A Note on the Analytical Treatment in Ferrohydrodynamics

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A new complete set of basic equations for magnetic suspensions is derived on the theory of micropolar fluids developed by Eringen and a solution for these basic equations is obtained analytically for a steady motion of magnetic suspensions in a circular pipe which is placed in a homogeneous magnetic field parallel to the flow direction. A phenomenological treatment is given for the specification of material constants of a micropolar fluid, which are vortex viscosity and spin viscosity. In order to clarify the dynamical flow characteristics in the presence of a magnetic field, six dimensionless parameters to show the magnetic effect, polar effect, wall surface effect and so on are introduced into the solution for the rotational Peclet number $\text{Per}<1$ and $\text{Per}=1$. Some discussions are presented on velocity profiles, distribution of microrotations and distribution of vorticities. In particular, the suspension is non-Newtonian even if a magnetic field does not exist.

Key Words: Non-Newtonian Fluid, Ferrohydrodynamics, Polar Fluid, Magnetic Suspension, Spin Angular Momentum, Intrinsic Rotation

1. Introduction

Ferrofluids are stable suspensions of fine particles of solid ferromagnetic materials such as magnetite (diameter =10nm, number density $=10^{23} \text{m}^{-3}$) in nonconducting liquids such as water or kerosene. Since the particles in ferromagnetic fluids are dispersed very stably due to surface treatment of the particles and adding of surface active agent such as oleic acid, neither aggregation nor sedimentation occurs in gravitational and magnetic fields, and concentration of the particles is constant. Therefore, ferrofluidic fluids exhibit apparently the same behavior as Newtonian fluid except responding to magnetic force. Magnetic properties in applied magnetic field, magnetic fluid seals, dampers, contrast media and levitation, and various studies for their practical applications have been performed. Ferrofluids will be applied to heat pumps, energy conversion and MHD generators because of its good heat conduction. Then a great deal of interest has been focused on magnetic fluids in view of the hydrodynamical aspects. However, satisfactory results have not been yet obtained in the theoretical studies because of the complexity of microstructure of magnetic fluids. In particular, it has been necessary to study the flow dynamics of magnetic fluids in a magnetic field in detail.

The previous studies have been mainly based on the basic equations proposed by Neuringer and Rosenweig(1964) or by Shilomits(1967). The former are the basic equations of quasi-steady ferrohydrodynamics. Namely, the main purpose is to clarify the effects of magnetic body force $\mathbf{M} \cdot \nabla \mathbf{H}$ without taking account of microrotational effects of spins, which show the effects of rotation of particles. While the latter takes account of both the effects of magnetic force and micro-rotational effects. The applicable range of the latter equations is therefore wider than that of the former.

On the other hand, Brenner(1970) and Brenner and Weisman(1972) performed the theoretical studies of an apparent increase in viscosity as non-Newtonian suspended fluids. McTague(1969), Kamiyama et al.(1979) and Tomita et al.(1981) investigated the flow behaviors experimentally.

By the way, the theory of polar fluids proposed by Eringen(1964,1966,1969) is considered a general theory which can
be applied to fluids which have internal freedom of degree. Ferromagnetic fluids are suspensions which have internal freedom of degree.

In the present paper, the authors will try to unify the theory of ferromagnetic fluids proposed by Shliomis and that of polar fluids proposed by Eringen to expand the theory of Eringen. Thereby, the material constants of polar fluids can be determined. Based on the fundamental equations derived from the fusion as described above, the authors analyze flows of ferromagnetic fluids in a circular pipe in longitudinal magnetic field and investigate the flow characteristics theoretically.

Nomenclature

\[ \text{A: pressure gradient } = \frac{\partial p}{\partial z} \]
\[ \alpha: \text{radius of a particle} \]
\[ \beta: \text{magnetic flux density} \]
\[ \delta: \text{wall surface coefficient} \]
\[ \mu: \text{rate of deformation tensor} \]
\[ \frac{g_1^2}{\gamma} = \tau_0 \frac{\partial H_0}{\partial y} = 3\eta_0 \text{EL}(\xi) / \gamma \]
\[ H: \text{magnetic field vector} \]
\[ H_0: \text{applied magnetic field strength} \]
\[ h: \text{Mg} / \mu_0 R_0 \]
\[ i: \text{idemvector} \]
\[ j: \text{sum of moments of inertia of particles per unit volume} \]
\[ k_0: \text{Bohr magneton constant} \]
\[ L(\xi): \text{Langevin's function} \]
\[ M: \text{magnetization vector} \]
\[ m: \text{magnetic moment of a particle} \]
\[ n: \text{number density of particles} \]
\[ \rho(\xi): \text{rotational Peclet number} \]
\[ p: \text{pressure, body force per unit mass} \]
\[ q: \text{flow rate} \]
\[ R: \text{radius of a pipe} \]
\[ Re(\xi): \text{Reynolds' number} \]
\[ R(\xi): \text{size effect parameter} \]
\[ T: \text{stress tensor} \]
\[ T_e: \text{absolute temperature} \]
\[ \mathbf{v}: \text{velocity vector} \]
\[ \alpha, \beta, \gamma: \text{spin viscosities} \]
\[ \epsilon: \text{ratio of viscosities} \]
\[ \delta: \text{volumetric concentration of particles in suspensions } = \frac{4}{3} \pi a^3 N \]
\[ k: \text{magnetic effect parameter} \]
\[ k = \frac{g_1^2}{\gamma} \frac{\mu_0}{4} \text{EL}(\xi) \]
\[ \mu_0: \text{magnetic permeability in vacuum} \]
\[ \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \]
\[ \rho: \text{mass density of suspensions} \]
\[ \rho_0: \text{mass density of a particle} \]
\[ \lambda: \text{couple stress tensor} \]
\[ \lambda: \text{pipe friction coefficient without particles} \]
\[ \Delta \lambda/\rho_0: \text{rate of inerese of pipe friction coefficient} \]
\[ \tau_B: \text{Brownian relaxation time of rotational diffusion} \]
\[ \tau_B = \frac{4 m_0^3 T}{\kappa T} = 3\eta_0 / kT \]
\[ \tau_B: \text{relaxation time of microrotation due to frictional drag of fluid} \]

2. Basic Equations

2.1 Governing equations

Particles of magnetite are much larger than fluid particles. Accordingly, considering the limit of a microscopic volume element in which a fluid can be assumed to be a continuous medium, particles of magnetite must be treated as discrete quantity. Let us consider a quite large cell, which is a macroscopic volume element whose size is much smaller than that of the equipment and in which many particles of magnetite are included. Considering a cell as the standard of scale, ferromagnetic fluids can be treated as continuous media. The authors deal with the cell which includes many microscopic volume elements, and consider rheology of the cell. Since the size of the cell is considered to be finite, the cell has microturbation, which is the sum of rotations of the particles in it, in addition to translation as a point mass. And average velocity of a point mass and average angular velocity of the cell are denoted by and \( \Phi \), respectively. Namely, spin angular momentum must be considered in regard to the cell. This spin angular momentum is considered to be the sum of spin angular momentum of each particle in the cell. As a result, antisymmetric part of stress tensor and couple stress tensor appears in regard to the cell model.

Eringen proposed the following governing equations for such fluids. That is,

\[ \frac{d\mathbf{u}}{dt} + \mathbf{v} \times \mathbf{u} = 0 \]  
\[ \rho \frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{T} + \rho \mathbf{a} \] 
\[ \rho \frac{d\mathbf{A}}{dt} = \mathbf{F} \times \mathbf{H} + e \mathbf{F} + \rho \mathbf{A} \]

\[ \rho \frac{d\mathbf{A}}{dt} = \mathbf{F} \times \mathbf{H} + e \mathbf{F} + \rho \mathbf{A} \]

where

\[ b: \text{external force per unit mass} \]
\[ l: \text{body couple per unit mass} \]
\[ s: \text{spin angular momentum per unit mass} \]
\[ e: \text{third order skew-symmetric tensor} \]

Refer to the nomenclature about the others. In the case of ferromagnetic fluids, it is necessary to add the following equations:

\[ \mathbf{F} \times \mathbf{H} = 0, \mathbf{F} \cdot \mathbf{B} = 0 \]

Here Eqs. (4a) and (4b) are Maxwell's equations in the static magnetic field, and Eq. (5) shown below is a relaxation equation of magnetization, taking account of the effect of rotational Brownian motion derived by Shliomis (1972).
\[
\frac{dM}{dt} = \Omega \times M - \frac{1}{\tau_s} (M - M_e H) \quad \cdots \cdots (5)
\]

2.2 Constitutive equations

Unifying the two theories of polar fluids of Eringen and of ferromagnetic fluids or Shliomis, three constitutive equations of stress tensor, couple stress tensor and magnetic field are respectively given by

\[
T = \rho l + \eta_2 (\tau \cdot D) / 2 + 2\eta_s \cdot (\omega - 2\Omega)
\]
\[+ B H - \frac{1}{2} (B \cdot H) \quad \cdots \cdots \cdots (6)\]

\[
A = \sigma \tau (\Omega \cdot F) \frac{\tau}{\tau} + \beta (\Omega \cdot F) + \tau (\Omega \cdot F) \quad \cdots \cdots (7)
\]

\[
B = \mu_0 H + \mu \cdot M \quad \cdots \cdots \cdots \cdots (8)
\]

where

\[\eta_2: \text{second viscosity},\]
\[\eta_1: \text{vortex viscosity},\]
\[\eta: \text{shear viscosity},\]
\[\sigma, \beta, \gamma: \text{spin viscosities},\]
\[\mu_0: \text{magnetic permeability in vacuum}.\]

The SI units system according to E-H correspondence is adopted here as a units system of electricity and magnetism. It is well known that this units system is symmetrical and practicable. When the fluid is incompressible, \(\tau \cdot D = 0\). If micro-rotation is rigid, the integration of spin angular momentum over a closed surface equals zero. The relation between spin angular momentum per unit mass \(s\) and angular velocity of the cell \(\Omega\) is expressed as follows:

\[s = i \Omega\]

where \(i\) is volume averaged radius of gyration. By means of Gauss's integral theorem, \(F \cdot \Omega = 0\) is obtained. These results indicate that the second viscosity \(\eta_2\) in Eq.(6) and two spin viscosities \(\sigma\) and \(\beta\) in Eq.(7) do not appear in a system of the basic equations.

2.3 Determination of material constants

From now on, the authors assume that external force \(b\) and body couple \(l\) can be neglected, that the fluid is incompressible and that microrotion is rigid. From these assumptions, Eqs.(2) and (3) become respectively as follows:

\[\rho \frac{d\nu}{dt} = \nu (\rho + \frac{1}{2} M \cdot H) \quad \eta \nu \frac{\partial \nu}{\partial \nu} + \eta \nu \times (2\Omega - \omega) + \nu \cdot \nu \cdot H \quad \cdots \cdots (9)\]

\[j \frac{d\Omega}{dt} = \nu \times (2\Omega - \omega) + \nu \cdot M \times H \quad \cdots \cdots (10)\]

Equations (9) and (10) agree with the basic equations of ferromagnetic fluids derived by Shliomis except notations. When there are no effects of magnetic field, the terms including magnetisation vector \(M\) in Eqs.(9) and (10) disappear. And the solution of Eq.(9) obtained in the case of \(\omega = 2\Omega\) is also one of the general solutions of Eq.(9). In this case, the third term on the right hand side of Eq.(9) disappears. Therefore, \(\eta\) means apparent viscosity of suspensions including particles. Considering interaction between the particles, Bedeaux et al. (1977) proposed the following expression for apparent viscosity of suspensions:

\[\eta = \eta_0 (1 + \frac{5}{2} \phi + 4.5 \phi^2) \quad \cdots \cdots (11)\]

where

\[\eta_0: \text{shear viscosity of solvent},\]
\[\phi: \text{volumetric concentration of particles in suspensions}.\]

The coefficient of the second term on the right hand side in Eq.(11) is Einstein's coefficient.

Considering the motion of the particles in a sufficiently large vessel, the first term on the right hand side of Eq.(10), which is a dispersive term of angular momentum, can be neglected unlike the second term which is a viscous damping term. In this case, neglecting torque due to magnetization, Eq.(10) is expressed as follows:

\[\frac{d\Omega}{dt} = \frac{1}{2 \tau_s} (2\Omega - \omega) \quad \cdots \cdots (12)\]

where \(\tau_s\) is relaxation time of micro-rotation due to frictional drag of fluid. Shliomis proposed the following equation for \(\tau_s\):

\[\tau_s = \frac{1}{4 \eta_1} \frac{1}{15 \eta_1} \quad \cdots \cdots (13)\]

On the other hand, the frictional torque acting on a sphere of radius \(a\) spinning uniformly with angular velocity \(\Omega\) in fluid is given by \(4\pi a^3 \omega (\omega - 2\Omega)\). Here \(\omega\) is the vorticity at an infinite distance. If a change takes place in a quasi-steady state, and if interaction between the particles is neglected, totalising the frictional torque yields the following balance equation of angular momentum:

\[\frac{d\Omega}{dt} = 4\pi a^3 N (\omega - 2\Omega) \quad \cdots \cdots (14)\]

where \(N\) is number density of particles. Comparing Eq.(12) with Eq.(14), vortex viscosity \(\eta_1\) is obtained as follows:

\[\eta_1 = \frac{8 \pi a^3}{15} \eta_0 = \frac{3}{2} \eta_0 \quad \cdots \cdots (15)\]

Because of \(j = \omega^2\), substituting Eq.(15) into Eq.(13), volume averaged radius of gyration is given by the positive square root of the following equation:

\[\sqrt{\frac{1}{15} + \frac{1}{15} \frac{3}{2} \rho a \omega} \quad \cdots \cdots (16)\]

The relation between spin viscosity \(\gamma\) and volume averaged radius of gyration \(i\) is obtained by Allen and kline(1970), that is,

\[\gamma = \rho a^2 \quad \cdots \cdots (17)\]

Therefore, spin viscosity can be calculated from shear viscosity and volume averaged radius of gyration, which have been already obtained in Eqs.(11) and (16). In consequence, the three material constants \(\eta_2\), \(\eta_1\) and \(\gamma\) have been expressed by known quantities.

For example, the authors determine the material constants of a ferromagnetic fluid called Ferricollolid W35 (water base, weight concentration of particles 32.7%). The results measured by Kamihana et al. are given the following values for Ferricollolid W35;

\[
\begin{align*}
\eta_2 & = 500 \\
\eta_1 & = 200 \\
\gamma & = 0.05
\end{align*}
\]
\( \varepsilon = 5.5 \text{ nm}, \quad \eta_0 = 1.14 \text{ m Pa s}, \quad \sigma = 1.37 \text{ kg m}^{-1} \text{ s}, \quad \beta = 8.68 \times 10^{-2} \),
\( \rho_0 = 5.16 \text{ kg m}^{-3}, \quad \eta = 5.16 \text{ kg m}^{-3}, \quad N = 1.25 \times 10^{23} \text{ m}^{-3} \)

When effective radius of a particle due to adsorption of surface active agent is \( \Delta_0 \approx 2.0 \AA \), the material constants are determined as follows.
\( m = 5.7 \text{ mPa s} \), \( \gamma = 4.8 \text{ Pa s} \),
\( \eta = 1.2 \text{ mPa s} \), \( \tau_0 = 37 \text{ Pa s} \),
\( \lambda = 11 \text{ mm} \), \( \gamma = 6.9 \times 10^{-9} \text{ Pa m}^2 \text{s} \),
where absolute normal temperature is \( T = 268 \text{K} \) and Boltzmann’s constant is \( k = 1.38 \times 10^{-23} \text{J/K} \).

3. Flow in a Circular Pipe

3.1 Basic equations

The authors analyze a steady flow of ferromagnetic fluids in a circular pipe with uniform longitudinal magnetic field (see Fig. 1). The authors assume that ferromagnetic fluids are incompressible, and that the flow is laminar and axisymmetric. So there is no change in both \( z \) and \( \theta \) directions. However, a constant pressure gradient exists in the \( z \) direction. Therefore, all physical quantities except pressure are constant and the following equations hold:

\[
\begin{align*}
\partial \psi &= 0, \\
\partial \phi &= 0, \\
\partial \omega &= 0
\end{align*}
\]

Considering these assumptions, the components of each vector are given by
\( v = (0, 0, u) \), \( \omega = (0, \omega_r, 0) \),
\( \Omega = (0, \Omega_r, 0) \), \( M = (M_r, 0, M_z) \)
\( H = (H_r, 0, H_z) \)

where the equation of continuity \( F \cdot r = 0 \) is satisfied automatically. Neglecting the inertia terms in Eqs. (4), (5), (9) and (10) yields the other governing equations. These equations are basic equations in this case, and should be solved on boundary conditions described in the next section.

3.2 Boundary conditions

Since the normal component of a magnetic vector and the tangent component of magnetic field are continuous, the boundary conditions for magnetic flux and magnetic field on the pipe wall are given by

\[
B_r = \mu_0 H_r + M_r = 0, 
H_r = H_0
\]  

where \( R \) is a pipe radius. And no-slip condition is applied to the boundary condition for fluid. That is

\[
r = R, \quad v_r = 0
\]

In regard to microrotation, no-spin condition used by Eringen (1966) and constant-spin condition used by Arman et al. (1974) are usually utilized. However, the boundary condition proposed by Condiff and Dahleen (1964) is suitable for a wide range of surface conditions. Then the authors adopt it here. That is, angular velocity of the cell is considered to be in proportion to vorticity, since velocity gradient of fluid causes micro-rotation of particles. Therefore, the following relation is applied to the boundary condition for microrotation:

\[
\Omega_r = \frac{1}{2} \omega_{r0}
\]

where \( \omega_{r0} \) is the vorticity on the wall and \( b \) is a constant determined by wall surface condition, shape of particle and state of flows. And the range of the values of \( b \) is considered to be \( 0 \leq b \leq 1 \). Let us call this condition proportional-spin condition.

3.3 Theoretical analysis

The authors solve the magnetic field equation (4) on the boundary condition (18). As a result, it can be verified that the same form as Eq. (16) is valid at an arbitrary \( r \) for interior of the pipe. The reason is that the field is axisymmetric. Secondly, the authors solve the equation of momentum (9) neglecting the inertia term. From the component of the \( r \) direction, the distribution of pressures is obtained as follows:

\[
\rho_r (r, z) = \rho_0 (r, z) - \frac{M_r}{2} \frac{dr}{d\Omega_r} \quad \text{...... (21)}
\]

From the component of the \( \theta \) direction, \( \Omega_r \) becomes zero and from the component of the \( z \) direction, the following equation is obtained:

\[
\begin{align*}
\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) + \frac{2 \beta}{r} \frac{1}{r} \frac{d}{dr} \left( r \Omega_r \right) \\
+ \frac{A_\theta}{\eta + \tau_0} = 0
\end{align*}
\]

(22)

Next, the authors decompose the equation of angular momentum (10) without the inertia term into each component. Components of \( r \) and \( z \) directions give \( \Omega_r = M_r = 0 \). The component of the \( \theta \) direction is expressed as follows:

\[
\begin{align*}
\frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) + \frac{2 \beta}{r} \frac{1}{r} \frac{d}{dr} \left( r \Omega_r \right) \\
- \frac{1}{\eta + \tau_0} \left( \mu_0 H_r + M_r \right) \Omega_r = 0
\end{align*}
\]

(23)

Finally, the authors examine each component of the relaxation equation of magnetization (5) without the inertia term. From the component of the \( \theta \) direction, \( M_r = 0 \). Components of \( r \) and \( z \) directions are expressed as follows:

\[
\begin{align*}
\tau_0 \Omega_r M_r \left( M_r - \frac{H_r}{\sqrt{H_r^2 + H_z^2}} \right) = 0 \quad \text{...... (24)}
\end{align*}
\]

\[
\begin{align*}
\tau_0 \Omega_r M_r \left( M_z - \frac{H_z}{\sqrt{H_r^2 + H_z^2}} \right) = 0 \quad \text{...... (25)}
\end{align*}
\]

The equations described above are the formal components of the basic equations to be analyzed. The authors solve Eqs. (22)-(25), which are simultaneous differential equations involving four unknown variables \( \psi, \Omega_r, M_r \) and \( M_z \) where \( H_r = H_0 = H_0 \). First, the authors substitute \( M_0 = M_z \) which is obtained from Eqs. (24) and (25) into Eq. (23), in order to eliminate them from Eq. (23) and next substitute \( dv/df \) into Eq. (23) which is given by integrating Eq. (22) taking account of finiteness of physical quantities at \( r = 0 \). Then Eq. (23) becomes a differential equation which has only a variable \( \psi \). Since the rewritten equation (23) is a complicated nonlinear differential equation, the authors analyze it approximately according to the following method. The induced magnetic field \( H_0 \) is much weaker than the applied magnetic field \( H_0 \), that is, \( H_0 < H_0 \) holds.
In this case, the term \((H_2/H_0)^2\) can be neglected in Eqs. (24) and (25). Then \(M_1\) and \(M_2\) are obtained by

\[
M_1 = \frac{H_1H_0}{\mu_0H_s + M_s} \left[1 + \frac{\mu_0H_s(x_0)^2}{\mu_0H_s + M_s} \right]^{-1}
\]

and

\[
M_2 = M_1 \left[1 + \frac{\mu_0H_s(x_0)^2}{\mu_0H_s + M_s} \right]^{-1}
\]

If \(Q_0\) is given in the above equations, the distribution of magnetization can be determined. Substituting \(M_1\) and \(M_2\) in Eqs. (26) and (27) into Eq. (23) and eliminating \(dQ_0/dr\), the following differential equation for \(Q_0\) is obtained:

\[
\frac{d^2Q_0}{dr^2} + \frac{1}{r} \frac{dQ_0}{dr} - \left(1 + \frac{1}{\gamma^2} \right)Q_0 = \frac{2}{\gamma} \left(1 + \frac{1}{\gamma^2} \right)Q_0 - \frac{2\gamma^2}{\gamma^2 - 1} \left(1 + \frac{1}{\gamma^2} \right)Q_0
\]

This is an important equation which should be solved first. Where

\[
f^2 = \frac{4\eta_0}{\gamma \left(\gamma^2 - 1\right)} = \frac{4\eta_0}{\gamma \left(\gamma^2 + 1\right)}
\]

\[
p^2 = \frac{\gamma \left(\gamma + 1\right)}{\gamma} Q_0
\]

\[
h = \frac{M_0}{\mu_0H_0}
\]

The physical meaning of each parameter is described as follows:

- \(f\): polar effect,
- \(g\): magnetic effect,
- \(\xi\): dimensionless magnetic field strength,
- \(\epsilon\): rate of magnetization.

Dimensionless parameter \(\epsilon\) is a ratio of viscosities, which means polar effects. Since Eq. (28) is still nonlinear, it is difficult to solve it analytically.

### 3.3.1 In the case of \(\gamma_0q\_0<<1\)

Since the order of Brownian relaxation time of rotational diffusion of angular momentum is \(10^{-6}\), \(\gamma_0q\_0<<1\) is valid in many cases. This condition means that rotational Frictional number which is defined by \(\nu_0 = \gamma_0q\_0 / \eta R\) is small. Therefore, neglecting the second order of \(\epsilon\), the analytical solution of Eq. (28) can be obtained. The dimensionless form of Eq. (28) is

\[
\frac{d^2Q^*}{dr^2} + \frac{1}{r} \frac{dQ^*}{dr} - \left(\gamma^2 + 1\right)Q^* = -\gamma^2 Q^*_r
\]

where

\[
r^* = r / R,
\]

\[
Q^* = Q \left(AR^4 / 4\eta\right),
\]

\[
\gamma = \frac{\gamma^2}{4} \left(\gamma + 1\right) Q^*_\xi
\]

and \(\gamma\) is a dimensionless parameter, that is,

\[
\gamma = \frac{3}{4} \left(\gamma_0 + 1\right) \left(\gamma^2 + 1\right)
\]

which indicates a ratio of polar effect to magnetic effect. If \(\gamma = 0\), both polar effect and magnetic effect disappear. The solution of Eq. (32) which takes a finite value at the center of a pipe and satisfies the boundary condition Eq. (29) is

\[
Q^* = -\frac{1}{4\epsilon^2} \left(1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right) \left(1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right)
\]

where \(Q^*_\xi\) is a modified Bessel function of the first kind of order \(n\), and

\[
\delta = \left[1 + \frac{1}{1 + \epsilon + x} \right]^{1/2}
\]

\(\delta\) is a dimensionless parameter which is determined by condition of the particles on the wall in the same way as \(b\). The authors call \(\delta\) wall surface condition coefficient. The solutions of velocity, vorticity and flow rate are expressed as the following equations, respectively:

\[
\frac{\nu}{\nu_0} = \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right] \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right]
\]

\[
\frac{\omega}{\omega_0} = \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right] \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right]
\]

\[
\frac{Q^*}{Q^*_0} = \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right] \left[1 + \epsilon + x - \frac{1}{1 + \epsilon + x} \right]
\]

where \(\nu_0, \omega_0\), and \(Q^*_0\) are average velocity, characteristic vorticity and flow rate for Newtonian fluid, respectively. They are given by

\[
\nu_0 = AR^4 / 4\eta, \omega_0 = AR^4 / 4\eta, Q^*_0 = \pi AR^4 / 8\eta
\]

In these solutions, \(\epsilon = 0\) means no effects of magnetic field. When \(\epsilon = 0\), the solutions are congruent with those of polar fluids. Therefore, the authors call \(\epsilon\) magnetic effect parameter. Furthermore \(\epsilon = 0\) satisfies no magnetic effect parameter. The rate of increase is expressed as given by Eringen.

From Eqs. (26) and (27), distributions of magnetization are expressed by the following equations:

\[
M^*_1 = M_1 / M_0 = \frac{P_0Q^*_1}{1 + h}
\]

\[
M^*_2 = M_2 / M_0 = \frac{P_0Q^*_2}{1 + h}
\]

Lastly, the authors consider an increment of a pipe friction coefficient. Pipe friction coefficient \(\lambda\) is defined by the following equation.

\[
-\frac{d\lambda}{dx} = \lambda^2 (\nu^2)^2 / 4R
\]

where \(\nu^2\) is an average velocity. Considering a flow of Newtonian fluid which has the same density and the same average velocity as ferromagnetic fluid, the pipe friction coefficient of the Newtonian fluid \(\lambda_0\) is given by

\[
\lambda_0 = 64 / R, \nu_0 = 2R (\nu^2) / \eta
\]

where \(Re\) is Reynolds' number. Therefore, when an increment of pipe friction coefficient is defined by \(\lambda^2 = \lambda_0\), the rate of increase is expressed as follows:

\[
\lambda^2 / \lambda_0 = 1 / Q^*_0 - 1
\]

### 3.3.2 In the case of \(\gamma_0q\_0<<1\)

The authors analyze Eq. (28) numerically by using central difference approximation. However, since the result is nonlinear, the following consideration is
necessary. Representing \( R_q \) simply by \( R_* \), the dimensionless difference equation is expressed as follows:

\[
(1+\frac{dr^*}{2r_0^*})Q_{\omega,0,0}-\left(1+\frac{dr^*}{r_0^*}\right)Q_{\omega,0,0}+\left(1+\frac{dr^*}{2r_0^*}\right)Q_{\omega,0,-1}+\left(1+\frac{dr^*}{r_0^*}\right)Q_{\omega,-1,0}+\delta \omega = 0 \quad (42)
\]

where \( \delta r^* = 1/n \) and \( n \) is division number;

\[
K_{\omega,0,0} = \frac{1}{1+k \left( \frac{r_0^*}{2r_0^*} \right)^2} \left( \frac{1+\frac{dr^*}{r_0^*}}{1+k \left( \frac{r_0^*}{2r_0^*} \right)^2} \right) \quad (43)
\]

First, assuming \( K_{\omega,0,0} = 1+k \) when \( j = 1 \), Eq. (42) becomes linear and can be solved for \( R_1^* \) considering the boundary conditions. Next, putting \( j = 2, K_{\omega,0,0} \) can be calculated from Eq. (43), and Eq. (42) can be solved again. Repeating this operation till converging, i.e. till the relative difference between \( R_1^* \) and \( R_n^* \) becomes less than \( 10^{-m} \), the converged value \( R_1^* \) is obtained as a solution.

3.4 Discussions

In the case of \( \bar{\Omega}_q \leq 1 \), the solutions (28), (35), (36), and (38) include four dimensionless parameters \( \epsilon, \kappa, \delta \), and \( \zeta \) which characterize flow behaviors. The effects of each parameter are expressed as follows:

- \( \epsilon \): ratio of viscosities (polar effects),
- \( \kappa \): magnetic effect,
- \( \delta \): wall surface condition effect,
- \( \zeta \): size effect.

The authors choose \( R/1 \) as size effect parameter in place of \( \zeta \).

3.4.1 Magnetic effect parameter

Figures 2, 3 and 4 show the change due to magnetic effect parameter \( \kappa \). Apparently, velocity profiles in Fig. 2 seem to be those of Poiseuille flow of a Newtonian fluid of which viscosity increases. However, non-Newtonian property appears clearly in distribution of vorticities in the neighborhood of the wall (see Fig. 4). This non-Newtonian property has been overlooked, and it appears due to polar effects of the particles. \( \kappa = 0 \) means no effects of magnetic field. In this case, each distribution of velocities, microrotations and vorticities also shows non-Newtonian property. Namely, ferromagnetic fluids have non-Newtonian property regardless of existence of magnetic fields.

When \( \kappa \rightarrow \infty \), microrotation is constrained and velocity profile reduces. The results are congruent with flow behaviors of Newtonian fluid of which viscosity increases to \( \eta(1+\epsilon) \).

3.4.2 Polar effect parameter

Figures 5, 6 and 7 show the effects due to the change of polar effect parameter \( \epsilon \). \( \epsilon = 0 \) means a decrease in the effects of viscosity of solvent which has on microrotation. Namely, the frictional drag between particles and solvent decreases, and polar effects are reduced. When \( \epsilon = 0 \), Eqs. (34), (36) and (37) become, respectively

\[
\omega = 1-\epsilon, \omega = 0, \omega = 2r^*
\]
Fig. 6 Distribution of microrotations 
(\(\varepsilon=2.0\), \(R/i=50.0\), \(\delta=0.0\))

Fig. 7 Distribution of vorticities 
(\(\varepsilon=2.0\), \(R/i=50.0\), \(\delta=0.0\))

Fig. 8 Velocity profiles 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(\delta=0.0\))

Fig. 9 Distribution of microrotations 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(\delta=0.0\))

Fig. 10 Distribution of vorticities 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(\delta=0.0\))

Fig. 11 Velocity profiles 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(R/i=50.0\), \(\delta=0.0\), \(h=0.2\))

Fig. 12 Distribution of microrotations 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(R/i=50.0\), \(\delta=0.0\), \(h=0.2\))

Fig. 13 Distribution of vorticities 
(\(\varepsilon=0.4\), \(\kappa=2.0\), \(R/i=50.0\), \(\delta=0.0\), \(h=0.2\))
These results are precisely in agreement with those of Poiseuille flow of Newtonian fluid. Generally, $\varepsilon$ is smaller than one and the value of Ferricollloid W35 is 0.21.

3.4.3 Size effect parameter
Figures 8, 9 and 10 are graphs in which size effect parameter $R/i$ changes. Velocity profiles go flat as $R/i$ decreases. Because the interaction between the particles becomes active and diffusion of angular momentum of the particles extends over a wide range. As a result, the condition of microrotation on the wall has great effects upon distribution of inner microrotations (see Fig.9). The wall effects vanish as $R/i$ increases, which is verified by the result that the wall surface condition coefficient $\delta$ is not included in $\Omega_i^*$, $\Omega_0^*$ and $\omega_i^*$ of Eqs. (34), (36) and (37) when $R/i=\infty$. And the flow corresponds to Poiseuille flow of a Newtonian fluid of which viscosity $\eta$ increases to $\eta(1+\varepsilon)(1+\kappa)/[(1+\varepsilon)+\kappa]$. In this case, the following relation between angular velocity of the cell $\Omega_0^*$ and vorticity $\omega_i^*$ holds.

$$\Omega_i^* = \frac{1}{2} \frac{1+\varepsilon}{1+\varepsilon+\kappa} \omega_i^*$$  \hspace{1cm} (44)

This result is in agreement with the case of $\delta=1$, and means $\beta=(1+\varepsilon)/(1+\varepsilon+\kappa)$. Further, if $\varepsilon=0$ in Eq.(44), twice the angular velocity of the cell is equal to vorticity in the whole flow field.

![Figure 14: Increase of pipe friction coefficient](R/i=50.0, \delta=0.0)

3.4.4 Nonlinear properties
Nonlinear properties appear as rotational Peclet number increases. Figs.11, 12 and 13 show the change of velocity profiles, distribution of microrotations and distribution of vorticities due to rotational Peclet number. These distributions extend slightly as $R/P$ increases. In particular, this tendency is conspicuous in distribution of microrotations near the wall in Fig.12. The change of curvatures also appears because of nonlinearity. This can be explained by the phenomena that since microrotation in the neighborhood of the wall is fast, the effects of magnetization are reduced. Namely, distributions of magnetization are reduced as $\Omega_i^*$ increases:

$$M_i = \left[ 1 + \left( \frac{P_i}{R_i} \right)^{\gamma-1} \right]^{-1} \hspace{1cm} (45)$$

If magnetized effect parameter $h$ is small, $h$ works such that it suppresses the effects of the rotational Peclet number.

3.4.5 Pipe friction coefficient
With the aid of Eq.(41), the authors investigate the rate of increase of pipe friction coefficient $\Delta \lambda/\lambda_0$. Microrotation is suppressed as $\kappa$ increases, and pipe friction coefficient increases. When $\kappa=0$, $\Delta \lambda/\lambda_0$ gradually approaches $\varepsilon$ (see Fig.14). On the other hand, Fig.15 shows that $\Delta \lambda/\lambda_0$ is almost in proportion to $\varepsilon$. When $R/i=\infty$ or $\delta=1$, the following equation is obtained.

$$\frac{\Delta \lambda}{\lambda_0} = \frac{\varepsilon}{1+\varepsilon+\kappa} = \frac{3}{2} \frac{\eta_0 A}{\eta} A = \frac{1}{2} \delta E(\varepsilon)$$  \hspace{1cm} (46)

This is in agreement with the analytical result of Shliomis.

4. Conclusions

The theory of polar fluids proposed by Eringen was applied to ferromagnetic fluids. As a result, a new treatment of ferromagnetic fluids was found. The basic equations obtained finally were in agreement with the theory of Shliomis, and a method of determination of material constants of polar fluids became clear. Further, the flow in a circular pipe in uniform longitudinal magnetic field was analyzed, and the authors obtained the following results:

1) In the case of $T_{\nu} NA<1$, the analytical solutions expressed by Eqs.(34)-(39) are obtained. These solutions are characterized by four dimensionless parameters, that is, polar effect parameter $\varepsilon$, magnetic effect parameter $\kappa$, wall surface condition coefficient $\delta$ and size effect parameter $R/i$. In the case of $T_{\nu} NA=1$, rotational Peclet number $P$ and magnetized effect parameter $h$ govern the solutions in addition to the above four parameters. And the solutions show nonlinearity.

2) The effects of dimensionless parameters expressing properties of ferromagnetic fluids and the flow behavior are as follows:

(a) Microrotation is suppressed as the magnetic effect parameter $\kappa$ in-
creases.
(b) Frictional drag between particles and solvent increases as the polar effect parameter ε increases.
(c) Condition of microrotation on the wall has great effects upon the flow as the size effect parameter decreases.
(d) Microrotation on the wall is constrained as the wall surface condition coefficient decreases.
(e) Angular velocity of the cell in the neighbourhood of the wall increases as the rotational Pelet number increases.
(f) The effects of rotational Pelet number are suppressed as the magnetized effect parameter increases.

Pipe friction coefficient λ increases as the magnetic effect parameter and the polar effect parameter increase, and it approaches ε gradually as k→∞. In particular, the analytical results obtained when size effect parameter R/ε→∞ are in agreement with those of Shliomis.

Acknowledgements

The authors acknowledge that editing and typing of the manuscript were made with the aid of "RUNOFF" of HITAC M-280H of Tokyo University Computer Center. This work has been partly supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, of Japan.

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