Minimum-Time Control of a Manipulator with Two-Degrees of Freedom*

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Accurate switching times of manipulated variables for a bang-bang control of a manipulator with two-degrees of freedom are easily calculated by approximating the velocity of a DC servomotor. Such bang-bang control is compared with a near-minimum-time control whose number of switching times of manipulated variables is two and the dynamic characteristics of the manipulator controlled by those methods are analyzed theoretically and also investigated experimentally.

Key Words: Control Theory, Control Devices, Minimum Time Control, Manipulator

1. Introduction

In general, a minimum-time control of a manipulator whose driving sources have characteristics of saturation is a bang-bang control, and it is difficult to calculate the switching times of manipulated variables by the maximum principle because the dynamical equations of the system are highly nonlinear.

The minimum-time control of the system has been reported to be obtained by the maximum principle⁽¹⁾⁽²⁾, but their calculations are complicated. And a suboptimal control has been obtained by approximating the nonlinear system to a linear system⁽³⁾, but the approximations for the effects of angular velocity forces are not enough. It is necessary to use compact driving sources and compact mechanisms for speedy handling work, and it is necessary to drive the system in consideration of the characteristics of saturation for the output torque and the velocity of a motor.

In this paper, the accurate switching times of the manipulated variables for the bang-bang control of the manipulator with two-degrees of freedom are easily calculated by using an approximated expression for the velocity of a DC servomotor, and speedy and accurate positioning handling works become possible.

2. Minimum-time control in Consideration of Characteristics for a Driving Source

2.1 Characteristics for a driving source

In general, the DC servomotor is in use to drive the system. For the armature of the servomotor, the output torque M_i is directly proportional to the electric current i_a , and their relation is expressed as

$$M_i = k_t i_a \dots\dots\dots (1)$$

And the input voltage for the armature is

$$e_i = k_v \dot{\theta} + R_a i_a + L_a \frac{di_a}{dt} \dots\dots\dots (2)$$

where

e_i : input voltage for armature
 $\dot{\theta}$: angular velocity of servomotor
 R_a : resistance of armature
 L_a : inductance of armature

A relation between the output torque of the armature and the output torque of the servomotor is expressed as⁽⁴⁾

$$M_i = J_m \frac{d\dot{\theta}}{dt} + D_m \dot{\theta} + M_l \dots\dots\dots (3)$$

where

M_l : output torque of servomotor
 J_m : moment of inertia for armature
 D_m : coefficient of viscous damping

When $L_a \ll 1$, the input voltage for the servomotor is

$$e_i = b_1 \dot{\theta} + b_2 \ddot{\theta} + b_3 M_l \dots\dots\dots (4)$$

where

$$b_1 = k_v + \frac{R_a}{k_t} D_m, \quad b_2 = \frac{R_a}{k_t} J_m, \quad b_3 = \frac{R_a}{k_t} \dots\dots (5)$$

When the reduction gear train is in use, the input voltage for the servomotor is

$$e = r b_1 \dot{\theta} + r b_2 \ddot{\theta} + \frac{1}{r} b_3 M_l \equiv c_1 \dot{\theta} + c_2 \ddot{\theta} + c_3 M_l \dots\dots (6)$$

where

$1/r$: reduction ratio
 c_j : coefficient for servomotor when reduction ratio is $1/r$

The output torque of the armature has the characteristics of saturation. When its maximum torque is M_{lmax} , the output torque, M_l , is

$$M_l = \frac{1}{c_3} e - \frac{c_1}{c_3} \dot{\theta} - \frac{c_2}{c_3} \ddot{\theta} \quad \text{for } M_l \leq M_{lmax} \dots\dots\dots (7)$$

$$M_l = M_{lmax} - D_m \dot{\theta} - J_m \ddot{\theta} \quad \text{for } M_l > M_{lmax} \dots\dots\dots (8)$$

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2.2 Dynamical equations of the manipulator with two-degrees of freedom

The parameters of the manipulator with two-degrees of freedom as shown in Fig.1 which is able to move in a perpendicular plane are as follows.

- l_j : length of link j
- m_j : mass of link j
- m_3 : mass of the load
- l_{j1} : length between an end of a link and its center of gravity
- I_j : moment of inertia about the center of gravity
- M_j : torque which acts on link j
- θ_j : angular displacement of link j

Then, the dynamical equations of the system are

$$M_1 = (a_1 + a_2 + 2a_3 \cos \theta_2) \ddot{\theta}_1 + (a_2 + a_3 \cos \theta_2) \ddot{\theta}_2 - a_3 \dot{\theta}_2^2 \sin \theta_2 - 2a_3 \dot{\theta}_1 \dot{\theta}_2 \sin \theta_2 + a_4 \cos \theta_1 + a_5 \cos (\theta_1 + \theta_2) \quad (9)$$

$$M_2 = (a_2 + a_3 \cos \theta_2) \ddot{\theta}_1 + a_2 \ddot{\theta}_2 + a_3 \dot{\theta}_1^2 \sin \theta_2 + a_5 \cos (\theta_1 + \theta_2) \quad (10)$$

where

$$\left. \begin{aligned} a_1 &= I_1 + m_1 l_{p1}^2 + (m_2 + m_3) l_1^2, & a_2 &= I_2 + m_2 l_{p2}^2 + m_3 l_2^2, & a_3 &= (m_2 l_{p2} + m_3 l_2) l_1 \\ a_4 &= (m_1 l_{p1} + (m_2 + m_3) l_1) g, & a_5 &= (m_2 l_{p2} + m_3 l_2) g \end{aligned} \right\} \quad (11)$$

By substituting M_j for M_i in Eqs. (7) and (8), the output torques and the input voltages of motors are as follows

$$M_{tj \max} = D_{mj} \dot{\theta}_j + J_{mj} \ddot{\theta}_j + M_j \quad \text{for} \quad M_t > M_{t \max} \quad (12)$$

$$e_j = c_{1j} \dot{\theta}_j + c_{2j} \ddot{\theta}_j + c_{3j} M_j \quad \text{for} \quad M_t \leq M_{t \max} \quad (13)$$

When the output torque of the motor is sufficiently large as compared with the torque of the load, the input voltages are as below

$$e_1 = c_{11} \dot{\theta}_1 + (c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \theta_2)) \ddot{\theta}_1 + c_{31}(a_2 + a_3 \cos \theta_2) \ddot{\theta}_2 - c_{31} a_3 \dot{\theta}_2^2 \sin \theta_2 - 2c_{31} a_3 \dot{\theta}_1 \dot{\theta}_2 \sin \theta_2 + c_{31} a_4 \cos \theta_1 + c_{31} a_5 \cos (\theta_1 + \theta_2) \quad (14)$$

$$e_2 = c_{12} \dot{\theta}_2 + (c_{22} + c_{32} a_2) \ddot{\theta}_2 + c_{32}(a_2 + a_3 \cos \theta_2) \ddot{\theta}_1 + c_{32} a_3 \dot{\theta}_1^2 \sin \theta_2 + c_{32} a_5 \cos (\theta_1 + \theta_2) \quad (15)$$

2.3 Bang-bang control

The minimum-time control between the positions where the handling work starts and where the handling work finishes is as follows.

By substituting φ_j for θ_j in Eqs. (14) and (15), we have the dynamical equations of the system as follows

$$\begin{aligned} \dot{\varphi}_3 &= [(c_{22} + c_{32} a_2)(e_1 - c_{11} \varphi_3 + c_{31} a_3 \varphi_4^2 \sin \varphi_2 + 2c_{31} a_3 \varphi_3 \varphi_4 \sin \varphi_2 - c_{31} a_4 \cos \varphi_1 - c_{31} a_5 \cos (\varphi_1 + \varphi_2)) \\ &\quad - c_{31}(a_2 + a_3 \cos \varphi_2)(e_2 - c_{12} \varphi_4 - c_{32} a_3 \varphi_3^2 \sin \varphi_2 - c_{32} a_5 \cos (\varphi_1 + \varphi_2))] / \alpha \quad (16) \end{aligned}$$

$$\begin{aligned} \dot{\varphi}_4 &= -[c_{32}(a_2 + a_3 \cos \varphi_2)(e_1 - c_{11} \varphi_3 + c_{31} a_3 \varphi_3^2 \sin \varphi_2 + 2c_{31} a_3 \varphi_3 \varphi_4 \sin \varphi_2 - c_{31} a_4 \cos \varphi_1 - c_{31} a_5 \cos (\varphi_1 + \varphi_2)) \\ &\quad - (c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \varphi_2))(e_2 - c_{12} \varphi_4 - c_{32} a_3 \varphi_3^2 \sin \varphi_2 - c_{32} a_5 \cos (\varphi_1 + \varphi_2))] / \alpha \quad (17) \end{aligned}$$

where

$$\varphi_1 = \theta_1, \quad \varphi_3 = \dot{\theta}_1, \quad \varphi_2 = \theta_2, \quad \varphi_4 = \dot{\theta}_2 \quad (18)$$

$$\alpha = (c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \varphi_2))(c_{22} + c_{32} a_2) - c_{32} c_{31}(a_2 + a_3 \cos \varphi_2)^2 \quad (19)$$

For use in the statement of the maximum principle, a Hamiltonian function⁽⁵⁾ H , is defined as

$$\begin{aligned} H &= \lambda_3 [(c_{22} + c_{32} a_2) e_1 - c_{31}(a_2 + a_3 \cos \varphi_2) e_2] / \alpha - \lambda_4 [c_{32}(a_2 + a_3 \cos \varphi_2) e_1 \\ &\quad - (c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \varphi_2) e_2) / \alpha + 1 + (\text{terms not depend on } e_1, e_2)] \quad (20) \end{aligned}$$

where the manipulated variables e_j have the characteristics of saturation as

$$-(e_j)_{\max} \leq e_j \leq (e_j)_{\max} \quad (j=1, 2) \quad (21)$$

According to the maximum principle, the optimal controls, e_j , $j=1, 2$, minimize Eq. (20), and by inspection we see that

$$e_1 = (e_1)_{\max} \text{ sign } [(\lambda_3(c_{22} + c_{32} a_2) - \lambda_4 c_{32}(a_2 + a_3 \cos \varphi_2)) / \alpha] \quad (22)$$

$$e_2 = (e_2)_{\max} \text{ sign } [-\lambda_3 c_{31}(a_2 + a_3 \cos \varphi_2) / \alpha + \lambda_4 (c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \varphi_2)) / \alpha] \quad (23)$$

Owing to the nonlinearity of the equations involved, a numerical approach is very difficult. We usually approximate the nonlinear system to a linear system, but, for the system whose driving source has characteristics of saturation about its angular velocity, the approximations for the effects of angular velocity forces are not enough.

2.4 Characteristics of saturation for torque and velocity

When the reduction ratio is large,

$$a_j \ll c_{2j}/c_{3j} = r^2 b_{2j}/b_{3j}$$

Then, equations (14) and (15) are approximated as follows.

$$e_1 = c_{11}\dot{\theta}_1 + \{c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \theta_2)\}\dot{\theta}_1 + c_{31}\{a_4 \cos \theta_1 + a_5 \cos(\theta_1 + \theta_2)\} \dots (24)$$

$$e_2 = c_{12}\dot{\theta}_2 + (c_{22} + c_{32}a_2)\dot{\theta}_2 + c_{32}a_3 \cos(\theta_1 + \theta_2) \dots (25)$$

And the angular velocity are approximated as follows.

$$\dot{\theta}_j = \frac{e_j - A_j}{c_{1j}} \left[1 - \exp\left(-\frac{t}{T_{mj}}\right) \right] \quad (j=1,2) \dots (26)$$

where

$$\left. \begin{aligned} A_1 &= c_{31}\{a_4 \cos \bar{\theta}_1 + a_5 \cos(\bar{\theta}_1 + \bar{\theta}_2)\}, & A_2 &= c_{32}a_3 \cos(\bar{\theta}_1 + \bar{\theta}_2) \\ T_{m1} &= \{c_{21} + c_{31}(a_1 + a_2 + 2a_3 \cos \bar{\theta}_2)\}/c_{11}, & T_{m2} &= (c_{22} + c_{32}a_2)/c_{12} \\ \bar{\theta}_j &= (\theta_{ij} + \theta_{fj})/2 & \dot{\theta}_{ij} &= (e_j - A_j)/c_{1j} \end{aligned} \right\} \dots (27)$$

Thus, when the output torque of the motor is sufficiently large as compared with the torque of the load, the angular velocity is expressed as an exponential function.

When the output torque of the motor is not sufficiently large, the output torque of the armature is saturated. When the output torque of motor 1 and the angular velocity of motor 2 are saturated, the angular velocity of link 1 is

$$\dot{\theta}_1 = (K_{11} + K_{12})t + T_{m2}K_{12} \exp(-t/T_{m2}) + c \quad (c: \text{constant}) \dots (28)$$

where

$$K_{11} = [M_{t1\max} - a_4 \cos \bar{\theta}_1 - a_5 \cos(\bar{\theta}_1 + \bar{\theta}_2) - (a_2 + a_3 \cos \bar{\theta}_2)(e_2 - c_{32}a_3 \cos(\bar{\theta}_1 + \bar{\theta}_2) + \bar{\theta}_2)] / (c_{22} + c_{32}a_2) / (a_1 + a_2 + 2a_3 \cos \bar{\theta}_2 + J_{m1}) \dots (29)$$

$$K_{12} = (a_2 + a_3 \cos \bar{\theta}_2) c_{12} \dot{\theta}_{12} / \{(a_1 + a_2 + 2a_3 \cos \bar{\theta}_2 + J_{m1})(c_{22} + c_{32}a_2)\} \dots (30)$$

Thus, when the angular velocity of motor 1 or motor 2 is saturated, the angular velocities of motors 1 and 2 are approximated to functions of one variable.

3. Simulations

We shall give the parameters of the manipulator as shown in Tables 1 and 2, and approximate $M_{j\max}$ to a constant. The simulations of the system are as follows.

3.1 Bang-bang control and near-minimum-time control

For the bang-bang control, the number of switching times of manipulated variables is two for the motor which arrives at the desired value before the other motor, and the number of switching times is one for the other motor. Then the sum of the number of switching times, n , is three. Switching time, t_{qj} , of the manipulated variable and the time, t_{fj} , when the link j arrives at the desired position are

$$t_{qj} = \frac{c_{1j}}{E_{j\max} - A_j} \left[\theta_{fj} - \theta_{ij} - \frac{E_{j\max} + A_j}{c_{1j}} T_{mj} \ln \frac{E_{j\max} + A_j}{2E_{j\max}} \right] \dots (31)$$

$$t_{fj} = t_{qj} - T_{mj} \ln \frac{E_{j\max} + A_j}{2E_{j\max}} \quad (j=1,2) \dots (32)$$

where

$$E_{j\max} = e_{j\max} \text{sign}(\theta_{fj} - \theta_{ij})$$

Then, we shall modify the switching time for the motor which arrives at the desired value before the other motor, and the time is

$$t_{qj} = t_{fk} + T_{mj} \ln \frac{E_{j\max} + A_j}{2E_{j\max}} \quad \{ j=1 (k=2) \text{ for } t_{f1} < t_{f2}, j=2 (k=1) \text{ for } t_{f1} > t_{f2} \} \dots (33)$$

And then, we shall calculate the switching time, t_{xj} , from the equation.

$$\begin{aligned} & \frac{E_{j\max} - A_j}{c_{1j}} \left[t_{qj} - t_{xj} - T_{mj} \left\{ 1 - \exp\left(-\frac{t_{qj} - t_{xj}}{T_{mj}}\right) \right\} \right] + T_{mj} \frac{E_{j\max} + A_j}{c_{1j}} \left[1 - \exp\left(-\frac{t_{xj}}{T_{mj}}\right) \right] \\ & \times \exp\left(-\frac{t_{qj} - t_{xj}}{T_{mj}}\right) - \frac{E_{j\max} + A_j}{c_{1j}} t_{xj} + \theta_{ij} - \theta_{fj} + \frac{E_{j\max} + A_j}{c_{1j}} T_{mj} \ln \frac{E_{j\max} + A_j}{2E_{j\max}} + T_{mj} \frac{E_{j\max} - A_j}{c_{1j}} = 0 \end{aligned} \dots (34)$$

Thus, we shall calculate the switching times and simulate the system from Eqs.(9), (10), (14) and (15). A flow chart of the simulation is shown in Fig.2.

A response of the manipulator for the bang-bang control is shown in Fig.3. When the output torques of the motors are saturated, the input voltages which are proportional to saturated output torques are shown. When $e_1 = e_{1\max}$ and $e_2 = e_{2\max}$, the link 2 arrives at the desired position before the link 1, then the number of switching times for the motor 2 is two.

Here, we shall calculate e_2 on the condition that the links 1 and 2 arrive at the same time. The relation is

$$t_{fk} - \frac{c_{1j}(\theta_{fj} - \theta_{ij})}{e_j - A_j} + \frac{2e_j}{e_j - A_j} T_{mj} \ln \frac{e_j + A_j}{2e_j} = 0 \quad (j=2 \text{ then } k=1, j=1 \text{ then } k=2) \dots (35)$$

We shall call it a near-minimum-time control, and simulate according to the flow chart in Fig.2, ($n=2$). The response of the system for the near-minimum-time control is shown in Fig.4. Thus, when the motors have the characteristics of saturation about angular velocity, the time that is necessary for bang-bang control is the same as the time that is necessary for near-minimum-time control.

3.2 Relation between reduction ratio and angular velocity

When $1/r_2=1/25, 1/50, 1/100$, the respective responses for near-minimum-time control are shown in Fig.5, 6 and 7. Here, we shall not repeat to calculate torques and voltages from Eqs.(9), (10), (14) and (15) in flow chart, but use an approximation of angular velocity which is

Fig.1 Mechanism of manipulator

Fig.2 Flow chart of simulation

Table 1 Parameters of mechanism

	Link 1	Link 2
l_j m	0.270 0	0.230 0
l_{qj} m	0.116 0	0.048 5
m_j kg	4.978	0.862
I_j kgm ²	0.126	0.075

Table 2 Parameters of motor

	Motor 1		Motor 2	
$1/r_j$	1/100	1/25	1/50	1/100
$e_{j\max}$ V	24.0	24.0	24.0	24.0
$M_{j\max}$ Nm	31.26	9.69	19.38	38.76
c_{1j} Vs/rad	8.29	1.699	3.398	6.796
c_{2j} Vs ² /rad	0.348 6	0.102 0	0.204	0.408
c_{3j} V/Nm	0.073 5	0.678 6	0.339 3	0.169 7

Fig.3 Bang-bang control

$$\dot{\theta}_1 = \frac{M_{1\max} - a_4 \cos \theta_{i1} - a_5 \cos(\theta_{i1} + \theta_{i2})}{a_1 + a_2 + 2a_3 \cos \theta_{i2}} t \dots (36)$$

When the reduction ratio of the motor 2 is small, the response of link 2 is influenced by the acceleration $\ddot{\theta}_1$, and, when the reduction ratio of the motor 2 is large, the output torque of motor 1 is saturated.

Here, we shall define a deviation of angular velocity as follows.

$$\delta_j = \frac{\int |\dot{\theta}_{ja} - \dot{\theta}_j| dt}{(\theta_{fj} - \theta_{ij})}$$

where

$\dot{\theta}_{ja}$: angular velocity which is calculated from Eqs.(26), (28) and (36)

$\dot{\theta}_j$: angular velocity which is calculated from Eqs.(9), (10), (14) and (15)

The relation between δ_j and $1/r_2$ is shown in Fig.8. When the mass of load is large, the δ_1 increases and δ_2 decreases as the reduction ratio of the motor 2 increases. Thus, when the mass of load is small and the reduction ratio of motor 2 is large, the switching times are easily calculated by approximating the velocity of servomotor to the exponential function.

4. Experiment

When $1/r_1=1/100$, $1/r_2=1/25$ and $e_{1\max}=e_{2\max}=10V$, the experimental results are as follows.

Fig.5 Relation between reduction ratio and angular velocity ($1/r_2=1/25$)

Fig.4 Near-minimum-time control

Fig.6 Relation between reduction ratio and angular velocity ($1/r_2=1/50$)

Figure 9 shows the response of the system when $m_3=0$ and $n=2$. The response of link 1 is influenced by the frictional torque of the gear train, and the response of link 2 is influenced by the centrifugal force.

Here, we shall define a positioning error as follows.

$$\epsilon_j = \frac{\theta_{fj} - \theta_j}{|\theta_{fj} - \theta_{ij}|} \quad (j=1, 2)$$

When the initial conditions and final conditions are the same as shown in Fig.9, the relation between ϵ_j and m_3 is as shown in Fig.10. If the power sources for the motors are sufficiently large, the positioning errors are smaller than 4 per cent.

Fig.7 Relation between reduction ratio and angular velocity ($1/r_2=1/100$)

Fig.8 Relation between reduction ratio and deviation of angular velocity

Fig.9 Experimental result ($n=2$)

Fig.10 Relation between positioning error and mass of load ($n=2$)

Fig.11 Experimental result ($n=3$)

Figure 11 shows the responses of the system when $m_3=0$ and $n=3$. The responses of links 1 and 2 are influenced by the frictional torque and the centrifugal force as the responses shown in Fig.9. With the same initial and final conditions, the relation between ε_1 and m_3 is as shown in Fig.12. An inertial force of link 2 increases as the mass of the load increases, and the error, ε_1 , changes to a negative value and ε_2 changes to a positive value.

From these results, it is clarified that the experimental responses of the system are nearly equal to the theoretical responses.

Fig.12 Relation between positioning error and mass of load ($n=3$)

5. Conclusions

The results obtained in this paper are summarized as follows.

(1) Accurate switching times of manipulated variables for the minimum-time control of the manipulator with two-degrees of freedom have been easily calculated using an approximated expression to the velocity of the DC servomotor.

(2) When the driving source of the manipulator has the characteristics of saturation for velocity, the time which is necessary for the bang-bang control is the same as the time for the near-minimum-time control whose number of switching times is two.

(3) It has been clarified that the experimental responses of this control method are effective.

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