**Numerical Computation of Unsteady
Laminar Boundary Layers with Separation
Using One-parameter Integral Method ***

By Michinori MATSUSHITA ** and Teruaki AKAMATSU ***

A two-moment one-parameter integral method is proposed which is generally applicable even to unsteady separated boundary layers over bodies of arbitrary shape in arbitrarily time-varying external flows. The governing equations of momentum and energy integrals are confirmed to be classified as a quasi-linear hyperbolic system and numerically solved by an up-wind differencing scheme. Then the method is applied to the laminar boundary layers over impulsively started circular and elliptic cylinders. The time-varying profiles of displacement thickness, shape factor and skin-friction coefficient are obtained. Especially, the loci on x - t diagram of the points of vanishing wall shear stress agree quite well with the known analytical or exact numerical solutions. The present method is found by no means inferior to exact numerical methods in accuracy and wide applicability, moreover it is excellently efficient in computer storage and computational time.

1. Introduction

Boundary layer separation is the controlling feature of wide range of flows of practical importance. Numerical integration of the full Navier-Stokes equations may be one of the promising approaches to solving the problems of separated flows of high Reynolds numbers. However, despite the recent advances made in computer technology and computational schemes, numerical solutions of Navier-Stokes equations for such flows are still impractical. Further, they might be unnecessary in many instances, for there are growing evidences that the boundary layer equations provide a reasonably accurate model of separated flows. The detailed discussion and reviews of regular solutions of boundary layer equations for separated flows are made in recent publications^{(5),(6),(10),(13)}.

There appears to be a great deal still to be learned about unsteady viscous flows within the more attractable framework of classical boundary layer theory. Further, for flow solutions about practical and complex geometries at reasonable cost, the advantages of integral method for boundary layer should never be overlooked.

One of the earliest and, until recently, most widely used approximate methods for the solution of the boundary layer equation is the momentum integral method developed by Pohlhausen. This method is based on the momentum integral equation and velocity profiles of quartic polynomial with

$$\lambda = \frac{\delta_2^2}{\nu} \frac{dU}{dx} \quad (1)$$

as the parameter. Though it gives a reasonably accurate solution in a region of accelerated flow, its adequacy in a region of retarded flow has been doubtful. Various attempts have been made to increase the accuracy of the method, and a monograph by Rosenhead⁽⁷⁾ discusses a large number of studies in detail. Some of them utilize higher order polynomials so as to improve the assumption of velocity profile. However, none of them gives satisfactory results so long as the velocity profile is expressed as a family of curves utilizing the parameter λ . Abandoning the boundary condition on $\partial^2 u / \partial y^2$ at the wall, Tani⁽⁸⁾ utilized mechanical energy integral equation instead and specified the velocity profiles of quartic polynomial in terms of a single independent parameter $a(x)$ in Eq.(14) proportional to the slope at the wall instead of λ . Thus he obtained excellent agreement with the exact solutions for prescribed retarded freestream velocity of

$$U = 1 - x^n \quad (n = 1, 2, 4, 8) \quad (2)$$

Though it takes extra computation because of utilizing two integral relations, Tani's method generally gives much better accuracy than other one-parameter integral methods.

Within the scope discussed above, a two-moment one-parameter integral method is proposed in this paper to solve the unsteady two dimensional laminar boundary layers including separation. The characteristic features of governing equations are discussed and a numerical method to solve them is described. Then the method is applied to the boundary layers over a circular cylinder and an elliptic cylinder impulsively started from rest.

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** Research student *** Professor
Department of mechanical engineering,
Kyoto University, Yoshida Honmachi, Sakyo-
ku, Kyoto

2. Governing Equations and Calculation Procedure

2.1 Governing equations

The momentum and mechanical energy integral equation are obtained as,

$$\frac{\partial}{\partial t}(U\delta_1) + \frac{\partial}{\partial x}(U^2\delta_1) + U\frac{\partial U}{\partial x}\delta_1 = \frac{\tau_w}{\rho} \quad (3)$$

$$\begin{aligned} \frac{\partial}{\partial t}(U^2\delta_2) + U^2\frac{\partial \delta_1}{\partial t} + \frac{\partial}{\partial x}(U^3\delta_3) \\ = \frac{2D}{\rho} \end{aligned} \quad (4)$$

which are also termed the zeroth and first moments of boundary layer equation. Here x is a distance measured along the surface of the body, t is time, and $U=U(x,t)$ is the freestream velocity just outside the boundary layer presumably known from a potential flow analysis. δ_1 , δ_2 and δ_3 are the displacement, momentum and energy thickness respectively defined as follows:

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \quad (5)$$

$$\delta_2 = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \quad (6)$$

$$\delta_3 = \int_0^\infty \frac{u^2}{U^2} \left(1 - \frac{u}{U}\right) dy \quad (7)$$

where y is the distance normal to the surface and u is the velocity component in the direction of x . The wall shear stress τ_w and the total dissipation D are defined as follows:

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \quad (8)$$

$$D = \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \quad (9)$$

The displacement thickness δ_1 is taken as a measure of the boundary layer thickness and the following four non-dimensional quantities are constructed from the physical quantities appearing in Eqs. (3) and (4).

$$E = \frac{\delta_2}{\delta_1} \quad B = \frac{2\delta_1\tau_w}{\mu U} \quad (10,11)$$

$$F = \frac{\delta_3}{\delta_1} \quad S = \frac{2\delta_1 D}{U^2} \quad (12,13)$$

Tani primarily adopted the momentum thickness δ_2 as the measure instead of δ_1 for the case without separation, whereas δ_2 is inconvenient in separated boundary layers as it may probably become zero. As mentioned in Sec. 2.2, the above four non-dimensional quantities are related to each other in a one-parameter family of velocity profiles. In the present paper the shape factor E defined in Eq. (10) is selected as the independent profile parameter. Though Lees and Reeves⁽³⁾ utilized different profile parameters for attached and separated boundary layers, the above mentioned shape factor E can deal with both attached and separated boundary layers without distinction.

2.2 Velocity profile assumptions

Following Tani⁽⁸⁾, we assume the velocity profile of quartic polynomial

$$\frac{u}{U} = (6-8\eta+3\eta^2)\eta^2 + a\eta(1-\eta)^3 \quad (14)$$

which satisfies the boundary conditions

$$\begin{aligned} \eta = 0; u = 0 \\ \eta = 1; u = U, \quad \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} = 0 \end{aligned} \quad (15)$$

where $\eta=y/\delta$ and δ is the boundary layer thickness and "a" is the profile parameter proportional to the wall shear stress. For positive and negative values of "a", Eq. (14) gives attached and separated velocity profiles respectively (Fig.1). The parameters E , F , S and B are now functions of "a" alone.

$$E = \frac{4/35 + a/105 - a^2/252}{2/5 - a/20} \quad (16)$$

$$F = \frac{876/5005 + 73a/5005 - 23a^2/5460 - a^3/2860}{2/5 - a/20} \quad (17)$$

$$S = 2\left(\frac{2}{5} - \frac{a}{20}\right)\left(\frac{48}{35} - \frac{4a}{35} + \frac{3a^2}{35}\right) \quad (18)$$

$$B = 2a\left(\frac{2}{5} - \frac{a}{20}\right) \quad (19)$$

1. $a=-7.06$ $E=-0.20$
2. $a=-4.30$ $E=0.0$
3. $a=0.0$ $E=0.286$
4. $a=2.0$ $E=0.392$
5. $a=4.0$ $E=0.444$

Fig.1 Velocity profiles for Tani's quartic polynomial.

Fig.2 Correlations of F , S and B with E for the quartic profile.

The value of E given by the above equation can never exceed 0.444. The parameters F , S and B are plotted versus E in Fig.2. Here the region $E > 0.444$ is extrapolated using the dotted three points of $E=0.286$, 0.392, 0.444 corresponding to $a=4, 2, 0$ respectively.

2.3 Characteristics of governing equations

In this section some mathematical characteristics of the governing equations are considered. The governing equations (3) and (4) may be rewritten in a non-conservation matrix form using non-dimensional variables as follows:

$$Au_x + Bu_t = g \quad (20)$$

Here subscripts denote partial differentiation, u and g are column vectors and A and B are 2×2 matrices obtained as:

$$\bar{u} = \begin{bmatrix} \delta_1 \\ E \end{bmatrix} \quad (21)$$

$$\bar{g} = \begin{bmatrix} B/2\delta_1 - \delta_1 U_t/U - (2E+1)\delta_1 U_x \\ S/\delta_1 - 2E\delta_1 U_t/U - 3F\delta_1 U_x \end{bmatrix} \quad (22)$$

$$A = \begin{bmatrix} UE & U\delta_1 \\ UF & U\delta_1 dF/dE \end{bmatrix} \quad (23)$$

$$B = \begin{bmatrix} 1 & 0 \\ E+1 & \delta_1 \end{bmatrix} \quad (24)$$

The characteristic equation of the above quasi-linear system is defined as $\det(A - \lambda B) = 0$ and we obtain

$$\begin{vmatrix} UE - \lambda & U\delta_1 \\ UF - \lambda(E+1) & U\delta_1 dF/dE - \lambda\delta_1 \end{vmatrix} = 0 \quad (25)$$

Rewriting it, we obtain

$$\lambda^2 - U\left(\frac{dF}{dE} - 1\right)\lambda + U^2\left(E\frac{dF}{dE} - F\right) = 0 \quad (26)$$

If the roots of this equation are real and distinct, then the system is classified as hyperbolic. Defining C_0 as the discriminant divided by U^2 of this quadratic equation, C_1 as the mean value of the roots divided by $U/2$ and C_2 as the products of the two roots, we obtain

$$C_0 = \left(\frac{dF}{dE}\right)^2 - 2(2E+1)\frac{dF}{dE} + 4F + 1 \quad (27)$$

$$C_1 = \frac{dF}{dE} - 1 \quad (28)$$

$$C_2 = E\frac{dF}{dE} - F \quad (29)$$

These variables are given in Fig.3 as functions of the shape factor E . As can be seen from the figure, C_0 is always positive in this region and the system (20) is hyperbolic in type. The characteristic direction dt/dx is given by the two roots λ_1 and λ_2 of Eq. (26) as $dt/dx = 1/\lambda_1, 1/\lambda_2$. One of the zero points of C_2 is $E=E_s=0.2857$ corresponding to separation profile. When E is less than E_s , at least one root or one characteristic direction is negative. And the mean value of the two roots becomes negative when E is less than $E_c=0.0699$, the zero point of C_1 .

2.4 Calculation procedure

The governing equations (3) and (4) may also be written in a so called conservation form as follows:

$$w_t + f_x = z \quad (30)$$

Here w , f and z are the following column vectors.

$$w = \begin{bmatrix} \delta_1 \\ (E+1)\delta_1 \end{bmatrix} \quad (31)$$

$$f = \begin{bmatrix} UE\delta_1 \\ UF\delta_1 \end{bmatrix} \quad (32)$$

$$z = \begin{bmatrix} B/2\delta_1 - \delta_1 U_t/U - (E+1)\delta_1 U_x \\ S/\delta_1 - 2E\delta_1 U_t/U - 2F\delta_1 U_x \end{bmatrix} \quad (33)$$

The simplest first order up-wind scheme is employed to solve Eq.(30). The time derivative is replaced by the forward difference and the spatial derivative is replaced either by backward or forward difference according as E is greater than or less than $E_c=0.0699$ mentioned in the last section. In addition, considering the effects of inhomogeneous term z the following two step scheme is proposed.

$$\begin{aligned} \tilde{w}_i^{n+1} &= w_i^n - \sigma \Delta t / \Delta x (f_i^n - f_{i-\sigma}^n) \\ &+ \Delta t z_i^n \end{aligned} \quad (34)$$

$$w_i^{n+1/2} = (w_i^n + \tilde{w}_i^{n+1})/2 \quad (35)$$

$$z_i^{n+1/2} = z(w_i^{n+1/2}) \quad (36)$$

$$\begin{aligned} w_i^{n+1} &= w_i^n - \sigma \Delta t / \Delta x (f_i^n - f_{i-\sigma}^n) \\ &+ \Delta t z_i^{n+1/2} \end{aligned} \quad (37)$$

Here, if both E_i^n and $E_{i-\sigma}^n$ are less than E_c , $\sigma=-1$ otherwise $\sigma=1$. The subscripts i and n are sequence numbers and Δx and Δt are grid sizes on the x - t coordinates respectively. The stability condition of this finite difference scheme is given by the well-known CFL condition

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{\max(|\lambda|)} \quad (38)$$

where λ is given by the equation (26) in the last section.

Fig.3 C_0 , C_1 and C_2 as functions of E .

3. Results and discussion

3.1 Circular cylinder

The unsteady two-dimensional laminar boundary layer over a circular cylinder impulsively started from rest is solved by the present method. For this problem there exist full numerical solutions of Navier-Stokes equations^{(2),(9)} as well as those of complete boundary layer equations^{(1),(11)}. Both solutions agree for the case of large Reynolds numbers and initial stages of motions. However, at larger times, the boundary layer solutions assuming an unchangeable freestream velocity distribution deviate from the Navier-Stokes solutions. The present results are compared with the series solutions by Blasius and Goldstein & Rosenhead⁽⁷⁾ and the numerical solutions by Cebeci⁽¹⁾. Cebeci utilized Keller's Box method modified by the zigzag differencing scheme when there were regions of reverse flow and he obtained a solution without singularity for $t < 1.4$.

The nondimensional freestream velocity around the circular cylinder is given by the potential flow theory as

$$U = 2\sin x \quad (39)$$

where x is the streamwise coordinate measured along the surface of the cylinder from the forward stagnation point and normalized by a radius of the cylinder. The freestream velocity U is normalized by the external flow velocity. Initially the cylinder is assumed to be covered entirely with the boundary layer of Rayleigh type. Then initial condition of the present problem is

$$t = \Delta t; \quad \delta_1 = \sqrt{B_R \Delta t}, \quad E = E_R \quad (40)$$

where B_R and E_R denote the values of B and E for Rayleigh flow. The numerical calculations are carried out up to $t=2$ and $x=\pi$ with the step sizes of $\Delta t=0.005$ and $\Delta x = \pi/100$ corresponding to 101 x -stations and 401 t -stations. The total CPU time was no more than four seconds of FACOM-M200 in Kyoto University Data Processing Center.

Figure 4(a) shows the dimensionless displacement thickness around the circular cylinder for different values of t ranging from 0.1 to 1.0. The results agree quite well with those of Cebeci⁽¹⁾ except near the rear stagnation point at larger values of t . These deviations are attributed to the fact that the velocity profile of an exact solution having a large back flow region near the rear stagnation point cannot be expressed accurately by the quartic profile. Fig.4(b) shows the time-varying profiles of the shape factor E and the horizontal fine line $E = 0.2857$ indicates the level of separation profile. Figure 4(c) shows the local skin-friction coefficient around the circular cylinder for different values of t ranging from 0.1 to 1.0. The present results agree quite well again with those of Cebeci not only in the attached flow region but also in the separated flow region. Figure 4(d) shows the trajectory of the point of vanishing wall shear stress as a function of time. Also shown in this figure are the results of Cebeci and first order

Fig.4 Computed results of (a) displacement thickness, (b) shape factor E and (c) skin friction for the circular cylinder.

Fig.4(d) Locus of the point of vanishing wall shear stress for the circular cylinder.

(Blasius) and second order (Goldstein & Rosenhead) series solutions given by the following equations⁽⁷⁾

$$t_B^{-1} = -U'(1 + 4/3\pi) \quad (41)$$

$$t_C^{-1} = -0.7122U'$$

$$+ \sqrt{0.7172U'^2 + 0.05975UU''} \quad (42)$$

where prime denotes differentiation with respect to x . The present result agrees quite well with that of Cebeci and Goldstein & Rosenhead. The first order expansion of Blasius underestimates the position of separation a little for larger times. Considering the fact that Cebeci terminated his computation at $t = 1.4$ because of an increasingly uneconomical situation, the present method is remarkably efficient.

3.2 Elliptic cylinder

The unsteady laminar boundary layer over an elliptic cylinder impulsively started from rest is investigated by the present method. In this problem, like the flows around the practical airfoils, leading edge separation and stall will occur at somewhat larger incidence angle. There exist numerical solutions of Navier-Stokes equations for this problem⁽⁴⁾ as well as analytical ones of matched asymptotic expansions⁽¹²⁾. Telionis & Tsahalis⁽¹¹⁾ obtained the numerical solutions of unsteady boundary layer equations.

The contour of an elliptic cylinder whose focus is $(+2,0)$ and thickness ratio is 0.5, is depicted in the Fig.5 given by

$$X = 4\cos\theta/\sqrt{3} \quad Y = 2\sin\theta/\sqrt{3} \quad (43)$$

When the incidence angle is α , the nondimensional freestream velocity around the cylinder is given by the potential flow theory as

$$U = |3\sin(\theta-\alpha)|/\sqrt{4\sin^2\theta + \cos^2\theta} \quad (44)$$

The nondimensional streamwise coordinate x along the body surface is given by

$$x = \left| \int_{\alpha+\pi}^{\theta} \sqrt{4(4\sin^2\theta + \cos^2\theta)}/3d\theta/L \right| \quad (45)$$

where the origin of this coordinate is the forward stagnation point $\theta = \alpha + \pi$. The freestream velocity distribution when the incidence angles are 0, 0.3π (upper and lower surface) and $\pi/2$ is shown in Fig.6. The initial and boundary conditions are the same as in the case of circular cylinder in the last section.

Figures 7, 8 and 9 show the results for the incidence angle $\alpha=0$, $\pi/2$ and 0.3π respectively. All the calculations are performed with the grid size of $\Delta x=0.01$ corresponding 101 x -stations. The total CPU time was 12 seconds of FACOM-M200. In Fig.9 as the flow field became asymmetrical, the calculations on the upper surface ($0 < x < 1$) and the lower surface ($1 < x < 2$) were performed separately and the results were plotted successively. All of them perfectly match in value and slope at the rear stagnation point $x = 1$, $\theta = 0.3\pi$.

Figures 7(a), 8(a) and 9(a) show the

Fig.5 Schema of an elliptic cylinder.

Fig.6 Potential flow velocity distribution on the elliptic cylinder.

time varying profiles of displacement thickness plotted at regular time intervals. Similarly in Fig.7(b), 8(b), 9(b) and Fig.7(c), 8(c), 9(c) the results of shape factor E and local skin friction coefficient are plotted respectively. The regions below the horizontal fine lines, $E=0.286$ and $C_f=0$, correspond to reversing flow. Figures 7(d), 8(d) and 9(d) show the locus of vanishing wall shear stress as a function of time, compared with the results of series solutions given in Eqs.(41) and (42).

Figures 7(a)-(d) show the results when the incidence angle is 0. Separation begins at the rear stagnation point $x = 1$ when $t = 0.045$ as indicated in Fig.7(d). The displacement thickness is minimum at the forward stagnation point and maximum at the rear stagnation point. The locus of the point of vanishing wall shear stress initially agrees with that of Goldstein & Rosenhead, but at larger times with that of Blasius.

Fig.8(a)-(d) show the results when the incidence angle $\alpha=\pi/2$. Separation first occurs at $x = 0.58$ when $t = 0.060$ as indicated in Fig.8(d) and then it moves apart and a separation bubble is formed. As shown by the slope of the curve, the reattachment point moves backward and the bubble becomes larger. The fine line shows the calculation limit where the shape factor E gets out of the available range shown in Fig.2.

Fig.7 Computed results of (a) displacement thickness, (b) shape factor E and (c) skin friction for the elliptic cylinder at the incidence angle $\alpha=0$.

Fig.7(d) Locus of the point of vanishing wall shear stress for the elliptic cylinder at the incidence angle $\alpha=0$.

Fig.8 Computed results of (a) displacement thickness, (b) shape factor E and (c) skin friction for the elliptic cylinder at the incidence angle $\alpha=\pi/2$.

Fig.8(d) Locus of the point of vanishing wall shear stress for the elliptic cylinder at the incidence angle $\alpha=\pi/2$.

Fig.9 Computed results of (a) displacement thickness, (b) shape factor E , (c) skin friction and (d) locus of the point of vanishing wall shear for the elliptic cylinder at the incidence angle $\alpha=0.3\pi$.

Fig.9(a)-(d) show the results when the incidence angle $\alpha=0.3\pi$. Separation first occurs at $x=1.2$ on the lower surface when $t=0.044$ as indicated by Fig.9(d), then the separation bubble spreads. When $t=0.123$ at $x=0.38$ on the upper surface, another bubble is formed. At the time $t=0.224$, two bubbles coalesce into a large wake, which means the onset of a stall. Thereafter the profiles in Fig.9(a)-(c) start unrealistic oscillations at $t=0.24$. Included further in Fig.9(d) are the results of Telionis & Tsahalis⁽¹¹⁾, which agree quite well with the present results.

4. Conclusions

A two-moment one-parameter integral method has been presented which is applicable to separated unsteady boundary layers. The governing equations, which consists of momentum and energy integral relations, were numerically analysed by a finite difference method so as to make the method applicable to the boundary layers on bodies with arbitrary shape and arbitrary varying freestream velocities. It was found that nothing but δ_1 and $E=\delta_1/\delta_2$ should be taken as the unknown variables of governing equation system for the identical treatment of both attached and separated flows. The governing equation system was confirmed to be classified as a quasi-linear hyperbolic system. Taking the inhomogeneous term of the system into account, a two-step up-wind scheme was constructed. Then the present method was

applied to the boundary layers over a circular and an elliptic cylinder impulsively started from rest. It was proved that the present method is excellently efficient as compared with the exact numerical methods and further it is by no means inferior to the latter in accuracy and applicability.

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