Generation of Curvilinear Coordinate System Having Arbitrary Line Densities*

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A fast method to generate curvilinear coordinate systems with arbitrary line densities in the physical region is presented. There are no restrictions on the shape of the surrounding boundaries. Two numerical examples are given to show the validity of the method; one is a rhomb as a typical shape of an inner boundary with many corners on its contour, and the other is a Karman-Trefftz airfoil as a typical case of a cusped body. In the latter case, in particular, an orthogonal curvilinear coordinate system can be obtained when a conformal mapping technique is used simultaneously. Moreover, it is expected that this grid generation procedure can be applied to a wide variety of other problems.

Key Words: Fluid Mechanics, Numerical Analysis, Generation of Grid Network, Arbitrary Coordinate Line Densities

1. Introduction

When flow problems such as flows around a body or internal flows in a passage are treated numerically, it is common practice to determine the solution at the nodal points of a grid that has been generated in the flow field in question. The use of a coarser grid is preferable if there are no restrictions on numerical stability and computational accuracy. Especially in cases of unsteady compressible flows or viscous-inviscid interaction problems, a coarser mesh is efficient for economization of computer time, because their solutions usually require a lot of iterations. If a potential flow approximation is valid, a sufficient accuracy may be expected from the use of a coarser grid, but in physically more complicated and important problems such as viscous flows with boundary-layer separation or compressible flows including shock waves, it becomes difficult to obtain reasonable results unless denser meshes are located in the region where the flow properties change rapidly and considerably.

Hitherto such a grid network has properly been provided by respective researchers as the case stood, but recently several generation methods of body-fitted grids most of which are applicable to flows around any number of bodies with arbitrary shapes have been proposed⁽¹⁾⁻⁽³⁾, and their applications to many practical cases have also been reviewed⁽⁴⁾⁽⁵⁾.

Essentially, such a method of grid generation should serve as an aid of the numerical analysis of the original flow

field, so that any method which requires a lot of computer time or needs a large size program for grid generation is itself inadequate. From this point of view, it seems few methods are available.

In the present paper, a method for numerical generation of a curvilinear coordinate system which has arbitrary line densities and contains no restrictions on the shapes of the boundaries is presented. The present grid generation method can be readily applied to a wide variety of problems, and a combination of this procedure with a conformal mapping technique can give orthogonal curvilinear meshes in the physical region.

2. Explanation of Method

2.1 Computational region

Let the flow field be restricted to a region bounded by inner and outer boundaries and two cutoff lines as shown in Fig.1. Especially if the flow field contains a single submerged body, its entire region is to be discussed, so that the two cutoff lines fall into a single line which is located at an appropriate position in the region. To illustrate the present method, the latter special case is supposed and the inner boundary is taken to coincide with the body surface. Then the physical region can be represented in a cylindrical polar coordinate system, (r, θ) , with its origin inside the body.

Let the geometries of the inner and outer boundaries be given by

$$\text{inner: } r = G, \quad \theta = \theta_G$$

$$\text{outer: } r = F, \quad \theta = \theta_F$$

respectively, then since θ_G and θ_F are generally independent and G and F depend on their own respective angles, the distances from the origin to the boundaries are assigned as

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$$G = G(\theta_G), \quad F = F(\theta_F)$$

At first the physical region surrounded by these boundaries is transformed onto a concentric circular region as seen in Fig.2. The radii of the inner and outer circles are determined so that the two circles have the same circumferential lengths as those of the corresponding boundaries in the physical plane.

Hereafter, the explanation will mainly be carried out for the inner boundary. Let the line segment along the body surface be ds_G , and the corresponding radius of the circle is given by

$$a_G = \frac{1}{2\pi} \oint_G ds_G \dots\dots\dots(1)$$

In this case, since the distance, s_G , from the datum point, say T_G in Fig.1, to any point along the body surface is coincident with the arc length of the inner circle, the angle ϕ in the plane of transformation (Fig.2) is given by

$$\phi = s_G / a_G \dots\dots\dots(2)$$

Thus the mutual correspondence of arbitrary points on the inner boundary in the circular plane to those in the physical plane is established as

$$\left. \begin{aligned} \theta_G &= \theta_G(s_G) = \theta_G(\phi) \\ G &= G(s_G) = G(\phi) \end{aligned} \right\} \dots\dots\dots(3)$$

The relations of Eqs.(1),(2) and (3) also hold on the outer boundary if the subscript G is changed to F.

For inverse mapping of the grid network from the concentric circular plane, (α, ϕ) , to the physical plane, (r, θ) , it is sufficient to use a simple formula such as

$$\frac{r - G(\phi)}{F(\phi) - G(\phi)} = \frac{\theta - \theta_G(\phi)}{\theta_F(\phi) - \theta_G(\phi)} = \frac{\alpha - a_G}{a_F - a_G} \dots\dots\dots(4)$$

When Eq.(4) is employed, the intersections of coordinate lines which consist of concentric circles and radial lines readily generate a corresponding node network on the physical plane. Therefore, it is preferable that a grid having arbitrary line densities is first designed on the mapping plane and then transformed onto the physical plane.

Fig.1 Physical plane

2.2 Transition functions

In order to designate the arbitrary line densities on the concentric circular region, putting first

$$\left. \begin{aligned} \tau &= \frac{1}{2\pi} \tau_b \phi \quad (\tau_b \text{ is a constant}) \\ \xi &= \frac{\alpha - a_G}{a_F - a_G} \end{aligned} \right\} \dots\dots\dots(5)$$

the region can be transformed onto a rectangular one, say (τ, ξ) -plane, as shown in Fig.3. Since τ and ξ are linear to the angle ϕ and the radius α , respectively, and have the domains of

$$\left. \begin{aligned} 0 &\leq \tau \leq \tau_b \\ 0 &\leq \xi \leq 1 \end{aligned} \right\} \dots\dots\dots(6)$$

Fig.2 Basic plane of transformation

Fig.3 Non-uniform spacing grid

Fig.4 Uniform spacing grid

if the lattice network with non-uniform spacing were once generated on the (τ, ξ) -plane, it might readily be transformed onto the physical plane through the concentric circular plane.

Now a uniform spacing network $[(\sigma, \eta)$ -plane in Fig. 4] which has the same number of coordinate lines as those in the non-uniform spacing one, is introduced. The actual calculations of a flow field are usually executed in this uniform spacing network. Both networks are related by transition functions, so the problem is reduced to one of estimating functions that can designate the arbitrary line densities at any locations in the (τ, ξ) -plane.

To do this, two series expansion formulae are introduced:

$$\left. \begin{aligned} \tau &= \sum_{m=1}^M A_m \sigma^m \\ \xi &= \sum_{n=1}^N B_n \eta^n \end{aligned} \right\} \dots\dots\dots (7)$$

where, A_m and B_n are transition constants, and they are determined automatically as will be seen in the subsequent sections when the line densities and their locations are assigned.

The domains of σ and η are

$$\left. \begin{aligned} 0 \leq \sigma \leq \sigma_b \quad (= \tau_b) \\ 0 \leq \eta \leq 1 \end{aligned} \right\} \dots\dots\dots (8)$$

respectively, and the mesh sizes are given by

$$\Delta\sigma = \frac{\sigma_b}{M-1}, \quad \Delta\eta = \frac{1}{N-1} \quad \dots\dots\dots (9)$$

where, M and N are numbers of coordinate lines in the directions of σ and η , respectively. Especially, if σ_b is taken as

$$\sigma_b (= \tau_b) = (M-1)/(N-1) \quad \dots\dots\dots (10)$$

the uniform spacing mesh becomes square. The practical application of these transition functions to the basic equations of the fluid dynamics is not so troublesome, and a

detailed example using a simpler transition function can be seen in reference (6).

2.3 Conditions on transition coefficients

The correspondence between the uniform spacing network (σ, η) and the non-uniform spacing one (τ, ξ) is represented by Eq. (7), and the spacing of the latter varies according to the manner of determination of A_m and B_n .

Here, the procedure to estimate these coefficients will be illustrated only for one direction, say the abscissa direction.

When the position of $\sigma = \sigma_1$ (with spacing $\Delta\sigma$) in the uniform spacing grid corresponds to that of $\tau = \tau_1$ in the non-uniform grid, the size of spacing, $\Delta\tau_1$, at this location is mathematically represented by

$$\Delta\tau_1 = \left(\frac{d\tau_1}{d\sigma} \right) \Delta\sigma \quad \dots\dots\dots (11)$$

Since τ_1 is a function of σ alone, the reciprocal of its derivative

$$\alpha_1 = 1 / \left(\frac{d\tau_1}{d\sigma} \right) \quad \dots\dots\dots (12)$$

denotes the local density of coordinate lines normal to the abscissa. Therefore, the problem is reduced to one of determining the coefficients A_m which are supposed to yield the local densities previously assigned at arbitrary points τ_1 ($i=1, 2, \dots$) on the coordinate line. Moreover, because it seems desirable that the local line density $\alpha(\sigma)$ is stationary around the position of τ_1 , an additional condition:

$$\frac{d^2 \tau_1}{d\sigma^2} = 0 \quad \dots\dots\dots (13)$$

is set.

Prior to the estimation of the coefficients, the maximum limits of domains for τ and σ are taken to be $\tau_b = \sigma_b = 1$, which makes the treatment simpler with scarcely any sacrifice of generality. And also for convenience of application of the boundary conditions, Eq. (7) and its derivatives are written as follows:

$$\tau(\sigma) = A_1 \sigma + A_2 \sigma^2 + A_3 \sigma^3 + A_4 \sigma^4 + \dots + A_M \sigma^M, \quad (0 \leq \sigma \leq 1) \quad \dots\dots\dots (14-a)$$

$$\frac{d\tau}{d\sigma} = \tau_\sigma(\sigma) = A_1 + 2A_2 \sigma + 3A_3 \sigma^2 + 4A_4 \sigma^3 + \dots + M A_M \sigma^{M-1} \quad \dots\dots\dots (14-b)$$

$$\frac{d^2 \tau}{d\sigma^2} = \tau_{\sigma\sigma}(\sigma) = 2A_2 + 6A_3 \sigma + 12A_4 \sigma^2 + \dots + M(M-1) A_M \sigma^{M-2} \quad \dots\dots\dots (14-c)$$

$$\frac{d^3 \tau}{d\sigma^3} = \tau_{\sigma\sigma\sigma}(\sigma) = 6A_3 + 24A_4 \sigma + 60A_5 \sigma^2 + \dots + M(M-1)(M-2) A_M \sigma^{M-3} \quad \dots\dots\dots (14-d)$$

2.4 Examples of simple application

Before explaining the principal part of this paper, two simpler cases are treated, in which arbitrary line densities are assigned at one and at both ends of the computational region. In both cases, the transition coefficients can readily be determined without any iterative procedures, moreover, these results may be applied to many practical problems such as the generation of a suitable grid in a boundary layer domain.

2.4.1 A case in which the line densities are assigned at both ends of the domain.

In this case, since the line densities α_a and α_b have previously been given at $\tau = 0$ and $\tau = 1$, respectively, the minimum number of conditions is $M=5$, so that they become as follows:

Conditions at $\tau = 0$

$\tau(0) = 0$: Eq.(14-a) is satisfied automatically.

$\tau_\sigma(0) = 1/\alpha_a$: from Eq.(14-b), $1/\alpha_a = A_1$ (15)

$\tau_{\sigma\sigma}(0) = 0$: from Eq.(14-c), $0 = 2A_2$ (16)

Conditions at $\tau = 1$

$\tau(1) = 1$: from Eq.(14-a), $1 = A_1 + A_2 + \dots + A_M$ (17)

$\tau_\sigma(1) = 1/\alpha_b$: from Eq.(14-b), $1/\alpha_b = A_1 + 2A_2 + \dots + MA_M$ (18)

$\tau_{\sigma\sigma}(1) = 0$: from Eq.(14-c), $0 = 2A_2 + 6A_3 + \dots + M(M-1)A_M$ (19)

Thus these five equations, (15)-(19), yield the following five coefficients:

$$\left. \begin{aligned} A_1 &= \frac{1}{\alpha_a} \\ A_2 &= 0 \\ A_3 &= 10 - \frac{6}{\alpha_a} - \frac{4}{\alpha_b} \\ A_4 &= -15 + \frac{8}{\alpha_a} + \frac{7}{\alpha_b} \\ A_5 &= 6 - \frac{3}{\alpha_a} - \frac{3}{\alpha_b} \end{aligned} \right\} \dots\dots\dots(20)$$

As an example, when $\alpha_a = 5$ and $\alpha_b = 2$ are assigned, the positions of the non-uniform spacing coordinate lines, τ , and the corresponding distribution of line densities, $\alpha (=1/\tau_\sigma)$, against the locations of the uniform spacing coordinate lines, σ , are plotted as shown in Fig.5.

2.4.2 A case in which the line density is assigned at one end of the domain.

If the line density α_a alone is assigned at one end ($\tau=0$) of the domain, the necessary conditions are Eqs.(15),(16),(17) and (19) in the preceding example, and further conditions which put the higher derivatives at another end as zero, i.e.

$$\left. \frac{d^n \tau}{d\sigma^n} \right|_{\sigma=1} = 0 \quad (n \geq 2) \dots\dots\dots(21)$$

may be added if necessary.

Therefore, the number of equations, M , for the transition coefficients $A_m (m=1,2,\dots,M)$ becomes $M=4,5,\dots$ corresponding to $n=2,3,\dots$ in the same expression of Eq.(21), and A_m are then determined by solving M

Fig.5 Example of the case where the line densities are assigned at both ends of the domain, ($\alpha_a=5$, $\alpha_b=2$)

($=n+2$)-elements linear equations. Table 1 shows the factors of the coefficient matrices and the values of right hand side of the equations for $M=4,5,6$ and 7, though the values for A_1 and A_2 are absent because they are self-evident from Eqs.(15) and (16). The solutions of the linear equations shown in Table 1 are given in Table 2. As an example, when $\alpha_a=5$ at $\tau=0$ is assigned, the distributions of line densities, α , versus σ are plotted as shown in Fig. 6 by taking M as a parameter.

2.5 Problem to assign arbitrary line densities at prescribed locations

Consider the problem which allocates the line densities $\alpha_n [=1/(\tau_\sigma)_n]$ at the N -discrete positions, $\tau_n (n=1,2,\dots,N)$, in the domain of $0 \leq \tau \leq 1$. Let σ_n be the locations on σ corresponding to τ_n . In practice, σ_n are yielded after all conditions have been satisfied completely.

The requisite conditions described in Section 2.3 are first written for every discrete location as

$$\tau_n = \sum_{m=1}^M A_m \sigma_n^m \dots\dots\dots(22)$$

$$\frac{1}{\alpha_n} = \sum_{m=1}^M m A_m \sigma_n^{m-1} \dots\dots\dots(23)$$

$$0 = \sum_{m=2}^M m(m-1) A_m \sigma_n^{m-2} \dots\dots\dots(24)$$

and, if necessary, some or all of the conditions for both ends of the domain stated in Section 2.4 may be provided [at least Eq.(17) must be included].

Here the explanation of the present procedure will be given by using a rather practical example. Suppose the case in which Eqs.(22)~(24) and both end conditions, Eqs.(15)~(19), are all taken into account. In this case, there are $(3N+5)$ -equations, and the total number of unknowns are $(M+N)$, namely $A_m (m=1 \sim M)$ and $\sigma_n (n=1 \sim N)$. Therefore, the number of transition coefficients which

Table 1 Coefficient matrices for A_m

M=	4	5	6	7	right hand side
A_3	A_4	A_5	A_6	A_7	
1	1	1	1	1	$1 - \frac{1}{\alpha_a}$
6	12	20	30	42	0
6	24	60	120	210	0
0	24	120	360	840	0
0	0	120	720	2520	0

Table 2 Transition coefficients $A_m(m=1\sim M)$, where $K=1-\frac{1}{\alpha_a}$

M	A_1	A_2	A_3	A_4	A_5	A_6	A_7
4	$\frac{1}{\alpha_a}$	0	$2K$	$-K$	—	—	—
5	$\frac{1}{\alpha_a}$	0	$\frac{10}{3}K$	$-\frac{10}{3}K$	K	—	—
6	$\frac{1}{\alpha_a}$	0	$5K$	$-\frac{15}{2}K$	$\frac{9}{2}K$	$-K$	—
7	$\frac{1}{\alpha_a}$	0	$7K$	$-14K$	$\frac{63}{5}K$	$-\frac{28}{5}K$	K

are sufficient in the series expansion formula, Eq. (14-a), becomes

$$M = 2N + 5 \quad \dots\dots\dots(25)$$

Since these $(3N+5)$ -equations include the unknowns σ_n in terms of power series formula, it is difficult to solve them directly, so σ_n are successively deduced by using the following method.

The method of solution

- (1) Noting that α_n at τ_n , α_a and α_b have been assigned in advance, σ_n are first appropriately assumed (for example putting $\sigma_n = \tau_n$).
- (2) Since the $(2N+5)$ -equations except Eq. (22), [(15)~(19), (23) and (24)], are a set of multi-elements linear equations including σ_n (assumed) for $A_m(m=1\sim 2N+5)$, then, after solving this set, the solutions are employed as new approximate values for these transition coefficients.
- (3) Then the right hand side of Eq. (22), $\sum_{m=1}^{2N+5} A_m \sigma_n^m$, yields approximations of τ_n , so that the errors E_n and their RMS value of the solutions in this stage can be obtained as follows:

$$E_n = \sum_{m=1}^{2N+5} A_m \sigma_n^m - \tau_n \quad (n=1\sim N) \quad \dots\dots(26)$$

$$RMS = \left(\frac{1}{N} \sum_{n=1}^N E_n^2 \right)^{1/2} \quad \dots\dots\dots(27)$$

- (4) When this RMS value is sufficiently small, the successive computations are

closed.

- (4)' If it still exceeds an allowable limit, σ_n are corrected by using either the expedient method described below or a regular Newton method.

- (5) After obtaining the perturbation quantities, say $\delta\sigma_n$, the solutions are revised as

$$\sigma_n + \delta\sigma_n \rightarrow \sigma_n \quad (n=1\sim N) \quad \dots\dots\dots(28)$$

and the procedure then returns to the stage (2) for the subsequent iteration.

The expedient method

Newton's method of predicting the perturbation quantities $\delta\sigma_n$ is orthodox but in general somewhat troublesome. Unless extraordinarily high line densities are assigned at numerous locations inside the domain, the following expedient method is very efficient.

Denote the values of σ_n to be revised, as

$$\sigma_n^{(+)} = \sigma_n - \omega E_n / \tau_{\sigma}(\sigma_n) \quad \dots\dots\dots(29)$$

where ω is a relaxation factor, and determined as follows.

At first, ω is set sufficiently small, and then successively increased by a small shred, $\delta\omega$. In each iteration, the total error, $E_R = \sum_{n=1}^N |E_n|$, and its rate of decrease [the ratio of E_R to the one obtained just previously, $E_R^{(-)}$], $R_T = E_R / E_R^{(-)}$, are stored into the computer memory. In the beginning, R_T decreases with an increasing ω , but soon it will start to increase. At this moment, the value of ω corresponding to the minimum R_T is obtained, and it is employed as an optimum ω for further iterations.

The convergency of this expedient method is exceedingly fast as far as an ordinary problem is treated. For example, in the case in which two kinds of line densities with values less than around four are assigned inside the domain, 10~30 iterations yield a convergency of $E_R < 10^{-10}$ (in practice, it is sufficient if $E_R < 10^{-4}$).

Table 3 shows a working example in which the line densities are assigned at both ends and two locations ($N=2$) inside the domain. The errors E_n in Eq. (26) were 0.16×10^{-5} and 0.19×10^{-5} at $\tau=0.4$ and $\tau=0.6$, respectively.

Fig.6 Example of the case where the line density is assigned at one end of the domain, ($\alpha_a=5$)

Table 3 Working example in which the line densities are assigned at both ends and two locations inside the domain

positions (τ)	0.0 0.4 0.6 1.0
line densities (α)	5.0 4.0 3.0 5.0
criterion of convergency	0.5×10^{-5}
status of convergency	NTRY = 19 RMS = 0.200×10^{-5}
transition coefficients ($N=2, M=9$)	
$A_1 = 0.2$	
$A_2 = 0.0$	
$A_3 = 0.209722958 \times 10^{-3}$	
$A_4 = -0.172004100 \times 10^{-4}$	
$A_5 = 0.600934013 \times 10^{-4}$	
$A_6 = -0.111690554 \times 10^{-5}$	
$A_7 = 0.115344912 \times 10^{-5}$	
$A_8 = -0.625010932 \times 10^{-4}$	
$A_9 = 0.138645148 \times 10^{-4}$	
solutions σ_n ($n=1,2$)	
$\sigma_1 = 0.33421894$	
$\sigma_2 = 0.66752139$	
evaluations of equation (22)	
$\tau(\sigma_1) = 0.4000016$	
$\tau(\sigma_2) = 0.6000019$	

3. Grid Network around a Rhomb

A numerical example is presented of a rhomb as a typical polygonal shape. As shown in Fig. 7, the contour of the inner boundary is taken as a rhomb which has the diagonal distances of $2A$ and $2B$ on the x and y axes, respectively, and the outer boundary is chosen as a circle of the radius $r = F$. Since the circumferential length of the inner boundary is then $\oint_G ds_G = 4\sqrt{A^2+B^2}$, the equivalent radius defined by Eq. (1) becomes $a_G = \frac{2}{\pi} \sqrt{A^2+B^2}$. Thus the computational region in the physical plane is transformed onto the region surrounded by the two concentric circles.

Hereinafter, the procedure is illustrated by a specific numerical example (the final result is given in Fig. 8).

Fig. 7 Schematic geometry for the region including a rhomb

Specifications

Inner boundary:

length of the horizontal diagonal line: $2A=1$,
aspect ratio: $B/A=0.5$

Outer boundary:

ratio of radii in the circular plane: $\alpha_F / \alpha_G = 12$

Uniform spacing grid:

number of coordinate lines: $M = 91$ in horizontal direction,
 $N = 46$ in vertical direction,

spacing (rectangular mesh, $\tau_b = \sigma_b = 1.0$): $\Delta\sigma = 1/90$, $\Delta\eta = 1/45$

Line densities and transition coefficients in the non-uniform spacing grid:

here, the values shown in the numerical example in Chapter 2 are used as they were obtained.

In horizontal direction:

positions ($\tau =$)	0.0 0.4 0.6 1.0	The transition coefficients A_m ($m=1\sim 9$) are previously given in Table 3.
line densities ($\alpha =$)	5.0 4.0 3.0 5.0	

In vertical direction:

the line density at the inner boundary is assigned as $\alpha_a = 10.0$, and from Table 2 seven transition coefficients, say B_n ($n=1\sim 7$), are used.

Transformations

Let a nodal point in the uniform spacing grid be (σ_i, η_j) , where $i=1-91$ and $j=1-46$. Using the coefficients A_m and B_n , the corresponding point (τ_i, ξ_j) in the non-uniform spacing grid is given by

$$\tau_i = \sum_{m=1}^9 A_m \sigma_i^m, \quad \xi_j = \sum_{n=1}^7 B_n \eta_j^n \dots (30)$$

Thus from Eq. (5) the angle ϕ_i and radius a_j in the region surrounded by the concentric circles become

$$\left. \begin{aligned} \phi_i &= 2\pi \tau_i \\ a_j &= a_G + \xi_j (a_F - a_G) \end{aligned} \right\} \dots (31)$$

Since the distances along the inner and outer boundaries are readily obtained by this ϕ_i as

$$s_{Li} = a_L \phi_i \quad (L \rightarrow G, F) \dots (32)$$

the angle ϕ_{Li} in the physical plane corresponding to s_{Li} can be determined from a simple geometry (referring to Fig. 7). Then the radial distances $G(\theta_{Gi})$ and $F(\theta_{Fi})$ are also confirmed. Therefore, from Eq. (4) the positions of the nodal points in the physical plane are given by

$$\left. \begin{aligned} r_{i,j} &= G_i + \xi_j (F_i - G_i) \\ \theta_{i,j} &= \theta_{Gi} + \xi_j (\theta_{Fi} - \theta_{Gi}) \end{aligned} \right\} \dots (33)$$

The numerical results are plotted in Fig. 8, and it is seen that there are concentrations of coordinate lines at the locations of $\tau=0.0$ (or 1.0), 0.4 and 0.6 in the peripheral direction, and also in the vicinity of the rhomb in the radial direction.

4. Grid Network around an Airfoil

One more numerical example is given, this time an airfoil as a typical shape which has a cusped edge and on which a conformal mapping is easily applicable.

The present method principally consist of two stages: the first is to transform the uniform spacing grid (Fig. 4) onto the region surrounded by two concentric circles through the non-uniform spacing grid (Fig. 3), and the second is to make the concentric region correspond to the physical plane by means of simple transition functions. The two kinds of coordinate lines ($\sigma=\text{constants}$, $\eta=\text{constants}$) in the first stage are mutually orthogonal, but in the second stage, their orthogonality in the physical plane will be lost usually. When a flow field is numerically treated by using such grid network, a considerable loss of the orthogonality is undesirable because the accuracy of the results is related with it, deeply and beyond expectation.

If the inner and outer boundaries in the physical plane can be mapped by means of some simple transition functions onto the concentric circles (it does not matter that they come to have only nearly circular contours), it is better to make the mapping through such transition functions. In particular, when the inner boundary can be approximated by an airfoil, the Joukowski transformation is useful.

Here an example on the grid generation around a Karman-Trefftz airfoil will be presented.

When the plane of concentric circles, (α, ϕ) , is described by using a complex plane, $\zeta = a e^{i\phi}$, it is well-known that the relation to the physical (airfoil) plane, $Z = X + iY$, is given by

$$\frac{Z - nc}{Z + nc} = \left(\frac{\zeta + \zeta_0 - c}{\zeta + \zeta_0 + c} \right)^n \dots (34)$$

where ζ_0 , n and c are constants which only depend on the characteristics of the airfoil section such as the thickness, camber, chord and trailing edge angle. In this case, the inner circle, radius $a = a_G$, corresponds to the contour of the airfoil, and the outer

Fig. 8 Resulting network for a rhomb

circle, $\alpha = \alpha_F$, becomes a near circle which has a radius of

$$r \approx \frac{\alpha_F}{\alpha_G} \times \frac{(\text{chord length})}{4} \dots\dots (35)$$

in the physical plane providing $\alpha_F \gg \alpha_G$.

Fig. 9 shows the final result for an airfoil of 15% thickness, 5% camber and 8° trailing edge angle, under the same conditions specified in the case of the rhomb except that the numbers of coordinate lines are taken as $M=74$ and $N=41$. It is clear that there is realized a completely orthogonal network of the lines throughout the entire region under consideration.

5. Concluding Remarks

A practical grid generation method which can arbitrarily control the coordinate line densities, and moreover, can rapidly generate a grid with no restrictions on the shape of the boundaries has been presented. To illustrate the essentials of the present method, the following three cases are considered: (1) line densities assigned at both ends of the computational domain, (2) assigned at one end of the domain, and (3) assigned at arbitrary discrete positions inside the domain. Furthermore, the effects of these line densities on the resulting distribution of densities inside the domain are also investigated.

In order to show the validity of the method in applications to more practical cases, two examples are given. One is a rhomb which is chosen as a typical shape having many corners on its contour, and the other is an airfoil chosen as a cusped body. In the latter case, in particular, an orthogonal grid can be obtained when a conformal mapping technique is used simultaneously.

In view of its easy application and less computing time, the present method may be judged sufficient to attain the purpose. It is expected that the procedure can be applied to a wide variety of other problems.

The numerical computations of this work were carried out by the HITAC M-200 computer of Hokkaido University.

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Fig.9 Resulting network for a Karman-Trefftz airfoil