Bifurcation of an Elastic Slab having Inclined Anisotropy-Axes under Axial Loads*

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Bifurcations from a homogeneous, finite, stretch-shear-combined deformation are studied for a transversely isotropic, compressible, elastic slab having inclined anisotropy-axes which is subjected to uniform axial loads, tensile and compressive. The bifurcations are analysed on the basis of the incremental deformation theory. For various wavelength-to-thickness ratios, the critical loads are shown in the form of curves as a function of anisotropy-axis inclination.

Key Words: Elasticity, Bifurcation, Instability, Transverse Isotropy, Inclined Anisotropy-Axis, Critical Load, Finite Deformation, Incremental Deformation

1. Introduction

There are many papers concerning bifurcation (instability) problems, even if our attention is called only to those for an elastic slab (or a thick plate or a rectangular block). Most of these papers considered a slab made of isotropic and incompressible materials.⁽¹⁻⁶⁾ The isotropic and compressible case was treated by Burgess and Levinson,⁽⁷⁾ Hill and Hutchinson,⁽⁸⁾ and Young⁽⁹⁾ considered a wide class of orthotropic and incompressible, incrementally linear materials. The author examined the bifurcation of a slab of a special, orthotropic (transversely isotropic) and compressible material.⁽¹⁰⁾

In the above orthotropic cases,⁽⁸⁻¹⁰⁾ the anisotropy-axes coincided with the geometrical axes. The present paper deals with the case where the principal anisotropy-axis inclines from the slab axis.

The slab considered here is made of a transversely isotropic elastic material having a special form of strain-energy function proposed in the previous paper.⁽¹⁰⁾ Its principal anisotropy axis inclines from the slab axis at the angle of $\pi/2 - \theta$. The slab is axially loaded by the uniform stress T_0 at infinity. In spite of this uniform stress loading, the initial finite deformation includes shear deformation as well due to the anisotropy-axis inclination. The bifurcation from this stretch-shear-combined deformation is analyzed on the basis of the incremental deformation theory. The slab is assumed to be in a plane-strain state prior to and after bifurcation.

Most of the basic equations associated with the initial finite deformations and the superposed infinitesimal deformations of the

transversely isotropic material are omitted in the present paper for the sake of brevity. All of these equations are given in the previous paper.⁽¹⁰⁾

2. Initial Homogeneous Finite Deformation

Let $\bar{X}(X,Y,Z)$ and $\bar{x}(x,y,z)$ denote, respectively, the material and spatial coordinates of a material point in a body as referred to the same Cartesian system. Consider an infinite elastic slab, whose center line

Fig.1 A transversely isotropic slab: (a) Natural state, (b) Homogeneously deformed state, and (c) Bifurcated state

coincides with the Y-axis in its natural state and with the y-axis in its deformed state, as shown in Figs.1 (a) and (b). The finite deformation due to an axial load, T_0 (Cauchy stress), changes the thickness, $2H$, of the slab and the direction, θ , of its anisotropy-axis into $2h$ and θ , respectively.

The deformation gradient F corresponding to the homogeneous plane deformation shown in Fig.1 (b) is given by

$$F = \begin{bmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1)$$

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where all components are constant. It follows from the definition of the left Cauchy-Green deformation tensor, $\underline{B} = \underline{F}\underline{F}^T$, that

$$\underline{B} = \begin{bmatrix} F_{11}^2 & F_{11}F_{21} & 0 \\ F_{11}F_{21} & F_{22}^2 + F_{21}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2)$$

whose inverse tensor is

$$\underline{B}^{-1} = \text{III}^{-1} \begin{bmatrix} F_{22}^2 + F_{21}^2 & -F_{11}F_{21} & 0 \\ -F_{11}F_{21} & F_{11} & 0 \\ 0 & 0 & \text{III} \end{bmatrix} \quad (3)$$

with

$$\text{III} = \det \underline{B} = (F_{11}F_{22})^2. \quad (4)$$

If $\underline{H}$ denotes a unit vector in the principal anisotropy-axis direction in the natural state, that is,

$$\underline{H} = [\cos \Theta \quad \sin \Theta \quad 0]^T, \quad (5)$$

then the deformation carries this unit vector into

$$\underline{h} = \underline{F}\underline{H} = \begin{bmatrix} F_{11}\cos \Theta \\ F_{21}\cos \Theta + F_{22}\sin \Theta \\ 0 \end{bmatrix} \quad (6)$$

The second, third, and fourth principal invariants of $\underline{B}$ under rotation about the preferred direction $\underline{H}$ are easily obtained as

$$\left. \begin{aligned} \text{II} &= (F_{11}F_{22})^2, & \text{III} &= (F_{11}F_{22})^2, \\ \text{IV} &= (F_{11}\cos \Theta)^2 + (F_{21}\cos \Theta + F_{22}\sin \Theta)^2, \end{aligned} \right\} \quad (7)$$

respectively.

Let us assume that the slab is made of a transversely isotropic elastic material having a special form of strain-energy density function proposed in the previous paper.⁽¹⁰⁾ The proposed form was

$$W = (\mu_0/2) \{ (\text{II}/\text{III} - 3) + 2(\sqrt{\text{III}} - 1) + (k/2)(\text{IV} - 1)^2 \}, \quad (8)$$

where μ_0 is the shear modulus in the natural state, and k is a nondimensional material constant characterizing the strength of the anisotropy: when $k=0$, the material is isotropic and in the limit $k \rightarrow \infty$, the material reduces to ideal fiber-reinforced composites under certain conditions.⁽¹¹⁾ It follows from the above strain-energy function that

$$\underline{T} = \mu_0 \{ [1 - (1/\sqrt{\text{III}})\underline{B}^{-1} + (k/\sqrt{\text{III}})(\text{IV} - 1)\underline{h}\underline{h}^T], \quad (9)$$

where tensor $\underline{1}$ and symbol $\otimes$ denote the identity transformation and the tensor product, respectively. Referring to boundary conditions $T_{11} = T_{12} = 0$ and loading condition $T_{22} = T_0$, Eq. (9) yields

$$\left. \begin{aligned} 1 - (1/\sqrt{\text{III}})(\underline{B}^{-1})_{11} + (k/\sqrt{\text{III}})(\text{IV} - 1)h_1^2 &= 0, \\ - (1/\sqrt{\text{III}})(\underline{B}^{-1})_{12} + (k/\sqrt{\text{III}})(\text{IV} - 1)h_1h_2 &= 0, \\ 1 - (1/\sqrt{\text{III}})(\underline{B}^{-1})_{22} + (k/\sqrt{\text{III}})(\text{IV} - 1)h_2^2 &= T_0/\mu_0. \end{aligned} \right\} \quad (10)$$

These are equations for F_{11} , F_{22} , and F_{21} with the help of Eqs. (3), (6), and (7).

3. Bifurcation Analysis

In order to examine whether the slab bifurcates at a certain value of the applied load from the homogeneous finite deformation obtained in the previous section, let us

superpose a nonhomogeneous infinitesimal plane deformation on a finite one. The displacement vector associated with this infinitesimal deformation is denoted by $\underline{u}(u, v, 0)$ where $u = u(x, y)$ and $v = v(x, y)$.

From the stress-deformation relation (9), the incremental stress-incremental strain relation of the transversely isotropic elastic material under consideration is easily obtained as

$$\underline{s} = (\mu_0/\sqrt{\text{III}}) \{ \underline{e}\underline{B}^{-1} + \underline{B}^{-1}\underline{e} + \underline{B}^{-1}\text{tr} \underline{e} + k(2\underline{h}\underline{h}^T \text{tr}(\underline{e}\underline{h}\underline{h}^T) + (\text{IV} - 1)(\underline{e}\underline{h}\underline{h}^T + \underline{h}\underline{e}\underline{h}^T - \underline{h}\underline{h}^T \text{tr} \underline{e})) \}, \quad (11)$$

where incremental stress tensor $\underline{s}$ and incremental strain tensor $\underline{e}$ are defined as

$$\underline{s} = \frac{\underline{T}}{\Delta t}, \quad \underline{e} = \frac{1}{2}((\partial \underline{u}/\partial \underline{x}) + (\partial \underline{u}/\partial \underline{x})^T), \quad (12)$$

with $\underline{T}$ and Δt denoting the Jaumann stress rate and a time increment, respectively. In the particular circumstances under consideration, Eq. (11) reduces to

$$\left. \begin{aligned} s_{11} &= \phi_{11}e_{11} + \phi_{12}e_{22} + \phi_{13}e_{12}, \\ s_{22} &= \phi_{21}e_{11} + \phi_{22}e_{22} + \phi_{23}e_{12}, \\ s_{12} &= \phi_{31}e_{11} + \phi_{32}e_{22} + \phi_{33}e_{12}, \end{aligned} \right\} \quad (13)$$

where ϕ_{11} , ϕ_{12} , etc. are given in the Appendix.

Since the nonvanishing component of $\underline{T}$ is only $T_{22} = T_0$, the equilibrium equations for the incremental stress components are

$$\left. \begin{aligned} \partial s_{11}/\partial x + \partial s_{12}/\partial y + T_0 \omega_{12}/\partial y &= 0, \\ \partial s_{12}/\partial x + \partial s_{22}/\partial y + T_0 \omega_{12}/\partial x &= 0, \end{aligned} \right\} \quad (14)$$

where ω_{12} is the local rotation about the z -axis associated with the incremental deformation and is given by

$$\omega_{12} = \frac{1}{2}(\partial u/\partial y - \partial v/\partial x). \quad (15)$$

Since the slab is free from traction at its surfaces, the boundary conditions for the incremental deformation are expressed as

$$\left. \begin{aligned} \Delta f_1 = s_{11} &= 0, \\ \Delta f_2 = s_{12} - T_0 e_{12} &= 0, \end{aligned} \right\} \text{ at } x = \pm h, \quad (16)$$

where h is a half thickness of the slab after the initial finite deformation.

By using Eqs. (12) and (13), the governing Eqs. (14) and boundary conditions (16) are expressed in terms of u and v as

$$\left. \begin{aligned} \partial^2 u/\partial x^2 + \psi_{11}\partial^2 u/\partial x\partial y + \psi_{12}\partial^2 u/\partial y^2 \\ + \psi_{13}\partial^2 v/\partial x^2 + \psi_{14}\partial^2 v/\partial x\partial y + \psi_{15}\partial^2 v/\partial y^2 &= 0, \\ \partial^2 v/\partial x^2 + \psi_{21}\partial^2 v/\partial x\partial y + \psi_{22}\partial^2 v/\partial y^2 \\ + \psi_{23}\partial^2 u/\partial x^2 + \psi_{24}\partial^2 u/\partial x\partial y + \psi_{25}\partial^2 u/\partial y^2 &= 0, \\ \phi_{11}\partial u/\partial x + \phi_{12}\partial v/\partial y + \frac{1}{2}\phi_{13}(\partial u/\partial y + \partial v/\partial x) &= 0, \\ \phi_{31}\partial u/\partial x + \phi_{32}\partial v/\partial y + \frac{1}{2}(\phi_{33} - T_0)(\partial u/\partial y + \partial v/\partial x) &= 0, \end{aligned} \right\} \text{ at } x = \pm h, \quad (18)$$

where ψ_{11} , ψ_{12} , etc. are given in the Appendix.

Let us assume solutions for a system of differential equations (17) in the following form:

$$u = A \exp[i\pi(\zeta x + y)/\ell], \quad v = B \exp[i\pi(\zeta x + y)/\ell], \quad (19)$$

where A and B are constants, and ℓ , the half wavelength in the y -direction. ζ is a constant to be determined so as to satisfy the above system. Substituting these assumed solutions into Eqs. (17) and eliminating A and B yields

$$\zeta^4 + a\zeta^3 + b\zeta^2 + c\zeta + d = 0, \quad (20)$$

whose coefficients a , b , etc. are given in the Appendix. Let its four roots be $\zeta = \zeta_i$ ($i=1,2,3,4$), and we obtain general solutions for the differential equations (17) as follows:

$$\left. \begin{aligned} u &= \sum_{j=1}^4 A_j \exp\{i\pi(\zeta_j x + y)/\ell\}, \\ v &= \sum_{j=1}^4 A_j \kappa_j \exp\{i\pi(\zeta_j x + y)/\ell\}, \end{aligned} \right\} \quad (21)$$

where A_j are integral constants, and

$$\kappa_j = -(\zeta_j^2 + \psi_{11}\zeta_j + \psi_{12})/(\psi_{13}\zeta_j^2 + \psi_{14}\zeta_j + \psi_{15}). \quad (22)$$

Referring to the boundary conditions (18), the above solutions yield the following homogeneous system of linear algebraic equations with respect to A_j :

$$\left. \begin{aligned} \sum_{j=1}^4 A_j \{\phi_{11}\zeta_j + \phi_{12}\kappa_j + \frac{1}{2}\phi_{13}(1+\zeta_j\kappa_j)\} \exp(\pm i\pi\zeta_j h/\ell) &= 0, \\ \sum_{j=1}^4 A_j \{\phi_{31}\zeta_j + \phi_{32}\kappa_j + \frac{1}{2}(\phi_{33} - T_0)(1+\zeta_j\kappa_j)\} \exp(\pm i\pi\zeta_j h/\ell) &= 0. \end{aligned} \right\} \quad (23)$$

For a nontrivial set of solutions of these equations, the determinant of their coefficients, say D_{mn} ($m,n=1,2,3,4$), must vanish, resulting in the following equation:

$$|D_{mn}| = 0. \quad (24)$$

This gives the critical load, T_{0c} , for the given Θ and ℓ/h .

4. Numerical Computations and Results

For the given values of T_0/μ_0 and Θ , a system of Eqs.(10) was solved with respect to F_{11} , F_{22} , and F_{21} by the Newton-Raphson method. Then based on the following bifurcation analysis, the critical loads, T_{0c}/μ_0 , at which the homogeneously deformed slab bifurcates, were determined by solving Eq. (24) by the trial and error method. The results are shown in Figs.2-5, where the critical loads are given as functions of Θ for various values of the anisotropy strength parameter k . When $k=0$, that is, in an isotropic case, they do not depend on Θ . The results for $k \rightarrow \infty$, i.e., for an idealized fiber-reinforced case are borrowed from Ref. (12).

Fig.2 represents a case of tensile load and of $\ell/h=3$. For various other values of ℓ/h as well, the critical loads were calculated. They, however, are not depicted here, because the curves are similar in essence to those for $\ell/h=3$.

When $k=\infty$ and $\Theta=0$, $T_{0c}/\mu_0=0.68$ ⁽¹¹⁾ (indicated by a solid circle in the vertical axis). When $k=\infty$ and $\Theta>0$, the critical load for $\Theta \rightarrow 0$ (indicated by a hollow circle) does not coincide with $T_{0c}/\mu_0=0.68$ but reaches $T_{0c}/\mu_0=0.81$ ⁽¹²⁾. When $k<\infty$ and $\Theta=0$, $T_{0c}/\mu_0=0.9$, which is reached by the critical load as $\Theta \rightarrow 0$ for all $k<\infty$. It may be understood from these facts that the idealized fiber-reinforced composite theory⁽¹³⁾ ($k=\infty$) does not appropriately estimate the critical loads, for small Θ , of a fiber-reinforced elastic slab, even if the fibers are very rigid.

It is also shown in Fig.2 that, as Θ increases from 0° , the critical load decreases slightly in the stages of small Θ and then increases significantly. This increase in

Fig.2 Critical loads for the case of tensile load, $\ell/h=3.0$

T_{0c}/μ_0 is the more remarkable for the larger k .

For the case of compressive loads, the dependence of the critical loads on the fiber inclination is shown in Figs.3-5 for $\ell/h=1.5$, 3.0, and 6.0, respectively. Curves are drawn for the anisotropy strength parameter $k=0.0$, 0.5, 2.0, 8.0, and ∞ .

Fig.3 Critical loads for the case of compressive load, $\ell/h=1.5$

Fig.4 Critical loads for the case of compressive load, $\ell/h=3.0$

Fig.3 ($\ell/h=1.5$) shows that the critical loads are the higher for the stronger anisotropy. For all $k>0$, the critical loads increases as Θ departs from 0° , and their curves fluctu-

Fig.5 Critical loads for the case of compressive load, $\ell/h=6.0$

ate near $\Theta=90^\circ$. In contrast to this, Fig.5 ($\ell/h=6.0$) shows that the critical loads decrease monotonically as Θ varies from 0° to 90° . The critical loads are the higher for the stronger anisotropy for small Θ , but vice versa near $\Theta=90^\circ$. Fig.4 ($\ell/h=3.0$) shows a transition stage from $\ell/h=1.5$ (Fig.3) to $\ell/h=6.0$ (Fig.5).

5. Conclusions

The bifurcation of an elastic slab having inclined anisotropy-axes under compressive and tensile loads has been examined with the following conclusions:

1. The slab can bifurcate not only under a compressive load but also under a tensile load from the uniform finite deformation including the shear.
2. For a tensile load, the idealized fiber-reinforced composite theory does not appropriately estimate the critical loads of the slab for very small Θ (fiber inclination), even if the fibers are very rigid.
3. For a compressive load, and for a large buckling wavelength, the critical loads increase as Θ departs from 0° , and their curves fluctuate near $\Theta=90^\circ$.
4. For a compressive load, and for a small wavelength, the critical loads decrease monotonically with Θ .

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Appendix

The coefficients appearing in Eqs.(13) are given as follows:

$$\begin{aligned}\phi_{11} &= (\mu_0/\sqrt{III}) \{3(B^{-1})_{11} + kh_1^2 \{2h_1^2 + (IV-1)\}\}, \\ \phi_{12} &= (\mu_0/\sqrt{III}) \{ (B^{-1})_{11} + kh_1^2 \{2h_1^2 - (IV-1)\}\}, \\ \phi_{13} &= 2(\mu_0/\sqrt{III}) \{ (B^{-1})_{12} + kh_1 h_2 \{2h_1^2 + (IV-1)\}\}, \\ \phi_{21} &= (\mu_0/\sqrt{III}) \{ (B^{-1})_{22} + kh_2^2 \{2h_1^2 - (IV-1)\}\}, \\ \phi_{22} &= (\mu_0/\sqrt{III}) \{3(B^{-1})_{22} + kh_2^2 \{2h_2^2 + (IV-1)\}\}, \\ \phi_{23} &= 2(\mu_0/\sqrt{III}) \{ (B^{-1})_{12} + kh_1 h_2 \{2h_2^2 + (IV-1)\}\}, \\ \phi_{31} &= 2(\mu_0/\sqrt{III}) \{ (B^{-1})_{12} + kh_1^3 h_2 \}, \\ \phi_{32} &= 2(\mu_0/\sqrt{III}) \{ (B^{-1})_{12} + kh_1 h_2^3 \}, \\ \phi_{33} &= (\mu_0/\sqrt{III}) \{ (B^{-1})_{11} + (B^{-1})_{22} \\ &\quad + k \{4h_1^2 h_2^2 + (IV-1)(h_1^2 + h_2^2)\} \}.\end{aligned}$$

The coefficients appearing in Eqs.(17) are expressed as follows:

$$\begin{aligned}\psi_{11} &= \frac{1}{2}(\phi_{13} + 2\phi_{31})/\phi_{11}, \\ \psi_{12} &= \frac{1}{2}(\phi_{33} + T_0)/\phi_{11}, \\ \psi_{13} &= \frac{1}{2}\phi_{13}/\phi_{11}, \\ \psi_{14} &= \frac{1}{2}(2\phi_{12} + \phi_{33} - T_0)/\phi_{11}, \\ \psi_{15} &= \phi_{32}/\phi_{11}, \\ \psi_{21} &= (\phi_{23} + 2\phi_{32})/(\phi_{33} - T_0), \\ \psi_{22} &= 2\phi_{22}/(\phi_{33} - T_0), \\ \psi_{23} &= 2\phi_{31}/(\phi_{33} - T_0), \\ \psi_{24} &= (2\phi_{21} + \phi_{33} + T_0)/(\phi_{33} - T_0), \\ \psi_{25} &= \phi_{23}/(\phi_{33} - T_0).\end{aligned}$$

The coefficients appearing in Eq.(20) are given by

$$\begin{aligned}a &= (\psi_{11} + \psi_{21} - \psi_{13}\psi_{24} - \psi_{23}\psi_{14})/(1 - \psi_{13}\psi_{23}), \\ b &= (\psi_{12} + \psi_{22} + \psi_{11}\psi_{21} - \psi_{13}\psi_{25} \\ &\quad - \psi_{15}\psi_{23} - \psi_{14}\psi_{24})/(1 - \psi_{13}\psi_{23}), \\ c &= (\psi_{11}\psi_{22} + \psi_{12}\psi_{21} - \psi_{14}\psi_{25} - \psi_{15}\psi_{24})/(1 - \psi_{13}\psi_{23}), \\ d &= (\psi_{12}\psi_{22} - \psi_{15}\psi_{25})/(1 - \psi_{13}\psi_{23}).\end{aligned}$$