Fluid Inertia Effects on the Characteristics of Spherical Hydrostatic Bearings (Part 1 Bearing Forces) *

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Based on a first-order perturbation solution in a modified Reynolds number, an analysis is presented to determine the fluid inertia effects on the dynamic characteristics of a spherical hydrostatic bearing that has a continuous spherical surface. This analysis demonstrates that fluid inertia influences the load capacity and dynamic properties. The corrections to the conventional elastic and damping coefficients derived on the assumption that inertia forces are negligible, are found not too small to neglect when inertia parameter $\delta = \omega^2 h^2 / \rho a > 0.1$ ($\omega =$lubricant density, $\omega =$angular velocity, $\rho =$radius of spherical bearing, $a =$ambient pressure). And the acceleration coefficients may become significant when the displacement of a rotor from the bottom of the bearing is small.

Key Words: Lubrication, Vibration of Rotating Body, Spherical Hydrostatic Bearing, Inertia Effect, Dynamic Characteristics

1. Introduction

Classical lubrication theory assumes a laminar flow and neglects inertia terms in the equation of motion governing a lubricant film flow. Although these assumptions are justified for small values of the Reynolds number, they are valid in the majority of bearing applications. However, inertia effects may become noticeable at high operating speeds without affecting the assumption of laminar flow. So far several investigators have examined the fluid inertia effects on the static characteristics of spherical hydrostatic bearings. Dowson and Taylor have carried out theoretical and experimental works and concluded that the inertia force alters the pressure distribution considerably while the inertia effects on load capacity are rather small.

This paper describes the inertia effects on the dynamic characteristics of spherical hydrostatic bearings. This analysis, which employs a first order perturbation expansion in Reynolds number, demonstrates that fluid inertia effects on the dynamic properties appear even for small values of Reynolds number.

2. Nomenclature

$\alpha_{ac}$ : acceleration coefficient

$\delta = \omega^2 h^2 / \rho a$ 

$b_{ii}(i,j=x,y) :$ damping coefficients

$B_{ii} = b_{ii}(e_0,0)/p^{*}$$e_0$ : radial clearance

$e_{r,a} :$ radial and axial displacements

$h = \text{film thickness} (R = h/c)$

$h_{ii}(i,j=x,y) :$ elastic coefficients

$(c_{ii} = h_{ii}(e_0,0)/p^{*})$ 

$p :$ pressure distribution ($P = p/p^{*}$)

$p_{i} :$ pressure at inlet ($P_i = p_i/p^{*}$)

$p^{*} :$ ambient pressure

$r :$ bearing radius

$R = r_1/r_2$ 

$r, \theta, \phi :$ spherical coordinates (Fig. 1)

$R_1, \theta_1, \phi_1 :$ spherical coordinates (Fig. 1)

$\tau :$ time ($\tau = \omega t$)

$v_r, v_{\theta}, v_{\phi} :$ velocity components in $r, \theta, \phi$

$V_r = v_r/r_2$, $V_{\theta} = v_{\theta}/r_2$, $V_{\phi} = v_{\phi}/r_2$

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Fig. 1 Spherical hydrostatic bearing
3. Analysis

3.1 The governing equations

This paper is concerned with a spherical hydrostatic bearing of the form shown in Fig.1. With the assumptions of usual lubrication theory, the governing equations for an incompressible fluid are written in dimensionless form as:

\[ \frac{\partial p}{\partial z} = 0 \]

\[ R_s \frac{\partial^2 V_x}{\partial z^2} - V_x \frac{\partial^2 V_y}{\partial z \partial \theta} - V_x \frac{\partial V_y}{\partial \theta} = \frac{A_s}{\sin \theta} \frac{\partial^2 V_x}{\partial \theta^2} \]

\[ R_s \frac{\partial^2 V_y}{\partial z^2} + V_x \frac{\partial^2 V_y}{\partial z \partial \theta} + V_x \frac{\partial V_y}{\partial \theta} + V_y \frac{\partial^2 V_y}{\partial \theta^2} = \frac{A_s}{\sin \theta} \frac{\partial^2 V_y}{\partial \theta^2} \]

\[ \frac{\partial V_x}{\partial z} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( V_y \sin \theta + \frac{\partial V_y}{\partial \theta} \right) = 0 \]

where \( A_s = \frac{c_s}{\rho \omega^2} \) and \( P_s = c_s \frac{\partial p}{\partial \theta} \).

The left hand sides in Eqs. (1) and (3) are the contributions from inertia forces which have been ignored in the classical theory. A parameter \( Re_s \) accounts for the inertia forces, and it is generally a small value. Equation (2) only expresses that the pressure does not vary across the film, and is of no interest in further analysis. To investigate the inertial effects, a first-order perturbation solution in \( Re_s \) is carried out by setting \( \psi, V_{x0}, V_{y0} \). Substituting Eqs. (5) and (6) into Eqs. (2) through (4) and rearranging the terms of zeroth and first order in \( Re_s \) yields the following equations:

\[ (R_s)^{*} = -\frac{c_s}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( V_{y0} \sin \theta \right) = 0 \]

\[ \frac{\partial}{\partial \theta} \left( V_y \sin \theta \right) = 0 \]

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The velocity boundary conditions are:

\[ \zeta = 0; \quad V_{x0} = 0; \quad V_{y0} = 0; \quad \zeta = H; \quad V_{x0} = \psi \sin \theta; \quad V_{y0} = -\frac{\partial H}{H^2} \frac{\partial \psi}{\partial \theta}; \quad \psi = 0 \]

By substituting these expressions into Eqs. (6), the boundary conditions are written as:

\[ (R_s)^{*} = \frac{\partial V_{y0}}{\partial \theta} + V_{y0} \frac{\partial \psi}{\partial \theta}; \quad \psi = 0 \]

\[ (R_s)^{*} = \frac{\partial V_{y0}}{\partial \theta} + V_{y0} \frac{\partial \psi}{\partial \theta}; \quad \psi = 0 \]

Integrating Eqs. (7) and (8) twice yields the velocity profiles of classical lubrication theory:

\[ V_{x0} = \frac{c_s}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) \]

\[ V_{y0} = \frac{A_s}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) \]

Inserting \( V_{x0} \) and \( V_{y0} \) into the continuity equation (9), and applying the boundary condition at \( \zeta = H \), we obtain \( V_{y0} \) and a differential equation for \( \psi \) as:

\[ \frac{\partial V_{y0}}{\partial \theta} = \frac{A_s}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) = \frac{2}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) \]

Since \( \psi \) is independent of \( \zeta \), integrating Eqs. (10) and (11) twice with respect to \( \zeta \), and using Eqs. (16) through (18), we obtain \( V_{y0} \) and \( V_{y0} \). Substituting \( V_{y0} \) and \( V_{y0} \) into Eq. (12) and applying the boundary conditions Eqs. (15), we get a differential equation for \( \psi \) as:

\[ \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) = \frac{1}{\rho \omega^2} \frac{\partial}{\partial \theta} \left( \frac{1}{2} \psi \left( \psi - \frac{H^2}{2} \right) \right) \]

3.2 Bearing forces

We will treat radially unloaded bearings. Then a shaft axis coincides with the z-axis in a static equilibrium state. We assume that the pressure distributions by small vibrations about z-axis can be written in the following forms:

\[ P_{z} = P_{z0} + r \psi + r \frac{\partial \psi}{\partial r} \]

\[ P_{r} = P_{r0} + r \psi + r \frac{\partial \psi}{\partial r} \]
where: $\text{ed/dt}$. Substituting these expressions into Eqs. (19) and (20), together with equations:

\[
\begin{align*}
H &= H_0 e^{i \theta} \cos \theta / \sin \theta, \\
H &= 1 - e^{i \theta} \cos \theta 
\end{align*}
\]  

and collecting terms of like order results in zeroth and first order equations for $P_j$ ($1 \leq 0, 1, j = 0 \sim 7$).

The differential equations for $P_0$ and $P_0$ are written as:

\[
\begin{align*}
\sin \theta \frac{\partial}{\partial \theta} \left[ H_0 \sin \theta \frac{\partial P_0}{\partial \theta} \right] &= 0 \\
\sin \theta \frac{\partial}{\partial \theta} \left[ H_0 \sin \theta \frac{\partial P_0}{\partial \theta} \right] &= \frac{12}{A} \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta S_a \right)
\end{align*}
\]

where

\[
S_a = - \frac{1}{120} \left[ H_0 \frac{\partial P_0}{\partial \theta} \frac{\partial P_0}{\partial \theta} + 2H_0 \frac{\partial P_0}{\partial \theta} \left( \frac{\partial P_0}{\partial \theta} \right)^2 \right] - \frac{H_0}{40} \sin \theta \cos \theta
\]

Integrating Eqs. (24) and (25) twice with respect to $\theta$ and applying the boundary conditions:

$\theta = \theta_0 : P_0 = P_a, \quad P_0 = 0, \quad \theta = \theta_0 : P_a = 1, \quad P_a = 0$

we obtain

\[
\begin{align*}
P_0 &= \frac{3}{30} \int_0 \left( \sin \theta \sec \theta \right) - \left( \sin \theta \sec \theta \right) \frac{K(\theta)}{K(\theta)} \\
K(\theta) &= \int_0^1 \left( \sin \theta \sec \theta \right) \frac{P_0}{H_0 \sin \theta}
\end{align*}
\]

where

$P_0$ is the pressure distribution due to inertia effects. When we consider only centrifugal effects, the second term in the right hand side of Eq. (28) vanishes \cite{2}.

The axial load capacity $F_a$ is calculated from the following expressions:

\[
F_a = \int_0^1 \left( P_a - 1 \right) \sin \theta \cos \theta \, d\theta
\]

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\]

where

\[
F_a = \frac{3}{30} \int_0^1 \left( \sin \theta \sec \theta \right) - \left( \sin \theta \sec \theta \right) \frac{K(\theta)}{K(\theta)} \\
K(\theta) &= \int_0^1 \left( \sin \theta \sec \theta \right) \frac{P_0}{H_0 \sin \theta}
\]

In this paper we set $\theta_0 = 10^\circ$ and $\theta_0 = 30^\circ$. The influence of these values on the load capacity and the dynamic properties will be discussed in the following paper.

The dependence of load capacity $F_a$ on $\epsilon_3$ at different values of $\Delta$ is shown in Fig. 3. The load capacities are dependent on $F_a$ and not on $\Lambda$. The load capacity increases with the axial displacement, i.e., $\Lambda(1-\epsilon_3)$. Fig. 3 shows $F_a$ as a function of

![Fig. 2 F_a versus $\epsilon_3$](image1)

![Fig. 3 F_a versus $\epsilon_3$](image2)
As the axial displacement increases, \( F_{ax} \) becomes positive from negative. Thus, for large values of axial displacement, i.e., \( \varepsilon_z < 0 \), fluid inertia effects increase load capacity \( F_{ax} \) since \( F_{ax} \) is positive. When the rotor speed is high, the value of \( \lambda \) is generally low. Under such a condition, the second term in the right hand side of Eq. (34) is much smaller than the first term which is a contribution from the centrifugal force only. In other words, the effect of centrifugal force on load capacity is much greater than that of the other inertial forces.

Considering \( E_{ij} \) (i=0,1, j=1~7) as functions with a period of 2\( \pi \) with respect to \( \varphi \), we set

\[ P_i = P_i(\theta) \cos \varphi + P_i(\theta) \sin \varphi \]  

.....(35)

Then, the differential equations for \( E_{ij} \) are written as

\[ L_i(P_i) = R_i(\theta) \]  

\[ (i=0, 1, j=1~7, k=\varepsilon, z) \]  

.....(36)

where

\[ L_i(P_i) = \frac{d}{d\theta}(R_i \sin \theta \frac{dP_i}{d\theta}) - \frac{R_i}{\sin \theta} P_i \]  

.....(37)

and \( R^e_i \) and \( R^z_i \) are functions of \( \theta \). Assuming the solution of Eqs. (36) to be written in the form:

\[ P_i = \sum_{n} \left( a_n P_{n\theta} + b_n P_{n\phi} \right) \sin \left( \theta - \delta_n \right) \]  

.....(38)

since the boundary conditions are

\[ \theta = 0; \quad P_i = 0 \]  

\[ \theta = \theta_0; \quad \delta = 0 \]  

we obtain \( P_{n\theta} \) and \( P_{n\phi} \) by Galerkin's method. The final solution for \( P \) is given in the form of Eq. (5).

The radial and tangential bearing forces \( F_r \) and \( F_\theta \) are calculated from the following expressions:

\[ F_r = \int_{0}^{2\pi} \int_{0}^{\infty} \rho \cos \varphi \sin \varphi \, r \, dr \, d\varphi \]  

.....(39)

By introducing an x-y coordinate system with origin in the bearing center as shown in Fig.4, the bearing forces in the x- and y-directions are:

\[ F_x = F_r \cos \phi - F_\theta \sin \phi \]  

\[ F_y = F_r \sin \phi + F_\theta \cos \phi \]  

.....(40)

Considering the relations:

\[ \varepsilon_z = \varepsilon \cos \phi \]  

\[ \varepsilon_z = \varepsilon \sin \phi \]  

.....(41)

we obtain \( F_{ax} \) and \( F_{ay} \) as:

\[ F_{ax} = K_{ax} \varepsilon_x + K_{ax1} \varepsilon_{x1} + K_{ax2} \varepsilon_{x2} \]  

\[ F_{ay} = K_{ay} \varepsilon_y + K_{ay1} \varepsilon_{y1} + K_{ay2} \varepsilon_{y2} \]  

.....(42)

where

\[ K_{ax} = K_{ax0} + R^e \varepsilon_{ax0} \]  

\[ K_{ax1} = K_{ax10} + R^e \varepsilon_{ax10} \]  

\[ K_{ax2} = K_{ax20} + R^e \varepsilon_{ax20} \]  

.....(43)

\[ B_{ax} = B_{ax0} + R^e \varepsilon_{ax0} \]  

\[ B_{ax1} = B_{ax10} + R^e \varepsilon_{ax10} \]  

\[ B_{ax2} = B_{ax20} + R^e \varepsilon_{ax20} \]  

.....(44)

\[ A_{ax} = R^e \varepsilon_{ax0} \]  

\[ A_{ax1} = A_{ax10} \]  

.....(45)

In Eqs. (43) through (45), the terms multiplied by \( \varepsilon \) are corrections due to fluid inertia effects. In addition to the usual elastic and damping coefficients, Eq. (42)

\[ K_{xy} \]  

.....(46)

\[ B_{xy} \]  

.....(47)

\[ A_{xy} \]  

.....(48)

\[ K_{yx} \]  

.....(49)

\[ B_{yx} \]  

.....(50)

\[ A_{yx} \]  

.....(51)

\[ K_{xx} \]  

.....(52)

\[ B_{xx} \]  

.....(53)

\[ A_{xx} \]  

.....(54)

Fig. 6 Elastic coefficient \( K_{xy} \) and damping coefficient \( B_{xx} \) (\( Pa=2 \))

Fig. 5 Elastic coefficient \( K_{xx} \) (\( Pa=2 \))

Fig. 4 Radial and tangential components of lubricant film forces
implies that the bearing forces also depend on the rotor accelerations. These acceleration coefficients $A_{xx}$ and $A_{yy}$ vanish when fluid inertia effects are neglected.

In Figs. 5 through 12 are shown the elastic, damping and acceleration coefficients as functions of $\xi_2$ and $P_0$. $K_{xx0}$ depends not on $\Lambda$ but on $P_0$. Although $K_{xy0}$ and $K_{yx0}$ depend on $\Lambda$, $M_{xy0}$ and $M_{yx0}$ are nearly constant for values of $\Lambda<0.1$. Similarly the coefficients $N_{xx0}$, $K_{xy0}$, $B_{xx0}$, $B_{xy0}$ and $M_{xx0}$, which are due to fluid inertia effects, are considered to be independent of $\Lambda$ for values of $\Lambda<0.1$. Fig. 5 shows that $K_{xx0}$ and $K_{xxz}$ increase with the axial displacement, i.e., $C_0(1-\xi_2)$. When the axial displacement is large, $K_{xxz}$ is positive. In other words, the inertia effects increase the radial stiffness $K_{xxz}$ for $\xi_2<0$. On the other hand, when the axial displacement is small, $K_{xxz}$ is negative. Thus, the inertia effects decrease the radial stiffness for $\xi_2>0$. In Fig. 6, as the axial displacement increases, $-K_{xy}$ increases while $M_{xy0}$ decreases. The cross coupling elastic coefficient $K_{xy}=K_{xy0}+R_e^2 K_{xy}$ decreases as the axial displacement increases. Fig. 7 shows that the cross-coupling damping coefficients $B_{xy}=B_{xy0}+R_e^2 B_{xy}$ appears when $Re^4/\Lambda$, that is, when the inertia effects are not negligible. The acceleration coefficient $M_{xxz}$ is shown in Fig. 8. As the axial displacement increases, $M_{xxz}$ decreases rapidly at first and then slowly at larger axial displacement.

Figs. 9 and 10 show that $K_{xx0}$, $-K_{xy}$ and $B_{xxz}$ increase linearly with $P_0$, while the other coefficients, $M_{xy0}$, $M_{xx0}$, $M_{xxz}$ and $M_{xxz}$ are independent of $P_0$ as shown in Figs. 11 and 12.

Rewriting Eqs. (43), (44) and (45), we get

$$B_{xx}=\frac{B_{xx0}+R_e^2 B_{xx0}}{\Lambda}, \quad B_{yy}=\frac{B_{yy0}}{\Lambda}$$

$$K_{xx}=K_{xx0}+\delta K_{xx1}$$

$$K_{xy}=\frac{K_{xy0}+R_e^2 K_{xy0}}{\Lambda}$$

$$A_{xx}=\delta A_{xx1}$$

Fig. 7 Damping coefficient $B_{xy}$

Fig. 9 $K_{xx0}$ versus $P_0$

Fig. 8 Acceleration coefficient $A_{xx}$

Fig. 10 $K_{xy}$ and $B_{xxz}$ versus $P_0$
where

\[ \delta = \frac{R_v}{A} \] .................................(47)

and \( \beta_{xx} = \lambda \beta_{xy}, \beta_{yy} = \lambda \beta_{yx}, \lambda = \frac{K_{xx}}{K_{xy}}, \lambda = \frac{K_{xy}}{K_{xx}}. \) \( \Delta \) and \( \Delta^* \) are small values, and as shown in Fig.6, \( \Delta \) and \( \Delta^* \) are much smaller than \( \lambda \beta_{xy} \) and \( \lambda \beta_{yx} \), respectively. Therefore we make approximations:

\[ \beta_{xx} = \lambda \beta_{xy}; \Delta = \beta_{xy} \] .................................(48)

Expressions (48) show that the fluid inertia effects on the coefficients \( \beta_{xx} \) and \( \beta_{yy} \) are small.

As mentioned above, \( K_{xy} = \lambda K_{xx}, \beta_{xy} = \lambda \beta_{yx}, \lambda = K_{xx}/K_{xy} \) and \( \lambda = K_{yy}/K_{xy} \) are independent of \( \Delta \). Thus, \( K_{xx}, \beta_{xy} \) and \( \beta_{yx} \) vary linearly with \( \delta \). Calculations are performed for a bearing whose bearing parameters are:

\( D = 10 \text{mm}, C = 15 \text{mm} \).

Setting \( \mu = 0.1 \text{MPa}, \rho = 0.9 \times 10^3 \text{kg/m}^3 \), and \( \mu = 0.01 \text{Pa.s}, \) we get \( \Delta = 0.01 \) and \( \Delta^* = 0.009 \) for \( \omega = 10000 \text{rad/s}, \) and \( \Delta = 0.001 \) and \( \Delta^* = 0.09 \) for \( \omega = 10000 \text{rad/s}. \) In Table 1, the values of the three coefficients are listed. It can be noted that the values of these coefficients vary considerably even when \( \Delta^* \) is small.

### Table 1: Changes of coefficients with \( \delta \) (\( \Delta^* = 2, \Delta^* = 1 \))

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>0</th>
<th>0.9</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>0.01</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>( \lambda^* )</td>
<td>0.009</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td>( K_{xx} )</td>
<td>0.475</td>
<td>0.555</td>
<td>8.50</td>
</tr>
<tr>
<td>( \beta_{xy} )</td>
<td>0.113</td>
<td>11.3</td>
<td></td>
</tr>
<tr>
<td>( \beta_{yx} )</td>
<td>0.227</td>
<td>22.7</td>
<td></td>
</tr>
</tbody>
</table>

### 4. Conclusions

1. The effect of centrifugal force on load capacity is much greater than that of the other inertial forces.
2. The coefficients \( K_{xx} \) and \( \beta_{xy} \) vary linearly with \( \delta = \omega \text{Re}^* \).
3. The inertial effects on \( K_{xx} \) and \( \beta_{xy} \) are quite limited.
4. When inertial forces are considered, the bearing forces depend not only on the rotor displacement and velocity but also on its acceleration. The acceleration coefficients \( \beta_{xx} \) and \( \beta_{yy} \) increase linearly with \( \delta \).

### Appendix

\[ S_1 = \sin \theta = \frac{-H/2}{T} + T, \quad S_2 = \frac{-H}{2} T - T_2. \]

\[ T_1 = \frac{A_1}{2} \begin{bmatrix} H & 2 \Pr & H & 2 \Pr & 2 \Pr & 3 \Pr & \frac{A_1}{2} & H & 2 \Pr & 2 \Pr & 3 \Pr & 2 \Pr \end{bmatrix} \]

\[ + \frac{A_2}{2} \begin{bmatrix} \frac{H^2}{2} & H^2 & 2 \Pr^2 & 2 \Pr^2 & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} \end{bmatrix} \]

\[ + \frac{A_3}{2} \begin{bmatrix} \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} \end{bmatrix} \]

\[ - \frac{H^2}{2} \begin{bmatrix} \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} \end{bmatrix} \]

\[ + \frac{A_4}{2} \begin{bmatrix} \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} & \frac{H^3}{3} \end{bmatrix} \]

\[ - \frac{H^2}{2} \begin{bmatrix} \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} \end{bmatrix} \]

\[ - \sin \theta \begin{bmatrix} \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} \end{bmatrix} \]

\[ - \cos \theta \begin{bmatrix} \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} & \frac{H^2}{2} \end{bmatrix} \]
References