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Stress Analysis around Circular Hole in Infinite Plate with Rigid Disk (Case of Load Applied to Disk)*

By Iwao MIZUSHIMA** and Minoru HAMADA***

A contact stress problem of an infinite plate with a rigid disk inserted in a circular hole is solved numerically. It is assumed that diameters of the disk and of the hole are identical in the unstressed state and that an in-plane load is applied to the disk. The method of stress analysis is a numerical one using recurrently the general form of a stress function expressed in the polar coordinates. Distributions of stresses and displacements around the hole are calculated and shown in some illustrated figures.

The presented results are compared with those obtained analytically and experimentally by other authors. The problem of a large circular plate fixed at the edge with a concentric circular hole where a rigid disk is inserted is also solved in order to check the condition of support of the infinite plate.

Key Words: Elasticity, Stress Analysis, Infinite Plate, Circular Hole, Rigid Disk, Stress Function, Iterative Method

1. Introduction

A contact stress problem of a plate with a disk inserted in a circular hole is one of the important and fundamental problems in machine design. The problems of an infinite plate subjected to uniaxial tension or compression as shown in Fig.1 were solved in authors' previous papers^{(1)~(3)}. The method of stress analysis adopted in these papers is a numerical one using a general solution of a stress function expressed in the polar coordinates, in which the values of coefficients of the stress function are adjusted recurrently so that the boundary conditions around the hole may be satisfied.

The case when an in-plane load is applied to the disk in the infinite plate as shown in Fig.2 is treated in this paper. Stresses and displacements around the hole are analyzed numerically by the method used in the previous papers. It is assumed that the disk is rigid and that the disk and the hole have identical diameters when unloaded. As shown in Fig.2, the disk is separated partially from the plate by the load. It is also assumed that friction does not exist in the hole-disk interface. Then, no shear stress acts around the hole. The stress analysis is carried out under the condition of plane stress. (4)~(7)

Many analytical papers^{(4)~(7)} and experimental papers^{(8)~(11)} have been published on such problems as mentioned above for the cases of an infinite plate, a semi-infinite plate and a strip. Solutions for the case of an infinite plate are especially important since they have a fundamental significance for a semi-infinite plate, a strip and a pin- or

rivet- joint of plates in practical cases. Since experiments for the infinite plate are difficult to carry out, analytical solutions only are available. In papers^{(4)~(5)} on the case of an infinite plate, approximate solutions were obtained by assuming load distri-

(a) Case of uniaxial tension

(b) Case of uniaxial compression

Fig.1 Infinite plate with a disk inserted in a circular hole

Fig.2 Infinite plate with a rigid disk inserted in a circular hole (Case of in-plane load applied to disk)

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** Associate Professor, Nara Technical College, Yamatokohriyama, Nara.

*** Professor, Faculty of Engineering, Osaka University, Yamadaoka, Suita.

bution in a suitable form, whereby the compatibility conditions of deformation between the hole and the disk were not considered. In the present paper, sufficiently accurate results are obtained on the case of the infinite plate considering the compatibility conditions.

It is necessary for the displacement analysis of an infinite plate to fix a point at some finite distance from the center of the circular hole. In this study the points A and A' are fixed at sufficiently long distances from the hole as shown in Fig.2. Various values are given to the distance in numerical calculations. When variations of stresses and displacements around the hole due to variation of the distance mentioned above are sufficiently small in results of calculations, the calculated stresses and displacements are considered to be valid in the case of an infinite plate fixed at infinity.

Next, the problem of a large circular plate fixed at the edge with a concentric circular hole where a rigid disk is inserted as shown in Fig.3 is also solved in order to check the condition of support of the infinite plate. The solutions for such a case are compared with those for the infinite plate fixed at two points A and A' as shown in Fig.2.

2. Method of Solution

A rigid circular disk inserted in a hole of an infinite plate is assumed to be subjected to an in-plane load in one direction as shown in Fig.2. Let a denote the radius of the hole, P the force per unit thickness applied to the disk, E the modulus of elasticity, ν Poisson's ratio, α the contact angle and u_* the radial clearance between the plate and the disk in the deformed state, respectively. Origins of the rectangular coordinates and the polar coordinates are taken at the center of the hole as shown in Fig.2. $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ and (u_r, u_θ) denote the stress components and the displacement components in the polar coordinates, respectively. The infinite plate is assumed to be fixed at a point A located at distance ka from the origin. Nondimensional stress and displacement components around the hole are defined as follows;

$$\left. \begin{aligned} \bar{\sigma}_r &= \frac{(\sigma_r)_{r=a}}{P/a}, & \bar{\sigma}_\theta &= \frac{(\sigma_\theta)_{r=a}}{P/a} \\ \bar{\tau}_{r\theta} &= \frac{(\tau_{r\theta})_{r=a}}{P/a}, & \bar{u}_r &= \frac{(u_r/a)_{r=a}}{P/(aE)} \\ \bar{u}_\theta &= \frac{(u_\theta/a)_{r=a}}{P/(aE)} \end{aligned} \right\} \dots\dots (1)$$

$$\bar{\sigma}_r^{00} = \frac{3+\nu}{4\pi} \cos \theta, \quad \bar{\sigma}_\theta^{00} = -\frac{1-\nu}{4\pi} \cos \theta, \quad \bar{\tau}_{r\theta}^{00} = -\frac{1-\nu}{4\pi} \sin \theta \dots\dots\dots (2)$$

Substituting Eq.(2) into the equations of relation between stress and strain of the plane stress state and integrating them under the condition that the point A does not move, the displacement components around the hole are obtained as follows:

$$\bar{u}_r^{00} = -\frac{(1+\nu)(3-\nu)}{4\pi} \log k \cos \theta, \quad \bar{u}_\theta^{00} = \frac{1+\nu}{4\pi} \{ (3-\nu) \log k - (1+\nu) \} \sin \theta \dots\dots\dots (3)$$

The general solution of stress function in the polar coordinates Φ is expressed by the following equation⁽¹³⁾:

Fig.3 Large circular plate with a rigid disk inserted in a concentric circular hole (Case of in-plane load applied to disk)

Fig.4 Infinite plate without a hole (Case of concentrated force at a point)

The method of stress analysis is essentially the same as used in the previous papers^{(1)~(3)}. A general solution of stress function expressed in the polar coordinates is adjusted recurrently so as to satisfy the boundary conditions. First, the stress state of the infinite plate without a hole subjected to a concentrated force P at the origin O as shown in Fig.4 is analyzed. Then, the stress state is modified recurrently so that the conditions $(\bar{u}_* = \bar{\tau}_{r\theta} = 0)$ at the contact surface between the plate and the disk and the conditions $(\bar{\sigma}_r = \bar{\tau}_{r\theta} = 0)$ at the non-contact surface are satisfied alternately. When such modifications are continued until all of the boundary conditions are numerically satisfied, the solution of the present problem is obtained to a desired accuracy.

The stress components⁽¹²⁾ around the hole in the stress state of the infinite plate without a hole subjected to a concentrated force P at the origin O as shown in Fig.4 are given by

$$\begin{aligned} \Phi = & a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2 \theta + a_0' \theta + \frac{1}{2} a_1 r \theta \sin \theta + (b_1 r^3 + a_1' r^{-1} + b_1' r \log r) \cos \theta \\ & - \frac{1}{2} c_1 r \theta \cos \theta + (d_1 r^3 + c_1' r^{-1} + d_1' r \log r) \sin \theta + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta \dots\dots\dots (4) \end{aligned}$$

where $a_0, b_0, \dots, c_n'$ and d_n' are undetermined coefficients. Stresses should be symmetric with respect to the x-axis and vanish at infinity. Also, stresses and displacements should be uniquely determined at any point. Considering these conditions and the conditions of equilibrium of resultant forces in the x- and y-directions in Eq.(4), Φ becomes as follows;

$$\Phi = a_0 \log r + \frac{1}{2\pi} Q r \theta \sin \theta + \left(a_1' r^{-1} - \frac{1-\nu}{4\pi} Q r \log r \right) \cos \theta + \sum_{n=2}^{\infty} (a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta \dots\dots\dots (5)$$

where Q is a resultant force in the x-direction. Nondimensional stress and displacement components around the hole are obtained from Eq.(1) and Eq.(5) as follows:

$$\left. \begin{aligned} \bar{\sigma}_r &= \bar{a}_0 - (2\bar{a}_1' - (3+\nu)\bar{q}) \cos \theta - \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + (n-1)(n+2)\bar{b}_n'\} \cos n\theta \\ \bar{\sigma}_\theta &= -\bar{a}_0 + \{2\bar{a}_1' - (1-\nu)\bar{q}\} \cos \theta + \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + (n-1)(n-2)\bar{b}_n'\} \cos n\theta \\ \bar{\tau}_{r\theta} &= -(2\bar{a}_1' + (1-\nu)\bar{q}) \sin \theta - \sum_{n=2}^{\infty} \{n(n+1)\bar{a}_n' + n(n-1)\bar{b}_n'\} \sin n\theta \\ \bar{u}_r &= (1+\nu) \left[-\bar{a}_0 + \{2\bar{a}_1' - (3-\nu)\bar{q}\} \log k \right] \cos \theta + \sum_{n=2}^{\infty} \left\{ n\bar{a}_n' + \left(n-2 + \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \cos n\theta \\ \bar{u}_\theta &= (1+\nu) \left[\{\bar{a}_1' + (3-\nu)\bar{q}\} \log k - (1+\nu)\bar{q} \right] \sin \theta + \sum_{n=2}^{\infty} \left\{ n\bar{a}_n' + \left(n - \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \sin n\theta \end{aligned} \right\} \dots\dots\dots (6)$$

where $\bar{a}_0, \bar{a}_1', \bar{a}_n'$ and $\bar{b}_n'$ are undetermined coefficients and $\bar{q}$ is a nondimensional force defined as $\bar{q} = Q / (4\pi P)$.

Now, stress and displacement distributions around the hole are solved numerically by using Eq.(6) through the following steps of calculations:

(1) The distributions of $\bar{\sigma}_r^{oo}$ and $\bar{\tau}_{r\theta}^{oo}$ expressed by Eq.(2) are shown for the case of $\nu = 0.3$ by full lines in Fig.5. $\bar{\tau}_{r\theta}$ should be zero because it is assumed that friction does not exist in the hole-disk interface and $\bar{\sigma}_r$ should not be positive. But Eq.(2) does not satisfy these conditions. Therefore $\bar{\tau}_{r\theta}^{oo}$ and $\bar{\sigma}_r^{oo}$ must be modified. The distributions of $\bar{\tau}_{r\theta}$ and $\bar{\sigma}_r$ shown by broken lines in Fig.5 should be added to those of $\bar{\tau}_{r\theta}^{oo}$ and $\bar{\sigma}_r^{oo}$, respectively. By expressing the distributions of $\bar{\tau}_{r\theta}$ and $\bar{\sigma}_r$ in forms of trigonometric series by harmonic analyses, the undetermined coefficients of Eq.(6) for $\bar{\tau}_{r\theta}$ and $\bar{\sigma}_r$ are determined. The stress and displacement components calculated by Eq.(6) are added to those of Eq.(2) and Eq.(3). The resulting components are denoted by $\bar{\sigma}_r^{io}, \bar{\sigma}_\theta^{io}, \bar{\tau}_{r\theta}^{io}, \bar{u}_r^{io}, \bar{u}_\theta^{io}$. The clearance arises between the plate and the disk after the modification and the radial clearance is denoted by $\bar{u}_*^{io}$. The distributions of $\bar{\sigma}_r^{io}$ and $\bar{u}_*^{io}$ are shown in Fig.6.

(2) It is unreasonable that a negative radial clearance exists between the plate and the disk. The part where $\bar{u}_*^{io}$ is negative may be the contact surface. Therefore, the stress distributions must be again modified so that $\bar{u}_*^{io}$ may vanish there. $\bar{\tau}_{r\theta}$ must be kept equal to zero while $\bar{u}_*^{io}$ is corrected. The procedures of this modification are the same as those in the step (1). The modified stress and displacement states are expressed by $\bar{\sigma}_r^{ii}, \bar{\sigma}_\theta^{ii}, \bar{\tau}_{r\theta}^{ii}, \bar{u}_r^{ii}, \bar{u}_\theta^{ii}, \bar{u}_*^{ii}$. The distributions of $\bar{\sigma}_r^{ii}$ and $\bar{u}_*^{ii}$ are shown in Fig.7.

(3) In the stress state obtained by the previous steps of modifications the radial stress at the non-contact surface may not vanish. Therefore, $\bar{\sigma}_r^{ii}$ must be modified so

that $\bar{\sigma}_r$ satisfies the condition of the non-contact part. $\bar{\tau}_{r\theta}$ must always be kept equal to zero. The modified state is expressed by $\bar{\sigma}_r^{2i}, \bar{\sigma}_\theta^{2i}, \bar{\tau}_{r\theta}^{2i}, \bar{u}_r^{2i}, \bar{u}_\theta^{2i}, \bar{u}_*^{2i}$. The distributions of $\bar{\sigma}_r^{2i}$ and $\bar{u}_*^{2i}$ are shown in Fig.8.

(4) After the above step of modifications the radial clearance may be positive on the whole circumference of the hole as shown in Fig.8(b); it means that the disk is completely separated from the plate. Therefore, $\bar{u}_*^{2i}$ must be modified so that $\bar{u}_*^{2i}$ may vanish on the contact part. The boundary point B between the contact part and the non-contact part is obtained from the distribution of $\bar{\sigma}_r^{2i}$ in Fig.8(a). A tangential line is drawn from the point B to the curve of $\bar{u}_*^{2i}$ as shown in Fig.8(c). Then, the hatched part in the figure is eliminated. The modified state is expressed by $\bar{\sigma}_r^{2z}, \bar{\sigma}_\theta^{2z}, \bar{\tau}_{r\theta}^{2z}, \bar{u}_r^{2z}, \bar{u}_\theta^{2z}, \bar{u}_*^{2z}$.

(5) The modifications in the steps (3) and (4) are repeated until the values of stresses and displacements converge to a desired accuracy.

Now, in order to check the condition of support of the infinite plate, the problem of a large circular plate fixed at the edge with a concentric circular hole as shown in Fig.3 is considered. A rigid disk is inserted in the hole and is subjected to an in-plane load. Stress and displacement components at the internal edge are expressed by Eq.(7) described below instead of Eq.(6) and those at the external one are expressed by Eq.(8). The conditions that the displacement components $\bar{u}_r$ and $\bar{u}_\theta$ at the external edge are zero should be added to those used in the analysis of the infinite plate mentioned above. Other equations and conditions are the same as those used in the analysis of the infinite plate.

Fig.5 Distributions of $\bar{\sigma}_r^{oo}$ and $\bar{\tau}_{r\theta}^{oo}$ Fig.6 Distributions of $\bar{\sigma}_r^{/o}$ and $\bar{u}_*^{/o}$ Fig.7 Distributions of $\bar{\sigma}_r^{''}$ and $\bar{u}_*^{''}$

$$\begin{aligned}
 \bar{\sigma}_r &= \bar{a}_0 + 2\bar{b}_0 + \{2\bar{b}_1 - 2\bar{a}_1' + (3+\nu)\bar{q}\} \cos \theta - \sum_{n=2}^{\infty} \{n(n-1)\bar{a}_n + (n+1)(n-2)\bar{b}_n \\
 &\quad + n(n+1)\bar{a}_n' + (n+2)(n-1)\bar{b}_n'\} \cos n\theta \\
 \bar{\sigma}_\theta &= -\bar{a}_0 + 2\bar{b}_0 + \{6\bar{b}_1 + 2\bar{a}_1' - (1-\nu)\bar{q}\} \cos \theta + \sum_{n=2}^{\infty} \{n(n-1)\bar{a}_n + (n+2)(n+1)\bar{b}_n \\
 &\quad + n(n+1)\bar{a}_n' + (n-1)(n-2)\bar{b}_n'\} \cos n\theta \\
 \bar{\tau}_{r\theta} &= \{2\bar{b}_1 - 2\bar{a}_1' - (1-\nu)\bar{q}\} \sin \theta + \sum_{n=2}^{\infty} \{n(n-1)\bar{a}_n + n(n+1)\bar{b}_n \\
 &\quad - n(n+1)\bar{a}_n' - n(n-1)\bar{b}_n'\} \sin n\theta \\
 \bar{u}_r &= (1+\nu) \left[-\bar{a}_0 + 2\frac{1-\nu}{1+\nu}\bar{b}_0 + \left\{ \frac{1-3\nu}{1+\nu}\bar{b}_1 + \bar{a}_1' - (3-\nu)\bar{q} \log k + \bar{f} \right\} \cos \theta \right. \\
 &\quad \left. - \sum_{n=2}^{\infty} \left\{ n\bar{a}_n + \left(n+2 - \frac{4}{1+\nu} \right) \bar{b}_n - n\bar{a}_n' - \left(n-2 + \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \cos n\theta \right] \\
 \bar{u}_\theta &= (1+\nu) \left[\left\{ \frac{5+\nu}{1+\nu}\bar{b}_1 + \bar{a}_1' + (3-\nu)\bar{q} \log k - (1+\nu)\bar{q} - \bar{f} \right\} \sin \theta \right. \\
 &\quad \left. + \sum_{n=2}^{\infty} \left\{ n\bar{a}_n + \left(n+2 - \frac{4}{1+\nu} \right) \bar{b}_n + n\bar{a}_n' + \left(n-2 + \frac{4}{1+\nu} \right) \bar{b}_n' \right\} \sin n\theta \right] \\
 \bar{\sigma}_r' &\equiv \frac{(\sigma_r)_{r=ka}}{P/a} = k^{-2}\bar{a}_0 + 2\bar{b}_0 + \{2k\bar{b}_1 - 2k^3\bar{a}_1' + (3+\nu)k^{-1}\bar{q}\} \cos \theta \\
 &\quad - \sum_{n=2}^{\infty} \{n(n-1)k^{n-2}\bar{a}_n + (n+1)(n-2)k^n\bar{b}_n + n(n+1)k^{n-2}\bar{a}_n' + (n+2)(n-1)k^{-n}\bar{b}_n'\} \cos n\theta \\
 \bar{\sigma}_\theta' &\equiv \frac{(\sigma_\theta)_{r=ka}}{P/a} = -k^{-2}\bar{a}_0 + 2\bar{b}_0 + \{6k\bar{b}_1 + 2k^3\bar{a}_1' - (1-\nu)k^{-1}\bar{q}\} \cos \theta \\
 &\quad + \sum_{n=2}^{\infty} \{n(n-1)k^{n-2}\bar{a}_n + (n+2)(n+1)k^n\bar{b}_n + n(n+1)k^{n-2}\bar{a}_n' + (n-1)(n-2)k^{-n}\bar{b}_n'\} \cos n\theta \\
 \bar{\tau}_{r\theta}' &\equiv \frac{(\tau_{r\theta})_{r=ka}}{P/a} = \{2k\bar{b}_1 - 2k^3\bar{a}_1' - (1-\nu)k^{-1}\bar{q}\} \sin \theta \\
 &\quad + \sum_{n=2}^{\infty} \{n(n-1)k^{n-2}\bar{a}_n + n(n+1)k^n\bar{b}_n - n(n+1)k^{n-2}\bar{a}_n' - n(n-1)k^{-n}\bar{b}_n'\} \sin n\theta \\
 \bar{u}_r' &\equiv \frac{(\bar{u}_r)_{r=ka}}{P/(aE)} = (1+\nu) \left[-k^{-1}\bar{a}_0 + 2\frac{1-\nu}{1+\nu}k\bar{b}_0 + \left\{ \frac{1-3\nu}{1+\nu}k^2\bar{b}_1 + k^2\bar{a}_1' + \bar{f} \right\} \cos \theta \right. \\
 &\quad \left. - \sum_{n=2}^{\infty} \left\{ nk^{n-1}\bar{a}_n + \left(n+2 - \frac{4}{1+\nu} \right) k^{n+1}\bar{b}_n - nk^{n-1}\bar{a}_n' - \left(n-2 + \frac{4}{1+\nu} \right) k^{n+1}\bar{b}_n' \right\} \cos n\theta \right] \\
 \bar{u}_\theta' &\equiv \frac{(\bar{u}_\theta)_{r=ka}}{P/(aE)} = (1+\nu) \left[\left\{ \frac{5+\nu}{1+\nu}k^2\bar{b}_1 + k^2\bar{a}_1' - (1+\nu)\bar{q} - \bar{f} \right\} \sin \theta \right. \\
 &\quad \left. + \sum_{n=2}^{\infty} \left\{ nk^{n-1}\bar{a}_n + \left(n+2 + \frac{4}{1+\nu} \right) k^{n+1}\bar{b}_n + nk^{n-1}\bar{a}_n' + \left(n-2 + \frac{4}{1+\nu} \right) k^{n+1}\bar{b}_n' \right\} \sin n\theta \right]
 \end{aligned}$$

.....(7)

.....(8)

Fig.8 Distributions of $\bar{\sigma}_r^{2/}$ and $\bar{u}_*^{2/}$

where $\bar{b}_0$, $\bar{b}_1$, $\bar{a}_n$, $\bar{b}_n$ and $\bar{f}$ are undetermined coefficients. Eqs.(7) and (8) are obtained by adding the uniform displacement $(1+\nu)\bar{f}$ in the x-direction to the displacements derived from the stress function of Eq.(4), thereby considering the conditions that stresses and displacements are uniquely determined at any point and symmetric with respect to the x-axis and the conditions of equilibrium in the x- and y-directions.

Fig.9 Distributions of stresses and clearance around hole

Table 1 Maximum radial stress, maximum circumferential stress, maximum radial clearance and contact angle

	infinite plate (case of $k=10, 20, 50$)	large circular plate (case of $k=20$)
$ \bar{\sigma}_r _{\max}$	0.601	0.602
$(\bar{\sigma}_\theta)_{\max}$	0.467	0.461
$(\bar{u}_*)_{\max}$	0.880	0.873
α	83.0°	83.2°

3. Numerical Results and Considerations

Numerical calculations are carried out under the condition of $\nu = 0.3$. With values of 10, 20 and 50 given to k which denotes the position of the fixed point of the infinite plate, stresses and displacements around the hole are calculated. The distributions of $\bar{\sigma}_r$, $\bar{\sigma}_\theta$ and the radial clearance $\bar{u}_*$ are shown in Fig.9. The maximum radial stress $|\bar{\sigma}_r|_{\max}$, the maximum circumferential stress $(\bar{\sigma}_\theta)_{\max}$, the maximum radial clearance $(\bar{u}_*)_{\max}$ and the contact angle α are indicated in Table 1. No change is noticed in the results in Fig.9 and Table 1 when k changes from 10 to 50.

In the displacement state expressed by Eq.(3) in which the point A is fixed, A', the symmetrical point of A with respect to the y-axis is also fixed as shown in Fig.2. Since the method of solution is one of iterative methods, the points A and A' fixed in the first step of calculation move in every step of modification and two other points are set in the neighborhood of the point A or A' in the final step. These points are situated at distances of $-1.92a$ and $-1.76a$ in the x-direction from A and A' in the case of $k = 20$, respectively. It is possible to make the displacements at the points A and A' zero in each step of modification, but for the simplicity of calculations such procedures are

not adopted in the treatments of this study.

The problem of a large circular plate fixed on the outer edge as shown in Fig.3 is also solved in order to discuss the condition of support of the infinite plate. The results for $k = 20$ are indicated in Table 1, and they almost coincide with the above results, i. e., those of the infinite plate fixed at two points. This fact means that the above results for the infinite plate fixed at two points can be regarded as the solutions of the infinite plate fixed at all its edges at infinity.

The results by R. C. Knight⁽⁵⁾ and by M. Nishida⁽¹¹⁾ are indicated by broken lines in Fig.9 in order to compare them with those in this study. Knight's results are approximate solutions obtained from functions expressed by 7 terms of the Fourier series which efface $\bar{\sigma}_r^{oo}$ and $\bar{\tau}_{r\theta}^{oo}$ of Eq.(2) in the region of $\theta = -90^\circ \sim 90^\circ$. Since the compatibility conditions of deformation between the hole and the inserted disk are not considered in Knight's study, some differences are found between his results and those in this paper. Nishida's results are those of photoelastic experiments for a semi-infinite plate having the edge at $x = -8a$ in Fig.2. His results are closer to the present results than Knight's results.

In numerical calculations 200 terms of the Fourier series are used. The appropriateness of this treatment was examined in the previous paper⁽¹⁾. After the cycles of the

Table 2 Decrease state of residual radial clearance on contact surface

number of cycles of steps of modification	1	2	3	4	5	6
ratio of maximum residual radial clearance to maximum radial clearance (%)	6.6	2.5	1.0	0.7	0.4	0.2

steps of modification of solutions for the contact surface and for the non-contact surface are repeated six times, the solutions of stresses and displacements with high accuracy are obtained. The residual radial clearance on the contact surface and the residual radial stress on the non-contact surface are estimated within 0.2% of $(\bar{u}_*)_{\max}$ and 0.8% of $|\bar{\sigma}_r|_{\max}$, respectively. The residual shear stress around the hole is within 0.1×10^{-5} , i. e., 0.0002% of $|\bar{\sigma}_r|_{\max}$. In order to show the state of convergence of the solution, the relation between the ratio of the maximum residual radial clearance to the maximum radial clearance and the number of cycles of the steps of modification is indicated in Table 2.

When the above-mentioned cycles of modification are repeated six times, the resultant force in the x-direction obtained by integrating numerically $\bar{\sigma}_r$ around the hole is 0.426P. The present results in Fig.9 and in Table 1 are, however, modified so that the force is equal to P. On the other hand, the force obtained from Knight's results shown in Fig.9(a) is less than P. The equilibrium of force is, however, satisfied in his treatments because some shear stresses act on the contact surface.

The curve of the present results in Fig.9(b) has a sharp point at the boundary between the contact part and the non-contact part, but those of the results in other researches⁽⁵⁾⁽¹¹⁾ have no sharp point. The existence of the sharp point in the present solution is confirmed by increasing gradually the number of terms of the Fourier series from 50 to 1500. Such a sharp point is also shown in the papers⁽¹⁾⁽³⁾⁽⁴⁾⁽¹⁶⁾ concerned with the problems shown in Fig.1. Since the number of terms of the Fourier series is very small in the analysis by Knight⁽⁵⁾, the curve of his results is considered to have no sharp point. Also the fact that the solution curve by Nishida⁽¹¹⁾ has no sharp point is considered to be due to the accuracy of experiments.

4. Conclusions

A contact stress problem of an infinite plate with a rigid disk inserted in a circular hole is numerically solved for the case when an in-plane load is applied to the disk. Distributions of stresses and displacements

around the hole are shown in some illustrated figures. The problem of a large circular plate fixed at the edge with a concentric hole where a rigid disk is inserted is also solved in order to check the condition of support of the infinite plate. It is found that the results of the infinite plate almost coincide with those of the circular plate. The present results are also compared with other approximate solutions and experimental data.

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