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Thermal Stress Concentrations in the Transient

Two-dimensional Temperature Field*

(2nd Report, Plate with a Semi-elliptic Notch)

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The transient thermal stress concentration of a notched semi-infinite plate is analysed numerically. The plate which is initially in a uniform temperature state is assumed to be heated along the straight and notched boundaries through a fluid touching directly on it. The notch shapes are semi-elliptic with various radius ratios. A finite-difference method is used both for temperature and stress analyses. The finite-difference mesh is formed on basis of the equi-potential and flow lines in the potential flow around an elliptic cylinder. It is found by this method that the thermal stress concentration increases with time and the Biot number of heat convection. An approximate method, by which the stress concentration factor of a semi-elliptically notched strip with finite width can be estimated, is proposed.

Key Words: Elasticity, Thermal Stress, Stress Concentration Factor, Transient State, Conformal Transformation, Numerical Analysis

1. Introduction

In general, many of structural members which compose machinery, structures and others have fillets, notches or holes, which cause stress concentrations. The applications of thermal loads to such members yield stress concentrations at the structural discontinuities just as those of mechanical loads. It was reported⁽¹⁾ that thermal stress fatigue cracks were observed in blade grooves and labyrinth seal grooves of steam turbines under steep temperature variation. Hence the analysis of thermal stress concentrations, especially in the transient temperature field, is of great practical importance. However, systematic studies in this field are very few, since thermal stress concentration problems must be analysed by taking account of thermal factors as well as shape factors. Only a

few studies^{(2),(3)} have been reported on a semi-infinite plate with a circular notch subjected to thermal loading (heating or cooling) along the free boundary.

In the previous paper⁽²⁾, a plate with a semi-circular notch heated suddenly along the free boundaries through a fluid touching directly on it was considered, and it was made clear that the stress concentrations at the root of the notch grow remarkably with an increasing time.

The analytical method which the authors proposed in the previous paper is modified here to solve the problem of a plate with a semi-elliptic notch. The effects of the notch depth on the stress concentrations are systematically examined comparing with the case of a semi-circular notch. It is recognized that the thermal stress concentrations are greatly affected by the notch shape. The numerical method and the results will be given here.

2. Conformal Transformation

Consider a semi-infinite plate with a semi-elliptic notch subjected to transient thermal loading (heating or cooling) through heat convection at the free boundary surfaces as shown in Fig.1. After sudden heating or cooling along the free boundaries by a thermal fluid, the environmental fluid is assumed to be kept at constant temperature θ . Fig.1(a) shows Model I, where the major axis of the semi-ellipse is on the $x(X)$ -axis and the model has a shallow notch compared with a semi-circle. Fig.1(b) shows Model II, where the minor axis is on the $x(X)$ -axis and this model has a deep notch.

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(a) Model I

(b) Model II

Fig.1 Semi-infinite plate with a semi-elliptic notch suddenly heated through heat-convection

By the way, it is considered that the temperature and stress fields are disturbed significantly near the notch, while those effects decrease gradually with an increasing distance from the notch, so that the stress state in the region sufficiently far from the notch can be assumed to approach the state of a semi-infinite plate without a notch. Hence, in a manner similar to that proposed previously, the solutions of the temperature and stress fields may be expressed as a sum of two components, respectively, that is, one is the solution of a semi-infinite plate without a notch and the other expresses the effects of the notch. The former solution is easily obtained analytically, while the latter one is not easily calculated and so it is solved numerically by means of the finite-difference method. The numerical analysis can be done accurately by mapping conformally the plate region with a notch into a semi-infinite region with straight boundary, so that the transformed domain can be easily divided into rectangular finite-difference meshes^[4].

Let the origin of the rectangular coordinates be the center of the notch and take the $x(X)$ -axis along the straight free boundary and $y(Y)$ -axis toward the depth of the notch as shown in Fig.1. To express all equations in dimensionless form, the major and minor axes are taken as a characteristic dimension in Model I and Model II, respectively. The following analytical functions^[5] are used to map the physical plane $z(Z)$ into the conformal plane (W) ;

$$Z = C_I \cosh W \quad (\text{Model I}) \dots\dots\dots (1)$$

$$Z = C_{II} \sinh W \quad (\text{Model II}) \dots\dots\dots (2)$$

where $Z = X + iY$, $X = x/a$, $Y = y/a$, $Z = z/a$, $W = \xi + i\eta$, $C_I = \sqrt{1 - \mu^2}$ (Model I: $\mu = b/a < 1$),
 $C_{II} = \sqrt{\mu^2 - 1}$ (Model II: $\mu = b/a > 1$).

Coordinates (ξ, η) are called elliptic coordinates and the following equations are obtained by equating the real and imaginary parts Eqs.(1) and (2), respectively;

$$X = C_I \cosh \xi \cos \eta, \quad Y = C_I \sinh \xi \sin \eta \dots (3)$$

$$X = C_{II} \sinh \xi \cos \eta, \quad Y = C_{II} \cosh \xi \sin \eta \dots (4)$$

Furthermore the curvilinear coordinates (ϕ, ψ) are introduced, which means the equi-potential lines and streamlines in the flow pattern around an elliptic cylinder placed in a uniform stream. ϕ and ψ for Model I and Model II are respectively given as follows^[6];

$$\phi = \cosh(\xi - \xi_{0I}) \cos \eta,$$

$$\psi = \sinh(\xi - \xi_{0I}) \sin \eta \quad (\text{Model I}) \dots\dots (5)$$

$$\phi = \cosh(\xi - \xi_{0II}) \cos \eta,$$

$$\psi = \sinh(\xi - \xi_{0II}) \sin \eta \quad (\text{Model II}) \dots\dots (6)$$

where $\xi_{0I} = 1/2 \cdot \log(1 + \mu)/(1 - \mu)$, ($\mu < 1$), $\xi_{0II} = 1/2 \cdot \log(\mu + 1)/(\mu - 1)$, ($\mu > 1$)

3. Temperature Field

3.1 Governing equations

Consider a plate to be insulated at both side surfaces and ignore heat generation in the plate to simplify the problem. Then the governing equations of the temperature field are given by

$$\partial g / \partial F = \nabla^2 g \dots\dots\dots (7)$$

where $\nabla^2 = \partial^2 / \partial X^2 + \partial^2 / \partial Y^2$, $g = T/\theta$, $F = \kappa t/a^2$: Fourier number, T : temperature, $\kappa = \lambda/(c\rho)$: thermal diffusivity, λ : thermal conductivity, c : specific heat, ρ : density and t : time.

3.2 Boundary and initial conditions

The boundary conditions at the heating surface and in the region sufficiently far from the notch are assumed as follows, respectively;

$$\partial g / \partial n = B(g - 1) \quad (\text{at the heating surface}) \dots\dots\dots (8)$$

$$\partial g / \partial X = 0 \quad (X = \infty),$$

$$\partial g / \partial Y = 0 \quad (Y = \infty) \dots\dots\dots (9)$$

where $B = ha/\lambda$: Biot number, h : surface convection heat-transfer coefficient, n : inward normal to the heating surface.

The initial condition is given by

$$g = 0, \quad (F = 0) \dots\dots\dots (10)$$

3.3 Separation of temperature field

Considering the linearity of Eqs.(7) to (10), temperature g may be resolved into the following two components;

$$g = g^{(1)} + g^{(2)} \dots\dots\dots (11)$$

where $g^{(1)}$: temperature of the semi-infinite plate without a notch, $g^{(2)}$: temperature component due to notch effect.

The solution for $g^{(1)}$ is given by⁷⁾

$$g^{(1)} = \operatorname{erfc}(Y/2\sqrt{F}) - \exp(BY + B^2F) \operatorname{erfc}(Y/2\sqrt{F} + B\sqrt{F}) \quad (12)$$

Consequently the remaining problem is reduced to calculation of $g^{(2)}$. It is convenient to introduce the curvilinear coordinates (ϕ, ψ) discussed in the previous chapter to specify the boundary condition of $g^{(2)}$. Substituting $g^{(2)}$ instead of g into Eq. (7) and performing the coordinates transformation, the governing equations of $g^{(2)}$ may be written as follows;

$$\partial g^{(2)} / \partial F = J^2 \nabla^2 g^{(2)} \quad (13)$$

where $\nabla^2 = \partial^2 / \partial \phi^2 + \partial^2 / \partial \psi^2$, and J is the stretch ratio of the conformal transformation and represented by

$$J^2 = \{ \sinh^2(\xi - \xi_{0I}) + \sin^2\eta \} / \{ C r^2 (\sinh^2\xi + \sin^2\eta) \} \quad (\text{Model I}) \quad (14)$$

$$J^2 = \{ \sinh^2(\xi - \xi_{0II}) + \sin^2\eta \} / \{ C r^2 (\sinh^2\xi + \cos^2\eta) \} \quad (\text{Model II}) \quad (15)$$

3.4 Boundary and initial conditions of $g^{(2)}$

Considering the regions ①②③④⑤ bounded by the boundaries of four types, that is, $\phi=0$ (Y -axis), $\psi=0$ (free boundary), equi-potential line ($\phi \gg 1$) sufficiently far from the notch and stream line ($\psi \gg 1$) as shown in Fig.2, the boundary conditions of $g^{(2)}$ are given below.

- (1) On the notched edge ①② ;

$$\partial g^{(2)} / \partial \psi = B/J (g^{(1)} + g^{(2)} - 1) - \partial g^{(1)} / \partial \psi \quad (16)$$

- (2) On the straight edge ②③ ;

$$\partial g^{(2)} / \partial \psi = B/J g^{(2)} \quad (17)$$

- (3) In the region sufficiently far from the notch ③④⑤ ;

$$g^{(2)} = 0 \quad (18)$$

- (4) On the Y -axis ①⑤ ;

$$\partial g^{(2)} / \partial \phi = 0 \quad (19)$$

From Eq.(10), the initial condition of $g^{(2)}$ becomes

$$g^{(2)} = 0, \quad (F=0) \quad (20)$$

3.5 Numerical approach to $g^{(2)}$

Because of difficulty in finding out a closed-form solution of $g^{(2)}$, the finite-difference method is employed here. Using a network with unequal increments in both the ϕ and ψ direction as shown in Fig.3, the governing equation (13) may be expressed as the following finite-difference approximation;

$$g_{0,k+1}^{(2)} = [1 - 2J_0^2 \Delta F \{ 1/(l_1 l_2 \Delta \phi^2) + 1/(m_1 m_2 \Delta \psi^2) \}] g_{0,k}^{(2)} + 2J_0^2 \Delta F \{ (l_2 g_{1,k}^{(2)} + l_1 g_{3,k}^{(2)}) / \{ l_1 l_2 (l_1 + l_2) (\Delta \phi)^2 \} + (m_2 g_{2,k}^{(2)} + m_1 g_{4,k}^{(2)}) / \{ m_1 m_2 (m_1 + m_2) (\Delta \psi)^2 \} \} \quad (21)$$

where $\Delta F = \kappa \Delta t / a^2$, Δt : time increment, J_0 : stretch ratio at the central node 0, $g_{0,k}^{(2)}, g_{1,k}^{(2)}, \dots, g_{4,k}^{(2)}$: temperatures of the nodes 0, 1, ..., 4 at the time $F = k \Delta F$ ($k=1, 2, \dots$)

Applying Eq.(21) and Eqs.(16) to (19) for each boundary node, the boundary conditions may be expressed by the so-called advancing finite-difference equations. As a representative boundary equation, the boundary condition along the edge ①② is given below in an explicit form.

$$g_{0,k+1}^{(2)} = [1 - 2J_0^2 \Delta F \{ 1/(l_1 l_2 \Delta \phi^2) + 1/(\Delta \psi)^2 + B/(J_0 \Delta \phi) \}] g_{0,k}^{(2)} + 2J_0^2 \Delta F \{ (l_2 g_{1,k}^{(2)} + l_1 g_{3,k}^{(2)}) / \{ l_1 l_2 (l_1 + l_2) (\Delta \phi)^2 \} + g_{2,k}^{(2)} / (\Delta \psi)^2 - B(g_{0,k}^{(1)} - 1)/(J_0 \Delta \phi) - B/\Delta \psi \exp(BY + B^2F) \operatorname{erfc}(Y/2\sqrt{F} + B\sqrt{F}) \} \quad (22)$$

To ensure stability of the numerical solutions of the above finite-difference equations, the time increment and the increments of space coordinates must satisfy the following relation⁸⁾;

$$1 - 2J_0^2 \Delta F \{ 1/(l_1 l_2 (\Delta \phi)^2) + 1/(\Delta \psi)^2 + B/(J_0 \Delta \phi) \} \geq 0 \quad (23)$$

Fig.2 Boundary to specify $g^{(2)}$, $\chi^{(2)}$ and $\zeta^{(2)}$

Fig.3 Finite-difference network

Calculation procedures are as follows: Determine ΔF for any chosen $\Delta \phi$, $\Delta \psi$ and the given B which satisfies Eq.(23). Starting from the initial condition, Eq.(20) and repeating calculations of Eqs.(21), (22), etc. with advancing time steps by ΔF at each step, then the values of $g^{(2)}$ at all node points for any desired time $F = k \Delta F$ are obtained. Adding $g^{(1)}$ to $g^{(2)}$, the desired temperature g is completely determined. As an example, the temperature distributions corresponding with $B=10$ and $F=0.4$ are shown in Fig.4 (Model I, $\mu=0.5$) and Fig.5 (Model II, $\mu=2.0$), respectively.

Fig.4 One example of temperature distributions

Fig.5 One example of temperature distributions

4. Stress Field

4.1 Governing equations

Consider such a problem that the stress distributions in the region sufficiently far from the notch approach gradually the state of a semi-infinite plate without a notch, which is completely constrained in the X -direction and heated along $Y=0$.

The material properties α , E (α : thermal expansion coefficient, E : Young's modulus) are really dependent on temperature, but the mean values of those material properties in the temperature range concerned may be used for simplicity of solution. Then the governing equation for the two-dimensional thermal stress field may be given as

$$\nabla^4 \chi + \nabla^2 g = 0 \quad (24)$$

where $\nabla^4 = \nabla^2 \nabla^2$, $\chi = \bar{\chi} / (\alpha E \theta a^2)$, $\bar{\chi}$: Airy stress function. Furthermore Eq.(24) is expressed by means of the curvilinear coordinates (ϕ, ψ) as follows;

$$\nabla^2 (J^2 \nabla^2 \chi) + \nabla^2 g = 0 \quad (25)$$

Fig.6 Stress components in the curvilinear coordinates (ϕ, ψ)

The stress components s_ϕ , s_ψ , $s_{\phi\psi}$ in the curvilinear coordinates (ϕ, ψ) are defined as shown in Fig.6 and they are expressed in terms of the stress function χ as follows, respectively⁽⁹⁾:

$$\left. \begin{aligned} s_\phi &= J^2 (\partial^2 \chi / \partial \psi^2) \\ &\quad + J \{ (\partial J / \partial \psi) (\partial \chi / \partial \psi) - (\partial J / \partial \phi) (\partial \chi / \partial \phi) \} \\ s_\psi &= J^2 (\partial^2 \chi / \partial \phi^2) \\ &\quad - J \{ (\partial J / \partial \phi) (\partial \chi / \partial \phi) - (\partial J / \partial \psi) (\partial \chi / \partial \psi) \} \\ s_{\phi\psi} &= -J^2 (\partial^2 \chi / \partial \phi \partial \psi) \\ &\quad - J \{ (\partial J / \partial \phi) (\partial \chi / \partial \phi) + (\partial J / \partial \psi) (\partial \chi / \partial \psi) \} \end{aligned} \right\} \quad (26)$$

where $s_\phi = \sigma_\phi / (\alpha E \theta)$, $s_\psi = \sigma_\psi / (\alpha E \theta)$, $s_{\phi\psi} = \tau_{\phi\psi} / (\alpha E \theta)$, σ_ϕ , σ_ψ , $\tau_{\phi\psi}$: normal stresses in the directions of streamlines and equi-potential lines and shear stress, respectively.

For the convenience of numerical analysis, Eqs.(24) and (25) are transformed into the following form, by introducing a parameter stress function ζ , respectively;

$$\nabla^2 \chi = \zeta, \quad \nabla^2 \zeta = -\nabla^2 g \quad (27)$$

$$J^2 \nabla^2 \chi = \zeta, \quad \nabla^2 \zeta = -\nabla^2 g \quad (28)$$

4.2 Separations of stress functions χ and ζ

In the same way as in the case of the temperature field, the stress functions χ and ζ may be separated into the following two components, respectively;

$$\chi = \chi^{(1)} + \chi^{(2)}, \quad \zeta = \zeta^{(1)} + \zeta^{(2)} \quad (29)$$

where $\chi^{(1)}$, $\zeta^{(1)}$: stress functions of the semi-infinite plate without a notch, $\chi^{(2)}$, $\zeta^{(2)}$: stress functions due to notch effect. $\chi^{(1)}$ and $\zeta^{(1)}$ are given below from the analysis in the previous paper, respectively;

$$\chi^{(1)} = Y \int_Y^\infty g^{(1)} dY + \int_0^Y Y g^{(1)} dY \quad (30)$$

$$\zeta^{(1)} = -g^{(1)} \quad (31)$$

Then the problem concerned is reduced to determination of the stress function components $\chi^{(2)}$ and $\zeta^{(2)}$. From Eqs.(27) and (28), the governing equations for these functions become the following equations, respectively;

$$\nabla^2 \chi^{(2)} = \zeta^{(2)}, \quad \nabla^2 \zeta^{(2)} = -\nabla^2 g^{(2)} \quad (32)$$

$$J^2 \nabla^2 \chi^{(2)} = \zeta^{(2)}, \quad \nabla^2 \zeta^{(2)} = -\nabla^2 g^{(2)} \quad (33)$$

4.3 Boundary conditions of $\chi^{(2)}$ and $\zeta^{(2)}$

The heating surface is assumed to be a free boundary. Then, the solutions must satisfy the boundary conditions that in the region of $X \rightarrow \pm\infty$ and $Y \rightarrow \infty$ the normal stresses in the X and Y directions are given by $s_x = -g^{(1)}$, $s_y = 0$, and the shear stress by $s_{xy} = 0$. Assuming that the above boundary conditions in the region of $X \rightarrow \infty$ and $Y \rightarrow \infty$ hold on the boundaries ③~④ and ④~⑤ as shown in Fig.2, then the boundary conditions of $\chi^{(2)}$ and $\zeta^{(2)}$ are given as follows;

(1) On the notch edge ①② ;
From the condition that it is a free boundary, the following relations are obtained;

$$\begin{aligned}\partial\chi^{(2)}/\partial\phi &= \int_0^Y g^{(1)} dY, \\ \chi^{(2)} &= Y \int_0^Y g^{(1)} dY - \int_0^Y Y g^{(1)} dY \dots\dots\dots (34)\end{aligned}$$

(2) On the straight edge ②③ ;
In the same way as the above condition (1),

4.4 Solutions of $\chi^{(2)}$ and $\zeta^{(2)}$

On account of the difficulty in finding out the closed-form solutions of $\chi^{(2)}$ and $\zeta^{(2)}$, the finite-difference method is employed here. In the same way as in the case of the temperature field, the governing equation (33) is transformed into the following finite-difference approximations by using an unequally spaced network as shown in Fig.3;

$$\begin{aligned}\chi_0^{(2)} &= \{l_1 l_2 m_1 m_2 (\Delta\phi)^2 (\Delta\psi)^2\} / \{l_1 l_2 (\Delta\phi)^2 + m_1 m_2 (\Delta\psi)^2\} [(l_2 \chi_1^{(2)} + l_1 \chi_3^{(2)}) / \{l_1 l_2 (l_1 + l_2) (\Delta\phi)^2\} \\ &\quad + (m_2 \chi_2^{(2)} + m_1 \chi_4^{(2)}) / \{m_1 m_2 (m_1 + m_2) (\Delta\psi)^2\} - \zeta_0^{(2)} / (2J_0^2)] \dots\dots\dots (38)\end{aligned}$$

$$\begin{aligned}\zeta_0^{(2)} &= \{l_1 l_2 m_1 m_2 (\Delta\phi)^2 (\Delta\psi)^2\} / \{l_1 l_2 (\Delta\phi)^2 + m_1 m_2 (\Delta\psi)^2\} [(l_2 (\zeta_1^{(2)} + g_1^{(2)}) + l_1 (\zeta_3^{(2)} + g_3^{(2)})) / \{l_1 l_2 (l_1 + l_2) (\Delta\phi)^2\} \\ &\quad + (m_2 (\zeta_2^{(2)} + g_2^{(2)}) + m_1 (\zeta_4^{(2)} + g_4^{(2)})) / \{m_1 m_2 (m_1 + m_2) (\Delta\psi)^2\} - g_0^{(2)}] \dots\dots\dots (39)\end{aligned}$$

The conditions of Eqs.(34) to (37) may also be written in terms of finite-difference as follows;

(1) On the notch edge ①② ;

$$\begin{aligned}\zeta_0^{(2)} &= 2J_0^2 [(l_2 \chi_1^{(2)} + l_1 \chi_3^{(2)}) / \{l_1 l_2 (l_1 + l_2) (\Delta\phi)^2\} + \chi_2^{(2)} / (\Delta\phi)^2 - \{l_1 l_2 (\Delta\phi)^2 \\ &\quad + (\Delta\psi)^2\} \chi_0^{(2)} / \{l_1 l_2 (\Delta\phi)^2 (\Delta\psi)^2\} - 1 / \Delta\phi \int_0^Y g^{(1)} dY] \dots\dots\dots (40)\end{aligned}$$

(2) On the straight edge ②③ ;

$$\chi_0^{(2)} = \chi_1^{(2)} = \chi_3^{(2)} = 0, \quad \zeta_0^{(2)} = 2J_0^2 \chi_2^{(2)} / (\Delta\phi)^2 \dots\dots\dots (41)$$

(3) At the root of the notch ① ;

$$\begin{aligned}\zeta_0^{(2)} &= 2J_0^2 [\chi_1^{(2)} / (\Delta\phi)^2 + \chi_2^{(2)} / (\Delta\psi)^2 - \{(\Delta\phi)^2 + (\Delta\psi)^2\} \chi_0^{(2)} / \{(\Delta\phi)^2 (\Delta\psi)^2\} - 1 / \Delta\phi \int_0^Y g^{(1)} dY] \\ &\dots\dots\dots (42)\end{aligned}$$

(4) In the region sufficiently far from the notch ③④⑤ ;

$$\chi_0^{(2)} = 0, \quad \zeta_0^{(2)} = 0 \dots\dots\dots (43)$$

(5) On the Y -axis ①⑤ ;

$$\begin{aligned}\chi_0^{(2)} &= m_1 m_2 (\Delta\phi)^2 (\Delta\psi)^2 / \{(\Delta\phi)^2 + m_1 m_2 (\Delta\psi)^2\} [\chi_1^{(2)} / (\Delta\phi)^2 + (m_2 \chi_2^{(2)} \\ &\quad + m_1 \chi_4^{(2)}) / \{m_1 m_2 (m_1 + m_2) (\Delta\psi)^2\} - \zeta_0^{(2)} / (2J_0^2)] \dots\dots\dots (44)\end{aligned}$$

$$\begin{aligned}\zeta_0^{(2)} &= m_1 m_2 (\Delta\phi)^2 (\Delta\psi)^2 / \{(\Delta\phi)^2 + m_1 m_2 (\Delta\psi)^2\} [(\zeta_1^{(2)} + g_1^{(2)}) / (\Delta\phi)^2 + (m_2 (\zeta_2^{(2)} + g_2^{(2)}) \\ &\quad + m_1 (\zeta_4^{(2)} + g_4^{(2)})) / \{m_1 m_2 (m_1 + m_2) (\Delta\psi)^2\} - g_0^{(2)}] \dots\dots\dots (45)\end{aligned}$$

Using the iterative technique established in the previous paper, the solutions of $\chi^{(2)}$ and $\zeta^{(2)}$ are obtained.

4.5 Results of calculation and considerations

Once the values of $\chi^{(2)}$ are determined according to the above-mentioned proce-

the boundary conditions are given by

$$\partial\chi^{(2)}/\partial\phi = 0, \quad \chi^{(2)} = 0 \dots\dots\dots (35)$$

(3) In the region sufficiently far from the notch ③④⑤ ;

The effects of the notch on the stress distributions are assumed to disappear there, hence, the conditions are

$$\chi^{(2)} = 0, \quad \zeta^{(2)} = 0 \dots\dots\dots (36)$$

(4) On the Y -axis ①⑤ ;

For the sake of symmetry, the conditions are given by

$$\partial\chi^{(2)}/\partial\phi = 0, \quad \partial\zeta^{(2)}/\partial\phi = 0 \dots\dots\dots (37)$$

Once the functions $\chi^{(2)}$ and $\zeta^{(2)}$ which satisfy Eqs.(32) and (33) and the conditions of Eqs.(34) to (37) are determined, then the solutions χ and ζ are completely obtained by adding $\chi^{(1)}$ to $\chi^{(2)}$ and $\zeta^{(1)}$ to $\zeta^{(2)}$, respectively. Clearly it follows that the solutions approximately satisfy the conditions that the displacements in the X -direction and the shear stress s_{xy} become zero in the region of $X \rightarrow \pm\infty$ and likewise $s_y = 0$, $s_{xy} = 0$ in the region of $Y \rightarrow \infty$.

dures, all stress components can be estimated with the values of χ by means of Eq. (26). To investigate the characteristic features of thermal stress concentrations due to notch shape, three types of a semi-

elliptic notch are chosen, that is, $\mu=0.5$, 0.75 for Model I and $\mu=2.0$ for Model II, respectively. Calculations of all the stress components are carried out for each type of notch under the conditions of $B=1$, 10 and 50 and $F=0.1$, 0.2, 0.3 and 0.4. In every case, the values of the stress s_ϕ , which is the normal stress component in the direction of streamline, are dominant in comparison with those of s_ψ and $s_{\phi\psi}$ in the whole region; hence, the stress s_ϕ is most significant. It is clearly recognized that the stress s_ϕ has a maximum value at the root of the notch and its stress gradient also becomes maximum there as shown in Figs.7 and 8. Therefore, the so-called thermal stress concentration phenomenon occurs there. The stress concentration factor K is defined by the following equation in the same way as in the previous paper;

$$K = s_{\phi_0} / s_0 \quad (46)$$

where s_{ϕ_0} : s_ϕ at the root of the notch,
 s_0 : longitudinal stress on the straight edge of a semi-infinite plate without a notch.

Fig.7 One example of stress distributions in terms of s_ϕ , normal stress component in the directions of streamlines

Fig.8 One example of stress distributions in terms of s_ϕ

Fig.9 Stresses at the root of the notch and at the straight edge

Fig.10 Stress concentration factor

Fig.9 shows the relation between s_{ϕ_0} and F , where B and μ are taken as parameters. A curve linking the round marks $\circ$ represents the hoop stress s_θ at the root of a semi-circular notch⁽²⁾, which corresponds to $\mu=1.0$. The curve of the reference stress s_0 is also indicated in the same figure. Fig.10 shows K - F relations with B and μ as parameters, where the values of a semi-circular notch are also plotted with the round marks $\circ$. It is clearly recognized from this figure that the K -value increases with an increase in F for all types of notch. Furthermore, it is also noticed that variations of B -value rapidly come to have little effects on the K -value with an increase in B . This fact was already pointed out in the previous paper. Then it follows that the values of K are much influenced by the shape of a notch and increase with an increase in μ -value. From the point of view in thermal stress concentrations, a semi-circular notch treated in the previous paper may be regarded as a considerably deep notch. The stress s_ϕ at the root of the notch, whose sign is not shown in the figure, is compressive in heating and tensile in cooling.

4.6 Stress concentration factors of a finite plate

Since plates for practical use have finite dimensions, it is necessary in designing to estimate the values of thermal stress concentration factors of such plates. However it is not easy to determine the values of K correctly in such cases. In what follows the values of K of a finite plate (a rectangular plate which is sufficiently long longitudinally compared with the width) may be estimated by using the approximate method proposed in the previous paper.

Fig. 11 Constraint force and moment

Let the semi-infinite plate be cut out at the ordinate Y_0 as shown in Fig. 11. Then the plate with the finite depth Y_0 is free from the constraint force F and the constraint moment M , of which the values may be calculated by the equations given in the previous paper. As the result the stresses at the root of the notch are released respectively. In the present case, it is assumed that the value of g at Y_0 is very small and that Y_0 is not so small compared with the dimension of the notch. Furthermore, assuming that the boundary $Y=Y_0$ is approximately free from stress, the effects on the stresses at the root of the notch caused by releasing the constraint force and moment may be estimated in terms of shape factors based on Neuber's method⁽¹⁾.

Calculations for the plates with various widths are carried out for the cases of $B=50$, 10 and 1 on the basis of the above consideration. The results are shown in Fig. 12. In the figure the reference stress s_0 is the edge fiber stress of the heating surface in a semi-infinite plate without a notch. These figures show that the values of K in the notched finite plates reduce with a decrease in Y_0 when B -value and F -value are kept constant. Clearly it follows that an increase of μ produces larger values of the shape factor

Fig. 12 Influence of effect of plate depth on K -value

at the root of the notch. Hence it also causes an increase of correcting quantity of K -value and consequently the features of K -value for notched finite plate do not necessarily correspond with μ -value. For example, Fig.(12)(d) shows the effects of the width on K -value in a finite plate for the cases $Y_0=4$ to 10000 when $B=50$ and $F=0.4$. From this figure it is found that K -value in the case $\mu=1.0$ is larger than that of $\mu=2.0$ in the range of $Y_0<19$.

Since the estimation method proposed here is not based on a strict analysis but simply derived from the solutions of a semi-infinite plate, it should be applied to a plate, which has relatively large width compared with the depth of the notch and little change of temperatures reaches the non-heated surface. Accordingly it is not adequate to apply such a simple correcting method to a plate with a deep notch such as $Y_0 \leq 6$ for Model II ($\mu=2.0$).

5. Conclusions

A semi-infinite plate with a semi-elliptic notch is treated and the effects of the shape of the notch on the stress concentrations in the transient temperature field are examined systematically. First a semi-infinite plate is investigated numerically and then the thermal stress concentration factors of a finite plate are approximately estimated. In the solving process the temperature and stress fields are respectively resolved into two components, that is, the solution for a semi-infinite plate without a notch and the solution considering the notch effect. Such two types of fields can be determined separately and the sum of them results in the desired solutions. The above-mentioned separation method is based on the consideration that the effects of the notch can be notably observed near the notch and disappear in the region sufficiently far from the notch. To obtain the solutions considering the notch effect, the curvilinear coordinates composed of equipotential lines and streamlines related to the flow around an elliptic cylinder placed in the uniform stream are introduced and then conformal transformation is performed, where the mediate elliptic coordinates are used for convenience of formulations. All the calculations are carried out by using the finite-difference method and reasonable solutions are obtained.

The results obtained are summarized below.

- (1) The pattern of the isotherms is similar to that of the streamlines.
- (2) The dominant stress is a normal stress component acting in the direction of streamlines and it becomes maximum (acting as compressive in heating and as tensile in cooling) at the root of the notch, where the stress gradient also shows a maximum value and a phenomenon of thermal stress concentrations can be clearly observed.
- (3) The effects of Fourier number F and Biot number B on the thermal stress concentration factor K are qualitatively the same as in the case of a semi-circular notch discussed in the previous paper. That is, the effect of B -value on K -value rapidly decrease when $B>10$. Furthermore K -value increases with F -value. K -value becomes smaller as the depth of the notch decreases, while it becomes larger with an increase of the notch depth. It may be noted that a semi-circular notch has the characteristics of a relatively deep notch.
- (4) It has been found that the K -value of a notched finite plate is remarkably influenced by the plate width as the notch depth increases in comparison with a semi-circular notch.

The numerical calculations of this research have been performed at the Computation Center of Osaka University.

References

- (1) Berry, W. R. and Johnson, I., Trans. ASME, Ser. A, Vol.86, No.3 (1964-7), p.361.
- (2) Furukawa, M., et al., Bull. of the JSME, No.129 (1976), p.239.
- (3) Tsuzi, M., et al., Trans. Japan Soc. Mech. Engrs. (in Japanese), Vol.43, No.369 (1977-5), p.1592.
- (4) Hamada, M. and Kitagawa, H., Bull. of the JSME, Vol.11, No.44 (1968), p.197.
- (5) Coker, E. G. and Filon, L. N. G., A Treatise on Photoelasticity, (1931), p.159, Cambridge Univ. Press.
- (6) Milne-Thomson, L. M., Theoretical Hydrodynamics, (1938), p.158, Macmillan.
- (7) Carslaw, H. S. and Jaeger, J. C., Conduction of Heat in Solids, (1959), p.72, Oxford Univ. Press.
- (8) Katayama, K. and Saito, A., Trans. Japan Soc. Mech. Engrs. (in Japanese), Vol.34, NO.268 (1968-12), p.2165.
- (9) p.163 of Ref.(5).
- (10) Neuber, H., Kerbspannungslehre, (1937), p.6, Springer-Verlag.