to approximate solutions and ones due to numerical solutions are shown in Fig. 7 with the solid and broken lines, respectively, for a small velocity. Moreover, they are shown in Figs. 8 and 10 for a large velocity. In

Fig.10 Stream lines by numerical solution (in case of large Reynolds number)

the former case, separation does not occur as yet, but, in the latter case, it occurs. In case of $R=20$ in Fig. 7, the approximate solution is in good agreement with the numerical one. Even in case of $R=100$ in Figs. 8 and 10, they are in good agreement. It is stated by Sobey that, in case of the flow in a channel whose walls are symmetric with respect to the center axis of the channel, the approximate solutions corresponding to only the first and second approximations are in good agreement with the numerical one for $\delta=0.25$, $k=0.125$ and $R=50$ (4). In this example, the flow contains separation points. Next, comparison between the approximate and numerical methods for the effect of the flow rate on the Reynolds number R and the wave number k is made in Figs. 4 and 5. Even if k increases up to 0.4, the difference in the flow rate between them is below 10 percent. Furthermore, when R becomes nearly equal to 100, good agreement is obtained in the range of k up to about 0.3. Thus, even in case of such a comparatively large amplitude as $\delta=0.5$, the approximate solution obtained here is regarded as one of the solutions of the Navier-Stokes equation. Then, this solution can be usefully applied to the practical engineering problems.

9. Conclusions

A steady viscous flow in a channel with a periodic wavy wall and a plane wall is theoretically solved under such a condition that the mean pressure gradient is externally given. Assuming the ratio of the mean width of the channel to the period of the wavy wall is small, an approximate solution is obtained up to the third order of the wave number. As a result, the general formula for the flow rate is found as a function of arbitrary profile of the wavy wall containing parametrically the Reynolds number, the ratio of the width to the period and the ratio of the amplitude of the wavy wall to the width. Feature of decrease in the flow rate due to the waviness of the wall is explained in connection with the stream lines. The approximate solution obtained here is ensured to hold in wide range, comparing it with the numerical solutions found by using the finite difference method.

Appendix

If the transformation is introduced as $\xi=x$ and $\eta=y/h$, Eqs. (7)~(9) governing

the flow in a wavy channel are rewritten as follows:

$$kR \frac{1}{h} (\phi_{\xi} \omega_{\eta} - \phi_{\eta} \omega_{\xi}) + \nabla^2 \omega = 0 \quad \text{.....(i)}$$

$$\omega = -\nabla^2 \phi \quad \text{.....(ii)}$$

where,

$$\nabla^2 = k^2 \frac{\partial^2}{\partial \xi^2} - 2k^2 \frac{h_{\xi}}{h} \eta \frac{\partial^2}{\partial \xi \partial \eta} + \left(k^2 \frac{h_{\xi}^2}{h^2} \eta^2 + \frac{1}{h^2} \right) \frac{\partial^2}{\partial \eta^2} + k^2 \left(-\frac{h_{\xi\xi}}{h} + 2 \frac{h_{\xi}^2}{h^2} \right) \eta \frac{\partial}{\partial \eta}$$

$$\eta=0: \phi=\phi_{\eta}=0 \quad \text{.....(iii)}$$

$$\eta=1: \phi=Q, \quad \phi_{\eta}=0 \quad \text{.....(iv)}$$

$$\int_0^1 \left[\frac{1}{h} \int_0^1 \left(-\frac{1}{h^3} \phi_{\eta\eta\eta} + kR \left\{ \frac{1}{h^2} \phi_{\xi} \phi_{\eta\eta} - \frac{h_{\xi}}{h^3} \phi_{\eta}^2 - \frac{1}{h^2} \phi_{\eta\eta} \phi_{\xi} \right\} - k^2 \left\{ \frac{1}{h} \phi_{\xi\xi\eta} - 2 \frac{h_{\xi}}{h^2} \phi_{\xi\eta} - 2 \frac{h_{\xi}}{h^2} \eta \phi_{\xi\eta\eta} + \frac{1}{h^2} \left(-h_{\xi\xi} + 2 \frac{h_{\xi}^2}{h} \right) \phi_{\eta} + \frac{1}{h^2} \left(-h_{\xi\xi} + 4 \frac{h_{\xi}^2}{h} \right) \eta \phi_{\eta\eta} + \frac{h_{\xi}^2}{h^3} \eta^2 \phi_{\eta\eta\eta} \right\} h d\eta \right] d\xi = 12 \quad \text{.....(v)}$$

Here, Eq. (iv) means that the wavy wall is one of the stream lines and Q denotes the flow rate which is an unknown variable. Then, the three unknown variables, that is, the stream function ϕ , the vorticity ω and the flow rate Q , are simultaneously found by using the finite difference method.

References

- (1) Lebeck, A.O., et. al., Trans. ASME, J. Lub. Technol., 100-1 (1978), 81.
- (2) Elrod, H.G., Trans. ASME, J. Lub. Technol., 101-1 (1979), 8.
- (3) Moore, D.F., J. Roy. Aeron. Soc., 69-653 (1965), 337.
- (4) Sobey, I.J., J. Fluid Mech., 96-1 (1980), 1.
- (5) Lessen, M. and Huang, P.S., Phys. Fluids, 19-7 (1976), 945.
- (6) Hasegawa, E. and Hukuoka, N., Bull. JSME, 23-186 (1980), 2042.
- (7) Smith, F.T., Q.J. Mech. Appl. Math., 29-3 (1976), 343.
- (8) Cameron, A., Principles of Lubrication, (1966), 54, Longmans.