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Contact Stresses of an Elastic Layer Indented by a Concave Rigid Punch*

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In this paper, the authors analyze exactly the stress state of an elastic layer indented by an axisymmetric concave rigid punch when the indented load is small and the contact region is an annulus. Numerical examples are presented for parabolic and conical concave punches. For each case, we show the distributions of surface displacements and contact stresses and clarify the relationships among the parameters of the indented load, the indented depth, the punch face configuration and the layer's thickness.

Key words: Elasticity, Contact Stresses, Elastic Layer

1. Introduction

When an axisymmetrical concave rigid punch is indented to an elastic layer by a small load, the contact region will be an annulus. The outer circumference of this contact region coincides with the edge of the punch, but the inner circumference shrinks with an increasing load and finally the entire punch face will be in contact with the elastic layer. One of the authors has presented a method of solution for such a punch indented to an elastic half space⁽¹⁾, and obtained the distributions of surface displacements and contact stresses.

In the present paper, this method is applied to the problem of a thick layer indented by a punch, and the effects of the layer's thickness on the distributions of surface displacements and contact stresses are clarified.

2. Stress Analysis

A cylindrical coordinate system (r, θ, z) and Boussinesq's stress function $\varphi_0(r, z)$ and $\varphi_3(r, z)$ are used to analyze the axisymmetrical state of an elastic layer without torsion. The components of the displacement (u_r, v_θ, w_z) and the components of the stress $(\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}, \tau_{\theta z}, \tau_{r\theta})$ are expressed in the forms of

$$\left. \begin{aligned} 2GU_r &= \frac{\partial \varphi_0}{\partial r} + z \frac{\partial \varphi_3}{\partial r}, & v_\theta &= 0 \\ 2Gw_z &= \frac{\partial \varphi_0}{\partial z} + z \frac{\partial \varphi_3}{\partial z} - (3-4\nu)\varphi_3 \\ \sigma_r &= \frac{\partial^2 \varphi_0}{\partial r^2} + z \frac{\partial^2 \varphi_3}{\partial r^2} - 2\nu \frac{\partial \varphi_3}{\partial z} \\ \sigma_\theta &= \frac{1}{r} \frac{\partial \varphi_0}{\partial r} + \frac{z}{r} \frac{\partial \varphi_3}{\partial r} - 2\nu \frac{\partial \varphi_3}{\partial z} \end{aligned} \right\} \dots\dots\dots (1)$$

$$\left. \begin{aligned} \sigma_z &= -\frac{\partial^2 \varphi_0}{\partial z^2} + z \frac{\partial^2 \varphi_3}{\partial z^2} - 2(1-\nu) \frac{\partial \varphi_3}{\partial z} \\ \tau_{rz} &= \frac{\partial^2 \varphi_0}{\partial r \partial z} + z \frac{\partial^2 \varphi_3}{\partial r \partial z} - (1-2\nu) \frac{\partial \varphi_3}{\partial r} \\ \tau_{\theta z} &= \tau_{r\theta} = 0 \end{aligned} \right\}$$

where G = shear modulus, ν = Poisson ratio, and φ_0, φ_3 = cylindrical harmonic functions.

Fig.1 The geometry of the problem

As is shown in Fig.1, an infinite elastic layer with thickness h lies on a rigid base and is indented by a rigid cylindrical punch with concave face. The indented load and the indented depth are denoted by P and ϵ_0 respectively. The contact region is an annulus with inner radius r_i and outer radius r_0 . Assuming that the lower surface of the layer is entirely in contact with the rigid base and both surfaces of the layer are so smooth that the contact without friction is realized. The boundary conditions to this problem are formulated as follows:

$$\left. \begin{aligned} (i) & (w_z)_{z=0} = 0, & (\tau_{rz})_{z=0} &= (\tau_{rz})_{z=h} = 0 & (0 \leq r < \infty) \\ (ii) & (\sigma_z)_{z=h} &= 0 & (0 \leq r < r_i, r_0 < r < \infty) \\ (iii) & (w_z)_{z=h} &= -\epsilon_0 + u_g(r) & (r_i \leq r \leq r_0) \\ (iv) & (\partial w_z / \partial r)_{z=h, r=r_i \pm 0} &= u_g'(r) & \end{aligned} \right\} \dots\dots\dots (2)$$

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where $g(r)$ is a function which describes the configuration of the punch face, and $g(0)=1$ and $g(r_0)=0$, and u denotes the concavity. It should be noticed that the condition (iv) of Eq.(2) represents a smooth separation of the punch face from the layer at the circumference $r=r_i$.

Taking into account the fact that the layer occupies the domain $0 \leq z \leq h$ and $0 \leq r < \infty$, the stress functions φ_0 and φ_3 are put as follows:

$$\left. \begin{aligned} \varphi_0(r, z) &= \int_0^\infty [A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z] J_0(\lambda r) d\lambda \\ \varphi_3(r, z) &= \int_0^\infty [C(\lambda) \cosh \lambda z + D(\lambda) \sinh \lambda z] J_0(\lambda r) d\lambda \end{aligned} \right\} \dots\dots\dots (3)$$

where $A(\lambda)$ to $D(\lambda)$ are unknown functions of λ , which will be determined so that Eq.(2) is satisfied, and $J_n(\lambda r)$ is the Bessel function of the first kind in order n .

Condition (i) of Eq.(2) gives the relations

$$A(\lambda) = \frac{(1-2\nu) \sinh \lambda h - \lambda h \cosh \lambda h}{\lambda \sinh \lambda h} D(\lambda), \quad B(\lambda) = C(\lambda) = 0 \dots\dots\dots (4)$$

and $(w_z)_{z=h}$, $(\sigma_z)_{z=h}$ and $(\sigma_z)_{z=0}$ are expressed by the unknown function $D(\lambda)$ in the forms

$$\left. \begin{aligned} (w_z)_{z=h} &= -\frac{1-\nu}{G} \int_0^\infty \sinh \lambda h D(\lambda) J_0(\lambda r) d\lambda \\ (\sigma_z)_{z=h} &= -\int_0^\infty \frac{\lambda h + \sinh \lambda h \cosh \lambda h}{\sinh \lambda h} \lambda D(\lambda) J_0(\lambda r) d\lambda \\ (\sigma_z)_{z=0} &= -\int_0^\infty \frac{\lambda h \cosh \lambda h + \sinh \lambda h}{\sinh \lambda h} \lambda D(\lambda) J_0(\lambda r) d\lambda \end{aligned} \right\} \dots\dots\dots (5)$$

If it is assumed that

$$E(\lambda) = -\frac{\lambda h + \sinh \lambda h \cosh \lambda h}{\sinh \lambda h} D(\lambda), \quad t(\lambda) = \frac{\sinh^2 \lambda h}{\lambda h + \sinh \lambda h \cosh \lambda h} \dots\dots\dots (6)$$

then the conditions (ii), (iii) and (iv) become

$$\left. \begin{aligned} (w_z)_{z=h} &= \frac{1-\nu}{G} \int_0^\infty t(\lambda) E(\lambda) J_0(\lambda r) d\lambda = -\varepsilon_0 + u g(r) \quad (r_i \leq r \leq r_0) \\ (\sigma_z)_{z=h} &= \int_0^\infty \lambda E(\lambda) J_0(\lambda r) d\lambda = 0 \quad (0 \leq r < r_i, \quad r_0 < r < \infty) \end{aligned} \right\} \dots\dots\dots (7)$$

$$\left(\frac{\partial w_z}{\partial r} \right)_{z=h} = -\frac{1-\nu}{G} \int_0^\infty \lambda t(\lambda) E(\lambda) J_1(\lambda r) d\lambda = u g'(r) \quad (r = r_i \pm 0) \dots\dots\dots (8)$$

Thus this problem is reduced to a dual integral equation (7) for $E(\lambda)$ with the complementary condition (8).

To simplify the analyses, a new variable ϕ relating to r is introduced as follows:

$$r = (r_e^2 + b^2 - 2r_e b \cos \phi)^{1/2} \quad (0 \leq \phi \leq \pi), \quad r_e = \frac{1}{2}(r_0 + r_i), \quad b = \frac{1}{2}(r_0 - r_i) \dots\dots\dots (9)$$

Further a continuous function $F(\phi)$ is introduced and it is assumed that the unknown function $E(\lambda)$ is expressed as

$$E(\lambda) = \int_0^\pi F(\phi) J_0(\lambda R) d\phi, \quad R = (r_e^2 + b^2 - 2r_e b \cos \phi)^{1/2} \dots\dots\dots (10)$$

Substitution of Eq.(10) into the expression for $(\sigma_z)_{z=h}$ and the application of the formula

$$\int_0^\infty \lambda J_0(\lambda R) J_0(\lambda r) d\lambda = \frac{\delta(R-r)}{\sqrt{Rr}} \dots\dots\dots (11)$$

yields

$$(\sigma_z)_{z=h} = \int_0^\pi F(\phi) d\phi \int_0^\infty \lambda J_0(\lambda R) J_0(\lambda r) d\lambda = \int_0^\pi \frac{F(\phi) \delta(R-r)}{\sqrt{Rr}} d\phi \dots\dots\dots (12)$$

where $\delta(R-r)$ denotes Dirac's delta function. Now the second of Eq.(7) is satisfied by an arbitrary form of $F(\phi)$, since $R-r \neq 0$ in the regions $0 \leq r < r_i$ and $r_0 < r < \infty$ noticing $r_i \leq R \leq r_0$. Next the continuous function $F(\phi)$ in Eq.(12) without the loss of generality is assumed as

$$F(\phi) = \sum_{n=0}^\infty a_n \cos n\phi \dots\dots\dots (13)$$

and $(\sigma_z)_{z=h}$ is expressed in the form

$$(\sigma_z)_{z=h} = \frac{1}{r_0 b} \sum_{n=0}^{\infty} a_n \frac{\cos n\phi}{\sin \phi} \quad (r_i < r < r_0) \quad (14)$$

where a_n denotes an unknown constant. Eq.(8), which is equivalent to the condition of the finite contact stress at $r=r_i$ ⁽¹⁾, leads to the following relation on a_n

$$\sum_{n=0}^{\infty} a_n = 0 \quad (15)$$

On the other hand, substituting Eq.(13) into Eq.(10) and making use of the formula

$$\int_0^\pi J_0(\lambda R) \cos n\phi d\phi = \pi J_n(\lambda r_c) J_n(\lambda b) \quad (n=0, 1, 2, \dots) \quad (16)$$

yields

$$E(\lambda) = \sum_{n=0}^{\infty} a_n \int_0^\pi J_0(\lambda R) \cos n\phi d\phi = \pi \sum_{n=0}^{\infty} a_n J_n(\lambda r_c) J_n(\lambda b) \quad (17)$$

So the substitution of Eq.(17) into the first of Eq.(7) gives

$$\sum_{n=0}^{\infty} a_n \int_0^\pi t(\lambda) J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda = -\frac{G}{\pi(1-\nu)} \{\varepsilon_0 - u g(r)\} \quad (r_i \leq r \leq r_0) \quad (18)$$

If the following unknown constants a_n' and a_n'' relating to a_n are introduced

$$a_n = -\frac{Gr_0}{\pi(1-\nu)} \{\varepsilon_0 a_n' - u a_n''\} \quad (n=0, 1, 2, \dots) \quad (19)$$

then Eq.(18) can be divided into two equations:

$$\sum_{n=0}^{\infty} (a_n', a_n'') r_0 \int_0^\pi t(\lambda) J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda = \{1, g(r)\} \quad (r_i \leq r \leq r_0) \quad (20)$$

Hence the unknown constants a_n' and a_n'' should be determined such that Eqs.(20) hold over the range $r_i \leq r \leq r_0$. Noticing that the range $r_i \leq r \leq r_0$ corresponds to the range $0 \leq \phi \leq \pi$, the functions $J_0(\lambda r)$ and $g(r)$ are developed into the Fourier series with respect to ϕ over this range as follows

$$\left. \begin{aligned} J_0(\lambda r) &= J_0(\lambda r_c) J_0(\lambda b) + 2 \sum_{m=1}^{\infty} J_m(\lambda r_c) J_m(\lambda b) \cos m\phi \\ g(r) &= b_0 + 2 \sum_{m=1}^{\infty} b_m \cos m\phi, \quad b_m = \frac{1}{\pi} \int_0^\pi g(r) \cos m\phi d\phi \quad (m=0, 1, 2, \dots) \end{aligned} \right\} \quad (21)$$

Substituting them into Eq.(20), and equating the coefficients of $\cos m\phi$ in both sides, an infinite system of simultaneous equations for unknown coefficients of a_n' and a_n'' is obtained in the forms

$$\sum_{n=0}^{\infty} (a_n', a_n'') r_0 \int_0^\pi t(\lambda) J_m(\lambda r_c) J_m(\lambda b) J_n(\lambda r_c) J_n(\lambda b) d\lambda = (\delta_{0m}, b_m) \quad (m=0, 1, 2, \dots) \quad (22)$$

where δ_{0m} denotes Kronecker's delta.

Solving Eqs.(22) and obtain a_n' and a_n'' , the relation between ε_0 and u is deduced from Eq.(15) as

$$\frac{u}{\varepsilon_0} = \sum_{n=0}^{\infty} a_n' / \sum_{n=0}^{\infty} a_n'' \quad (23)$$

The surface displacement $(w_z)_{z=h}$, the contact stresses $(\sigma_z)_{z=h}$ and $(\sigma_z)_{z=0}$ and the indented load P are expressed as

$$\left. \begin{aligned} (w_z)_{z=h} &= -\sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') r_0 \int_0^\pi t(\lambda) J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda \\ (\sigma_z)_{z=h} &= -\frac{Gr_0}{\pi r_0 b (1-\nu)} \frac{1}{\sin \phi} \sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') \cos n\phi \quad (r_i \leq r < r_0) \\ (\sigma_z)_{z=0} &= -\frac{Gr_0}{1-\nu} \sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') \int_0^\pi \frac{\lambda h \cosh \lambda h + \sinh \lambda h}{\lambda h + \sinh \lambda h \cosh \lambda h} \lambda J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b) d\lambda \\ P &= -2\pi \int_{r_i}^{r_0} (\sigma_z)_{z=h} dr = \frac{2\pi Gr_0}{1-\nu} (\varepsilon_0 a_0' - u a_0'') \end{aligned} \right\} \quad (24)$$

Moreover, use of the following formula

$$\left. \begin{aligned} \frac{d}{dr} \{J_0(\lambda r) J_n(\lambda r_c) J_n(\lambda b)\} &= -J_1(\lambda r) \left[\lambda J_n(\lambda r_c) J_n(\lambda b) \right. \\ &\quad \left. - \frac{\cos \lambda r_i + (-1)^n \sin \lambda r_0}{\pi \sqrt{r_c b}} \right] + \frac{\cos \lambda r_i + (-1)^n \sin \lambda r_0}{\pi \sqrt{r_c b}} \end{aligned} \right\} \quad (25)$$

yields

$$\begin{aligned} \left(\frac{\partial w_z}{\partial r}\right)_{z=h} = & - \sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') r_0 \int_0^{\infty} \{t(\lambda) - 1\} \lambda J_1(\lambda r) J_n(\lambda r_0) J_n(\lambda b) d\lambda \\ & - \sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') r_0 \int_0^{\infty} J_1(\lambda r) \left\{ \lambda J_n(\lambda r_0) J_n(\lambda b) - \frac{\cos \lambda r_0 + (-1)^n \sin \lambda r_0}{\pi \sqrt{r_0 b}} \right\} d\lambda \\ & - \frac{1}{\pi \sqrt{r_0 b}} \frac{r_0}{r} \sum_{n=0}^{\infty} (\varepsilon_0 a_n' - u a_n'') \left\{ \left(1 - \frac{r_i H(r_i - r)}{\sqrt{r_i^2 - r^2}}\right) + (-1)^n \frac{r_0 H(r - r_0)}{\sqrt{r^2 - r_0^2}} \right\} \dots\dots\dots (26) \end{aligned}$$

Since every integrand appearing in Eq.(26) is finite and continuous and approaches zero in the order of $\lambda^{-3/2}$ as $\lambda \rightarrow \infty$, the infinite integrals are finite and continuous at $r=r_i$ and $r=r_0$. Further, the first term of the final term turns out to be equal to zero by virtue of Eq.(23). Thus it is seen that $(\partial w_z / \partial r)_{z=h}$ is continuous at $r=r_i$, and that the finiteness of $(\sigma_z)_{z=h}$ is equivalent to the continuity of the radial gradient of $(w_z)_{z=h}$ at the inner circumference $r=r_i$. On the other hand, since the second term of the final term in Eq.(26) does not vanish, it is seen that the radial gradient of $(w_z)_{z=h}$ diverges to infinity near the outer edge $r=r_0+0$.

In particular, the following are obtained for the parabolic concave punch

$$u=u_1, \quad g(r)=1-\frac{r^2}{r_0^2}, \quad b_m=\frac{r_0 b}{r_0^2} (2\delta_{0m}+\delta_{1m}) \quad (m=0, 1, 2, \dots) \dots\dots\dots (27)$$

and for the conical concave punch

$$u=u_2, \quad g(r)=1-\frac{r}{r_0}, \quad b_m=\frac{1}{\pi r_0} \int_0^\pi \{r_0 - \sqrt{r_0^2 + b^2 - 2r_0 b \cos \phi}\} \cos m\phi d\phi \quad (m=0, 1, 2, \dots) \dots\dots\dots (28)$$

Table 1 The values of $\{a_n' - (u/\varepsilon_0)a_n''\}$, ε_0/u and $P(1-\nu)/Gr_0u$

(a) parabolic concave punch

$\begin{matrix} r_i/r_0 \\ \delta/r_0 \\ n \end{matrix}$	0.25	0.5					0.75
	1.0	0.5	1.0	2.0	10.0	∞	1.0
0	1.0026350	1.2649584	0.9533817	0.7571161	0.5837918	0.5467539	0.8404521
1	-1.1916390	-1.3776045	-1.0912333	-0.8649547	-0.6665250	-0.6241657	-0.9088071
2	0.1679276	0.0964162	0.1272724	0.0989470	0.0758048	0.0709899	0.0651087
3	0.0149826	0.0105082	0.0085694	0.0072853	0.0056821	0.0053169	0.0023847
4	0.0038877	0.0044676	0.0014726	0.0011806	0.0009170	0.0008583	0.0001670
5	0.0013101	0.0009212	0.0003390	0.0002685	0.0002081	0.0001888	0.0000880
6	0.0004739	0.0001055	0.0000323	0.0000258	0.0000201	0.0000317	-0.0001370
7	0.0002556	0.0001397	0.0001003	0.0000796	0.0000614	0.0000234	0.0003225
8	0.0000048	-0.0001817	-0.0001464	-0.0001160	-0.0000894	-0.0000166	-0.0005697
9	0.0001615	0.0002686	0.0002116	0.0001677	0.0001292	0.0000196	0.0009913
ε_0/u_1	1.304	0.875	1.144	1.601	2.174	2.322	0.800
$P(1-\nu)/Gr_0u_1$	8.214	6.953	6.850	7.617	7.974	7.975	4.227

(b) conical concave punch

$\begin{matrix} r_i/r_0 \\ \delta/r_0 \\ n \end{matrix}$	0.25	0.5					0.75
	1.0	0.5	1.0	2.0	10.0	∞	1.0
0	1.1398974	1.3856392	1.0215664	0.7938122	0.6040578	0.5644867	0.8631494
1	-1.0450401	-1.2988157	-1.0125656	-0.7974924	-0.6101677	-0.5701304	-0.8752017
2	-0.0776330	-0.0910419	-0.0113090	0.0005874	0.0034665	0.0032348	0.0099525
3	-0.0145934	-0.0008422	0.0009291	0.0019369	0.0017374	0.0016209	0.0012780
4	-0.0027548	0.0036934	0.0008212	0.0007197	0.0005725	0.0005344	0.0001465
5	-0.0004357	0.0009648	0.0003214	0.0002536	0.0001950	0.0001759	0.0000874
6	0.0000248	0.0001507	0.0000556	0.0000419	0.0000316	0.0000422	-0.0001349
7	0.0001732	0.0001532	0.0001067	0.0000829	0.0000631	0.0000288	0.0003113
8	0.0000493	-0.0001646	-0.0001313	-0.0001037	-0.0000795	-0.0000111	-0.0005516
9	0.0003123	0.0002671	0.0002062	0.0001615	0.0001233	0.0000179	0.0009630
ε_0/u_2	1.295	0.620	0.845	1.215	1.671	1.789	0.479
$P(1-\nu)/Gr_0u_2$	9.278	5.402	5.423	6.058	6.344	6.345	2.595

3. Numerical Results

The present problem is characterized by six parameters of r_0 , r_i , h , ε_0 , u and P , among which four are independent parameters because of two relationships Eq.(23) and the fourth of Eqs.(24). It is desirable to determine r_i and ε_0 for given r_0 , h , u and P , but it is difficult to realize this. Hence firstly the relationships among u , ε_0 and P are determined by solving the infinite system of equations (22) with given r_0 , r_i and h . Table 1 shows these coefficients and the relationships among parameters for the parabolic and the conical concave punches. Although the convergence slows down as the layer becomes thin, generally excellent convergence of the coefficients is observed.

In Fig.2 are shown the displacement $(w_z)_{z=h}$ and the contact stress $(\sigma_z)_{z=h}$ on the upper surface of the layer, and the contact stress $(\sigma_z)_{z=0}$ on the lower surface, for the case of the parabolic concave punch, $r_i/r_0=0.5$ and various thickness ratios h/r_0 . The surface displacement $(w_z)_{z=h}$ for $0 \leq r \leq r_i$ is larger than that of the punch and there arises an opening (the two-dot chain lines represent the punch face). At the circumference $r=r_i$, the layer makes smooth contact with the punch face, and the deformation of the layer coincides with the configuration of the punch face for $r_i \leq r \leq r_0$. From the punch edge $r=r_0+0$, the displacement decreases with infinite gradient, and approaches zero as $r \rightarrow \infty$. The distribution of $(w_z)_{z=h}$ increases with the layer's thickness and approaches that of the half space. The contact stress $(\sigma_z)_{z=h}$ is always compressive. From the circumference $r=r_i$ it increases from zero with infinite gradient and tends to infinity as $r \rightarrow r_0-0$. The distributions of $(\sigma_z)_{z=h}$ are not so affected by the layer's thickness. The contact stress $(\sigma_z)_{z=0}$ varies smoothly. It is always compressive if the layer is thick, but it becomes tensile near the center $r=0$ if the layer is thin. This fact implies the layer separates from the rigid base, but it is assumed that $(w_z)_{z=0}=0$ and this phenomenon is not examined quantitatively.

Fig.3 shows $(w_z)_{z=h}$, $(\sigma_z)_{z=h}$ and $(\sigma_z)_{z=0}$ in the case of conical concave punch, $r_i/r_0=0.5$ and various h/r_0 . The distributions are similar to those of Fig.2 (the two-dot chain lines represent the punch face).

Fig.4 shows $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ on the upper surface of the layer with thickness $h/r_0=1.0$, in the case of the parabolic concave punch when the indented load P is increasing and r_i/r_0 takes the values of 0.75, 0.5 and 0.25. The two-dot chain lines represent the punch faces. Fig.5 shows similar distributions for the conical concave punch. In both of the above results, the larger the indented load P , the larger become the indented depth ε_0 and the contact region, and the distributions of $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ become the larger in general.

Fig.6 shows the relationships between the parameters $P(1-\nu)/Gr_0u$ and r_i/r_0 . The punch meets the layer at $r_i/r_0=1.0$ where $P=0$. As r_i/r_0 decreases and the contact region increases, the indented load P increases to a definite value in the case of parabolic concave punch, but it diverges to infinity in the case of the conical concave punch. This fact means that the entire face of the conical punch can not be in contact with the layer.

Fig.7 shows the relationships between the parameters ε_0/u and r_i/r_0 . The indented depth ε_0/u is zero when $r_i/r_0=1.0$, and it increases monotonously as r_i/r_0 decreases to zero. In that case, ε_0/u for the parabolic concave punch approaches a definite value. But ε_0/u for the conical concave punch diverges to infinity and the entire contact can

(a) $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ on the upper surface of the layer

(b) $(\sigma_z)_{z=0}$ on the lower surface of the layer

Fig.2 Displacements and stresses in the case of the parabolic concave punch ($r_i/r_0=0.5$)

(a) $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ on the upper surface of the layer

(b) $(\sigma_z)_{z=0}$ on the lower surface of the layer

Fig.3 Displacements and stresses in the case of the conical concave punch ($r_i/r_0=0.5$)

Fig. 4 $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ in the case of the parabolic concave punch ($h/r_0=1.0$)

Fig. 5 $(w_z)_{z=h}$ and $(\sigma_z)_{z=h}$ in the case of the conical concave punch ($h/r_0=1.0$)

(a) parabolic concave punch

(b) conical concave punch

Fig. 6 The relation between $P(1-\nu)/G_0u$ and r_i/r_0

(a) parabolic concave punch

(b) conical concave punch

Fig. 7 The relation between ϵ_0/u and r_i/r_0

not take place as has been stated about Fig. 6. In all cases the indented depth becomes greater as the layer becomes thicker, and approaches the results of the case $h/r_0=\infty$ (the half space).

The above numerical examples give the values of r_0 , r_i and h to obtain the relationships among P , ϵ_0 and u . However, it is easy to obtain the relationships among r_i , ϵ_0 and u for given r_0 , h and P if Figs. 6 and 7 are used.

4. Conclusions

When an axisymmetric concave rigid punch is indented to an elastic layer by a small load, the contact region is an annulus.

The stress state of this problem is analyzed by the use of the three-dimensional theory of elasticity. In the case of the parabolic and the conical concave punches, some numerical calculations are carried out, and the distributions of the surface displacements and the contact stresses are determined and the relationships among the indented load, the indented depth, the punch face configuration, the layer's thickness and the contact region are clarified.

Reference

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