Virtual Work Principles for Two Elastoiimpact

Contact Bodies*

By Naoki ASANO**

The virtual work principles for two elastoiimpact bodies are required to formulate the finite element method for elastoiimpact contact structures with translational and rotational motions.

At first, the virtual work principles and their subsidiary contact conditions for the two bodies are two-dimensionally formulated in the basic contact and separate states of the two bodies, based on the conventional principle for a body.

In the separate state of the two bodies, the virtual work principle and its subsidiary conditions are obtained by introducing a relative movement corresponding to clearance and regarding the bodies as a connective body. The relative movement in a slip direction is applied to the subsidiary conditions of a slip state in the same way as to the subsidiary conditions of the separate state.

Finally, the virtual work principles and their subsidiary conditions for the two bodies are three-dimensionally established on the basis of the two-dimensional principles and their subsidiary conditions.

Key word : Shock, Impact, Dynamic Contact, Virtual Work Principle, Problems of Two Bodies.

1. Introduction

Impact contact behavior arises in many machine parts such as meshing gears, rolling bearings clutches and so on. At present, it is very difficult to measure and/or calculate the contact stresses, their distributions, and contact states on impact surfaces. Recently, the increase of large-scale computational capabilities makes it much easier to use a finite element method (FEM), especially effective for geometrically complicated configurations in many areas of structural analysis. Ohte⁽¹⁾, Tsuta and Yamaji⁽²⁾, Okamoto and Nakazawa⁽³⁾, Yagawa et al.⁽⁴⁾ proposed the FEMs for solving elastostatic problems involving contact effects.

Timoshenko⁽⁵⁾ analyzed the impact behavior of a beam struck by a ball, using the Hertz theory for elastostatic contact. The contact and separate states between two bodies are dependent upon the propagation of stress waves in each body, and the contact area between the two bodies always changes. In addition, the contact stresses and their distributions are unknown. To analyze the dynamic contact conditions such as the impact contact behavior, the stick-slip behavior, and so on, it is necessary to formulate a dynamic contact theory based on the propagation of stress waves and on the treatment of the problems of the two bodies without assuming previously various contact states. The author has formulated the elastoiimpact contact theories with friction^{(6),(7)}, using, respectively, a conventional FEM for small strain and a new non-incremental FEM⁽⁸⁾ for flexible deformation.

Variational principles or energy principles are generally used in the formulation of FEMs. In order to formulate the FEM for elastoiimpact contact structures with translational and rotational motions, the author has presented an expanded Hamilton's principle⁽⁹⁾ and virtual work principles in various contact and separate states.^{(10),(11)} In addition, the author has reported an elastodynamic contact theory using the FEM for elastoiimpact contact structures with a translational motion⁽¹²⁾ and compared experimental results with the calculated results of the FEM⁽¹³⁾.

This paper summarizes some aspects of the virtual work principles and their subsidiary contact conditions in various contact and separate states of two elastoiimpact bodies. The author presents a new principle for two bodies in a slip state and the subsidiary conditions which introduce the relative movements between the two bodies required in a stick-slip behavior.

2. Statement of Problems

2.1 On the Relation of Coordinate System and Variational Principle

Let us consider an elastic body with translational and rotational motions. The body (α) is treated with a fixed coordinate system ($O-X_1, X_2, X_3$) shown in Fig. 1. The center of gravity $g(\alpha)$ for the body has a vector $r_g(\alpha)$ in the system. Any point $h(\alpha)$ of the body has a positional vector $r(\alpha)$ to the center in the moving coordinate system ($g(\alpha), x_1(\alpha), x_2(\alpha), x_3(\alpha)$). By action of a body force vector $F(\alpha)$ in the volume $V(\alpha)$ and a surface traction vector $P(\alpha)$ on the area $S_\sigma(\alpha)$, the point $h(\alpha)$ deforms to the point $p(\alpha)$ with a vector $V(\alpha)$. The vector $u(\alpha)$ at the point $p(\alpha)$ with respect to the origin O in the fixed system is expressed as

* Received October 13th, 1980.

** Associate Professor, Department of Mechanical Engineerings, Tamagawa University, Machida, Tokyo 194.

$$\mathbf{u}^{(\alpha)} = \mathbf{r}_g^{(\alpha)} + \mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)} \dots \dots \dots (1)$$

The velocity vector $\dot{\mathbf{u}}^{(\alpha)}$ at the point $p^{(\alpha)}$ in the fixed system is

$$\dot{\mathbf{u}}^{(\alpha)} = \dot{\mathbf{r}}_g^{(\alpha)} + \dot{\mathbf{v}}^{(\alpha)} + \dot{\boldsymbol{\theta}}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \dots \dots \dots (2)$$

where $\boldsymbol{\theta}$ denotes the rotational angle vector at the center $g^{(\alpha)}$ of the body (α) , and “.” is the differentiation with respect to time.

The acceleration vector $\ddot{\mathbf{u}}^{(\alpha)}$ at the point $p^{(\alpha)}$ is

$$\begin{aligned} \ddot{\mathbf{u}}^{(\alpha)} = & \ddot{\mathbf{r}}_g^{(\alpha)} + \ddot{\mathbf{v}}^{(\alpha)} + \ddot{\boldsymbol{\theta}}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \\ & + 2\dot{\boldsymbol{\theta}}^{(\alpha)} \times \dot{\mathbf{v}}^{(\alpha)} + \dot{\boldsymbol{\theta}}^{(\alpha)} \times \dot{\boldsymbol{\theta}}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \end{aligned} \dots \dots \dots (3)$$

where the fourth term is the gyrotory component, and the last term denotes the centrifugal component.

The virtual displacement vector $\delta \mathbf{u}^{(\alpha)}$ at the point $p^{(\alpha)}$ is

$$\delta \mathbf{u}^{(\alpha)} = \delta \mathbf{r}_g^{(\alpha)} + \delta \mathbf{v}^{(\alpha)} + \delta \boldsymbol{\theta}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \dots \dots \dots (4)$$

where δ means the variational operator.

Let $\rho^{(\alpha)}$ denote the density of the body (α) . The equilibrium equation, boundary conditions, and subsidiary conditions are written as follows:

$$\sigma_{ij,j}^{(\alpha)} + F_i^{(\alpha)} = \rho^{(\alpha)} \ddot{u}_i^{(\alpha)} \quad (\text{in } V^{(\alpha)}) \dots \dots \dots (5)$$

$$n_j^{(\alpha)} \sigma_{ji}^{(\alpha)} = P_i^{(\alpha)} \quad (\text{on } S^{(\alpha)}) \dots \dots \dots (6)$$

$$u_i^{(\alpha)} = \bar{u}_i^{(\alpha)} \quad (\text{on } S_u^{(\alpha)}) \dots \dots \dots (7)$$

$$P_i^{(\alpha)} = \bar{P}_i^{(\alpha)} \quad (\text{on } S_\sigma^{(\alpha)}) \dots \dots \dots (8)$$

$$\varepsilon_{ij}^{(\alpha)} = \frac{1}{2} (v_{i,j}^{(\alpha)} + v_{j,i}^{(\alpha)}) \quad (\text{in } V^{(\alpha)}) \dots \dots \dots (9)$$

where $n_j^{(\alpha)}$ denotes the unit outward normal to a surface $S^{(\alpha)}$ in a specified direction. $\sigma_{ij}^{(\alpha)}$ is the stress tensor, $\varepsilon_{ij}^{(\alpha)}$ the strain tensor, $F_i^{(\alpha)}$ the body force tensor, $u_i^{(\alpha)}$ the displacement tensor, and $P_i^{(\alpha)}$ the surface tensor. $\ddot{u}_i^{(\alpha)}$, $\sigma_{ij,j}^{(\alpha)}$ and $v_{i,j}^{(\alpha)}$ are

$$\ddot{u}_i^{(\alpha)} = \frac{d^2}{dt^2} (u_i^{(\alpha)}), \quad \sigma_{ij,j}^{(\alpha)} = \frac{\partial \sigma_{ij}^{(\alpha)}}{\partial x_j^{(\alpha)}},$$

$$v_{i,j}^{(\alpha)} = \frac{\partial v_i^{(\alpha)}}{\partial x_j^{(\alpha)}}$$

Let $E_{ijkl}^{(\alpha)}$ denote the modulus of elasticity from Hooke's law. Thus, the relation between the stress tensor and the strain tensor is

$$\sigma_{ij}^{(\alpha)} = E_{ijkl}^{(\alpha)} \varepsilon_{kl}^{(\alpha)} \dots \dots \dots (10)$$

If we represent the virtual displacement tensor by $\delta u_i^{(\alpha)}$ and use Eq. (4) to (10), the virtual work principle for the body (α) without damping effect is

$$\int_{\tau_1}^{\tau_2} (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) dt = 0 \dots \dots \dots (11)$$

$$\left. \begin{aligned} \delta E^{(\alpha)} &= \int_{V^{(\alpha)}} \sigma_{ij}^{(\alpha)} \delta(\varepsilon_{ij}^{(\alpha)}) dV \\ \delta T^{(\alpha)} &= \int_{V^{(\alpha)}} \rho^{(\alpha)} \ddot{u}_i^{(\alpha)} \delta(u_i^{(\alpha)}) dV \\ \delta W^{(\alpha)} &= \int_{V^{(\alpha)}} F_i^{(\alpha)} \delta(u_i^{(\alpha)}) dV \\ &\quad + \int_{S_\sigma^{(\alpha)}} \bar{P}_i^{(\alpha)} \delta(u_i^{(\alpha)}) dS \end{aligned} \right\} \dots \dots \dots (12)$$

Fig. 1 Various vectors of a three-dimensional elastic body with translational and rotational motions.

and τ_1 and τ_2 denote the time instants. $\delta E^{(\alpha)}$, $\delta T^{(\alpha)}$, and $\delta W^{(\alpha)}$ are, respectively, the virtual internal, kinetic, and external works.

Using Eqs. (7) to (12), we obtain Hamilton's principle as follows:

$$\delta \int_{\tau_1}^{\tau_2} (E^{(\alpha)} - T^{(\alpha)} - W^{(\alpha)}) dt = 0 \quad \dots\dots\dots (13)$$

where

$$\left. \begin{aligned} E^{(\alpha)} &= \int_{V^{(\alpha)}} \frac{1}{2} E_{ijkl}^{(\alpha)} \varepsilon_{ij}^{(\alpha)} \varepsilon_{kl}^{(\alpha)} dV \\ T^{(\alpha)} &= \int_{V^{(\alpha)}} \frac{1}{2} \rho^{(\alpha)} \dot{u}_i^{(\alpha)} \dot{u}_i^{(\alpha)} dV \\ W^{(\alpha)} &= \int_{V^{(\alpha)}} F_i^{(\alpha)} u_i^{(\alpha)} dV \\ &\quad + \int_{S_{\sigma}^{(\alpha)}} \bar{P}_i^{(\alpha)} u_i^{(\alpha)} dS \end{aligned} \right\} \dots\dots\dots (14)$$

In addition, the initial conditions of Eqs. (11) and (13) replace the virtual displacement by zero at the time instants, τ_1 and τ_2

The tensor representation of Eq. (12) is converted into the vector representation, so that

$$\left. \begin{aligned} \delta E^{(\alpha)} &= \int_{V^{(\alpha)}} \sigma_{ij}^{(\alpha)} \delta(\varepsilon_{ij}^{(\alpha)}) dV \\ \delta T^{(\alpha)} &= \int_{V^{(\alpha)}} \rho^{(\alpha)} \dot{u}_i^{(\alpha)} \delta(u_i^{(\alpha)}) dV \\ \delta W^{(\alpha)} &= \int_{V^{(\alpha)}} F_i^{(\alpha)} \delta(u_i^{(\alpha)}) dV \\ &\quad + \int_{S_{\sigma}^{(\alpha)}} \bar{P}_i^{(\alpha)} \delta(u_i^{(\alpha)}) dS \end{aligned} \right\} \dots\dots\dots (15)$$

The term $T^{(\alpha)}$ in Eq. (14) is, also, converted into

$$T^{(\alpha)} = \int_{V^{(\alpha)}} \dot{u}^{(\alpha)} \frac{\rho^{(\alpha)}}{2} \dot{u}^{(\alpha)} dV \quad \dots\dots\dots (16)$$

Using Eq (2), Eq. (16) is rewritten as

$$\begin{aligned} 2T^{(\alpha)} &= \int_{V^{(\alpha)}} (\dot{\mathbf{r}}_s^{(\alpha)} + \dot{\mathbf{v}}^{(\alpha)}) \{ \dot{\mathbf{r}}_s^{(\alpha)} + \dot{\mathbf{v}}^{(\alpha)} \\ &\quad + \dot{\boldsymbol{\theta}}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \} \rho^{(\alpha)} dV \\ &\quad + \int_{V^{(\alpha)}} \dot{\boldsymbol{\theta}}^{(\alpha)} (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \times \{ \dot{\mathbf{r}}_s^{(\alpha)} + \dot{\mathbf{v}}^{(\alpha)} \\ &\quad + \dot{\boldsymbol{\theta}}^{(\alpha)} \times (\mathbf{r}^{(\alpha)} + \mathbf{v}^{(\alpha)}) \} \rho^{(\alpha)} dV \quad \dots\dots (17) \end{aligned}$$

The FEM based on Hamilton's principle can be formulated using the matrix representation of Eqs. (14) and (17). However, a variational operation is carried out after completing the matrix representation of Eq. (13). Hence, it is impossible to formulate the nonlinear components such as the gyratory component and the centrifugal component, as is clear from Eq. (17). On the contrary, a variational operation has already been carried out in a virtual work principle. Using Eqs. (3) and (5), the nonlinear components of $\delta T^{(\alpha)}$ in Eq. (15) can be easily formulated in the virtual work principle, Eq. (11). In introducing the nonlinear components, a virtual work principle is more useful than Hamilton's principle.

The static behavior of structures is included in the dynamic behavior. The linear behavior of structures is also within the nonlinear behavior. The author has formulated the virtual work principle including the virtual works due to viscosity, mass transfer, pressure, a convective type of inertia, etc. and applied the principle to the formulation of the FEM for the analysis of viscous flow⁽¹⁴⁾. So, it is possible for a virtual work principle to adopt or reject the various virtual works such as the virtual kinetic and viscous works and a relaxation term of the Lagrange multiplier in a discontinuous field, etc. according to the problems to be analyzed: whether or not the problems are conservative, linear, steady, static, and so on. Moreover, the principle can be easily transformed into a hybrid type and/or an incremental representation.

2.2 On the Variational Principles for Two Elastostatic Bodies

There are variational principles and energy principles applicable to the formulation of a discontinuous field (e.g., the variational method with a weight parameter⁽¹⁵⁾ Hu-Washizu's principle with a Lagrange multiplier⁽¹⁶⁾ a hybrid type of potential energy principle⁽¹⁶⁾ etc.). These principle are all of hybrid type, and applicable to static problems.

If the Lagrange multiplier λ_1 is used for the continuous condition on a contact area S_c between two bodies, the virtual work principle for elastostatic contact problems without damping and thermal effects is

$$\sum_{\alpha=1}^2 (\delta E^{(\alpha)} - \delta W^{(\alpha)}) + \delta \int_{S_c} \lambda_1 (\mathbf{u}_c^{(1)} - \mathbf{u}_c^{(2)}) dS = 0 \quad \dots\dots\dots (18)$$

where $\mathbf{u}_c^{(1)}$ and $\mathbf{u}_c^{(2)}$ are the contact displacement vectors of both bodies (1) and (2), respectively. In addition, λ_1 corresponds to a contact force vector. $\delta E^{(\alpha)}$ and $\delta W^{(\alpha)}$ are calculated using Eq. (15).

The FEM for elastostatic contact problem⁽³⁾ is formulated using the incremental representation of Eq. (18). If the virtual kinetic work $\delta T^{(\alpha)}$ in Eq. (15) is introduced into Eq. (18), we obtain

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) \right. \\ \left. + \delta \int_{S_c} \lambda_1 (\mathbf{u}_c^{(1)} - \mathbf{u}_c^{(2)}) dS \right\} dt = 0 \quad \dots\dots\dots (19) \end{aligned}$$

Dynamic contact problems require the subsidiary conditions of the contact velocity vector $\dot{\mathbf{u}}_c^{(\alpha)}$ and the acceleration vector $\ddot{\mathbf{u}}_c^{(\alpha)}$ on the contact area. Let us assume the subsidiary contact conditions on S_c . Then, we obtain

$$\dot{u}_c^{(1)} = \dot{u}_c^{(2)}, \quad \ddot{u}_c^{(1)} = \ddot{u}_c^{(2)} \quad (\text{on } S_c) \quad \dots (20)$$

If the Lagrange multipliers λ_2 and λ_3 are used for the relaxation of Eq. (20), the virtual work principle for elastoimpact contact problems is

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) \right. \\ & + \delta \int_{S_c} \lambda_1 (\dot{u}_c^{(1)} - \dot{u}_c^{(2)}) dS \\ & + \delta \int_{S_c} \lambda_2 (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)}) dS \\ & \left. + \delta \int_{S_c} \lambda_3 (\ddot{u}_c^{(1)} - \ddot{u}_c^{(2)}) dS \right\} dt = 0 \quad \dots (21) \end{aligned}$$

where the physical properties of λ_2 and λ_3 are undefined. The formulation of the relationship among λ_1 , λ_2 , and λ_3 is required, but it has not been published yet.

Hughes et al.⁽¹⁷⁾ analyzed the elastoimpact contact problems by using the incremental FEM, based on Eq. (19). For the subsidiary conditions in a contact state, they apply the relative inertia component to the incremental quantity of an unknown contact force, the conservation law of momentum to that of a contact node velocity, and the arithmetic mean value of the pair of contact node masses to that of a contact node acceleration. The contact force component is also used for the calculation of incremental velocities and accelerations in a separate state.

From the above description, it is very difficult to formulate the hybrid type of the virtual work principle for two elastoimpact bodies, such as Eq. (21). This paper formulates a non-hybrid type of the virtual work principles and their subsidiary contact conditions in various contact and separate states of two elastoimpact contact bodies.

Fig. 2 Various vectors of two two-dimensional elastoimpact contact bodies with a translational motion.

3. Virtual Work Principles for Two-dimensional Contact Structures

In order to explain clearly the virtual work principles and their subsidiary conditions for two elastoimpact bodies, let us consider a two-dimensional contact structure^{(10),(11)}. To make the analysis of a stick-slip behavior easy, we formulate the subsidiary conditions with a relative movement in a slip state. It is assumed that the two bodies deform in the range of an elastic limit, and their damping and thermal effects are neglected.

For simplification of treatment, let us discuss the bodies with translational motions, as shown in Fig. 2. The center of gravity $g^{(\alpha)}$ for the body (α) , $(\alpha = 1, 2)$, has a vector $r_g^{(\alpha)}$ in the fixed coordinate system $(0 - x_1, x_2)$. Any point $h^{(\alpha)}$ of the body has a positional vector $r^{(\alpha)}$ to the center in the moving coordinate system $(g^{(\alpha)} - x_1^{(\alpha)}, x_2^{(\alpha)})$. Let b denote the thickness of both bodies. By action of a body force vector $F^{(\alpha)}$ in the area $S^{(\alpha)}$, a surface traction vector $P^{(\alpha)}$ on a boundary line $L\sigma^{(\alpha)}$, and a displacement vector $u^{(\alpha)}$ on a line $L_u^{(\alpha)}$, the point $h^{(\alpha)}$ deforms to the point $p^{(\alpha)}$ with a vector $v^{(\alpha)}$. The vectors $u^{(\alpha)}$, $\dot{u}^{(\alpha)}$, $\ddot{u}^{(\alpha)}$, and $\delta u^{(\alpha)}$ at the point $p^{(\alpha)}$ with respect to the origin in the fixed system are

$$\left. \begin{aligned} u^{(\alpha)} &= r_g^{(\alpha)} + r^{(\alpha)} + v^{(\alpha)} \\ \dot{u}^{(\alpha)} &= \dot{r}_g^{(\alpha)} + \dot{v}^{(\alpha)} \\ \ddot{u}^{(\alpha)} &= \ddot{r}_g^{(\alpha)} + \ddot{v}^{(\alpha)} \\ \delta u^{(\alpha)} &= \delta r_g^{(\alpha)} + \delta v^{(\alpha)} \end{aligned} \right\} \dots (22)$$

The impact force vector R acts on the contact line L_c of the boundary line for each body, and is decomposed into the vector R_n in the n -direction perpendicular to L_c , and the vector R_t in the t -direction tangential to L_c . The action of R_n and R_t operates in the opposite directions between the two bodies. Since L_c and R change with configurations and deformations of the contact line, their values are always unknown. Then, L_c and R are determined by a trial calculation⁽⁶⁾.

The vectors $u_c^{(\alpha)}$, $\dot{u}_c^{(\alpha)}$, $\ddot{u}_c^{(\alpha)}$ and $\delta u_c^{(\alpha)}$ on L_c are decomposed, respectively, into the n -direction vectors and the t -direction vectors. Then, the vectors are expressed as

$$\left. \begin{aligned} u_c^{(\alpha)} &= \{u_{cn}^{(\alpha)}, u_{ct}^{(\alpha)}\} \\ \dot{u}_c^{(\alpha)} &= \{\dot{u}_{cn}^{(\alpha)}, \dot{u}_{ct}^{(\alpha)}\} \\ \ddot{u}_c^{(\alpha)} &= \{\ddot{u}_{cn}^{(\alpha)}, \ddot{u}_{ct}^{(\alpha)}\} \\ \delta u_c^{(\alpha)} &= \{\delta u_{cn}^{(\alpha)}, \delta u_{ct}^{(\alpha)}\} \end{aligned} \right\} \dots (23)$$

If R is regarded as $P^{(\alpha)}$, we can consider the virtual contact work $\delta Q^{(\alpha)}$. Then, Eq. (11) is rewritten as

$$\int_{\tau_1}^{\tau_2} (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)} - \delta Q^{(\alpha)}) dt = 0 \quad \dots (24)$$

The various virtual works of the body (α) are

$$\left. \begin{aligned}
 \delta E^{(\alpha)} &= b \int_{S^{(\alpha)}} \delta \mathbf{e}^{(\alpha)} \boldsymbol{\sigma}^{(\alpha)} dS \\
 \delta T^{(\alpha)} &= b \int_{S^{(\alpha)}} \delta \mathbf{u}^{(\alpha)} \rho^{(\alpha)} \ddot{\mathbf{u}}^{(\alpha)} dS \\
 \delta W^{(\alpha)} &= b \int_{S^{(\alpha)}} \delta \mathbf{u}^{(\alpha)} \mathbf{F}^{(\alpha)} dS \\
 &\quad + b \int_{L_c^{(\alpha)}} \delta \mathbf{u}^{(\alpha)} \bar{\mathbf{P}}^{(\alpha)} dL \\
 \delta Q^{(1)} &= b \int_{L_c} \delta \mathbf{u}_c^{(1)} (-\mathbf{R}) dL \\
 \delta Q^{(2)} &= b \int_{L_c} \delta \mathbf{u}_c^{(2)} \mathbf{R} dL
 \end{aligned} \right\} \dots\dots (25)$$

Equation (24) represents the virtual work principle of each body in a contact state of the two bodies. Since the various virtual work of both bodies are treated in the same time instant and the same space, the virtual work principle for the two bodies is expressed as

$$\left. \begin{aligned}
 \int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) + \delta Q \right\} dt = 0 \\
 \delta Q = b \int_{L_c} (\delta \mathbf{u}_c^{(1)} - \delta \mathbf{u}_c^{(2)}) \mathbf{R} dL
 \end{aligned} \right\} \dots\dots (26)$$

In Eq. (26), the two bodies are regarded as a connective body through a contact line. However, $\mathbf{R}$ and L_c are always unknown except in a separate state. Therefore, we must eliminate δQ in Eq. (26) in various contact states.

3.1 Stick State of Two Bodies

Let us consider the motion of two bodies not slipping on a contact line. The following relation between $\mathbf{R}_n$ and $\mathbf{R}_t$ is established by Coulomb's law.

$$|\mathbf{R}_t| < |\mu \mathbf{R}_n| \dots\dots (27)$$

where μ indicates a frictional coefficient. The two bodies are interconnected through the contact line. Here we can regard the two bodies as a connective body. Hence, the subsidiary condition between $\mathbf{u}_c^{(1)}$ and $\mathbf{u}_c^{(2)}$ is

$$\mathbf{u}_c^{(1)} = \mathbf{u}_c^{(2)} \dots\dots (28)$$

The subsidiary conditions of the velocities and the accelerations on the contact line are, respectively, expressed as

$$\dot{\mathbf{u}}_c^{(1)} = \dot{\mathbf{u}}_c^{(2)}, \quad \ddot{\mathbf{u}}_c^{(1)} = \ddot{\mathbf{u}}_c^{(2)} \dots\dots (29)$$

The subsidiary condition of the virtual displacements is

$$\delta \mathbf{u}_c^{(1)} = \delta \mathbf{u}_c^{(2)} \dots\dots (30)$$

Introducing Eq. (30) into Eq. (26), we obtain $\delta Q = 0$. Then, the virtual work principle for the two bodies⁽¹⁰⁾ is

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) \right\} dt = 0 \dots\dots (31)$$

where the subsidiary contact conditions are given by Eqs. (28) to (30).

3.2 Slip State of Two Bodies (Case 1)

Let us consider the motion of two bodies slipping on a contact line. The slipping condition in the t -direction is expressed as

$$|\mathbf{R}_t| = |\mu \mathbf{R}_n| \dots\dots (32)$$

Since the two bodies are interconnected in the n -direction, we can formulate the bodies as a connective body. From the continuity of contact displacements, the subsidiary condition is

$$\mathbf{u}_n^{(1)} = \mathbf{u}_n^{(2)} \dots\dots (33)$$

The subsidiary conditions of contact velocities and accelerations in the n -direction are represented as

$$\dot{\mathbf{u}}_n^{(1)} = \dot{\mathbf{u}}_n^{(2)}, \quad \ddot{\mathbf{u}}_n^{(1)} = \ddot{\mathbf{u}}_n^{(2)} \dots\dots (34)$$

The subsidiary condition of virtual contact displacements is

$$\delta \mathbf{u}_n^{(1)} = \delta \mathbf{u}_n^{(2)} \dots\dots (35)$$

On the other hand, δQ in Eq. (26) is composed of the following relations.

$$\left. \begin{aligned}
 \delta Q &= \delta Q_n + \delta Q_t \\
 \delta Q_n &= b \int_{L_c} (\delta \mathbf{u}_n^{(1)} - \delta \mathbf{u}_n^{(2)}) \mathbf{R}_n dL \\
 \delta Q_t &= b \int_{L_c} (\delta \mathbf{u}_t^{(1)} - \delta \mathbf{u}_t^{(2)}) \mathbf{R}_t dL
 \end{aligned} \right\} \dots\dots (36)$$

As δQ_n disappears using Eq. (35), Eq. (36) becomes $\delta Q = \delta Q_t$. Hence, Eq. (26) is rewritten as

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) + \delta Q_t \right\} dt = 0 \dots\dots (37)$$

where it is required for the calculation of δQ_t to obtain R_t or R_n . Both R_n and R_t fluctuate according to the slipping states. However, the direction of force application changes in R_t , and remains contact in R_n . This is because the two bodies are interconnected in the n -direction. Let δ^n indicate the variational operator in the n -direction. Using Eqs. (24) and (25), R_n is expressed as

$$\left. \begin{aligned} & \int_{\tau_1}^{\tau_2} \left\{ \delta^n E^{(1)} + \delta^n T^{(1)} - \delta^n W^{(1)} \right. \\ & \quad \left. + b \int_{L_c} \delta^n u_n^{(1)} R_n dL \right\} dt = 0 \\ \text{or} & \int_{\tau_1}^{\tau_2} \left\{ \delta^n E^{(2)} + \delta^n T^{(2)} - \delta^n W^{(2)} \right. \\ & \quad \left. - b \int_{L_c} \delta^n u_n^{(2)} R_n dL \right\} dt = 0 \end{aligned} \right\} \dots\dots (38)$$

Using Eq. (38), R_n of the body (1) or (2) can be calculated approximately. Consequently, R_t is determined approximately by using Eq. (32). Thus, δQ_t is obtained by Eq. (36). Hence, R_n and R_t can be determined in the correction of an iterative calculation process by using Eqs. (32) and (36) to (38).

From the above description, Eq. (37) represents the virtual work principle for the two bodies in a slip state, and the subsidiary conditions are given as Eqs. (32) to (36) and (38).

3.3 Separate State of Two Bodies (Case 1)

When two bodies are in separation, we have $R = 0$ on L_c . So the virtual contact work δQ in Eq. (26) leads to zero. Equation (31) in a stick state expresses the virtual work principle for the two bodies in a separate state. However, the relation between the two bodies is independent, because there exist no subsidiary conditions. Therefore, the expression of Eq. (31) can be converted to Eq. (11).

On the other hand, Eq. (11) can be derived immediately from Eq. (24), because $\delta Q^{(1)}$ and $\delta Q^{(2)}$ in Eq. (25) become zero.

From these descriptions, the conventional virtual work principle of a body, Eq. (11), is applicable to each body of the two bodies in a separate state.

3.4 Separate State of Two Bodies (Case 2)

There exists a behavior of alternation between contact and separation, such as observed on a beam struck with a rod, on meshing gears, and so on. Then, the use of the FEM for two bodies makes it easy to discriminate between contact and separation on the contact line, and makes it possible to decrease the computing time.⁽¹¹⁾

In a separate state of two bodies, Eqs. (11) and (31) are used, as is clear in section 3.3. On the other hand, the two bodies may be regarded as a connective body, using a relative movement vector r_c corresponding to the clearance between both bodies. Then, the subsidiary condition of displacements is

$$u_c^{(1)} - u_c^{(2)} = r_c \dots\dots\dots (39)$$

The subsidiary conditions of velocities, accelerations, and virtual displacements between the two bodies are

$$\left. \begin{aligned} \dot{u}_c^{(1)} - \dot{u}_c^{(2)} &= \dot{r}_c, & \ddot{u}_c^{(1)} - \ddot{u}_c^{(2)} &= \ddot{r}_c \\ \delta u_c^{(1)} - \delta u_c^{(2)} &= \delta r_c \end{aligned} \right\} \dots\dots (40)$$

From the above description, Eq. (31) represents the virtual work principle, and the subsidiary conditions are given by Eqs. (39) and (40).

3.5 Slip State of Two Bodies (Case 2)

Figure 3 shows a two dimensional element division near the contact line between two bodies by using triangular elements. Nodal points discretized on L_c are arranged with a proper distance which makes it possible to investigate the variation of configurations and stress distributions in various contact states. When the number of elements near the contact nodes is identical for each body, the contact nodes of the two bodies are paired in a contact state. In the stick-slip behavior, there appears a discrepancy of position in each pair of the contact nodes. If the discrepancy is not negligible compared with the distance between neighboring contact nodes, an alignment of contact nodes must be made by discretizing again the two bodies or interpolating the results obtained.

Introducing a relative movement vectors r_t in the t -direction in the same manner as in a separate state, it is very easy to investigate the discrepancy in each pair of contact nodes, and to discriminate between the stick and slip states.⁽¹¹⁾ In section 3.2, the subsidiary conditions in the n -direction are given by Eqs. (33) to (35). In this section, the subsidiary conditions in the t -direction are supplied referring to Eqs. (39) and (40). Then, we obtain

$$u_n^{(1)} = u_n^{(2)}, \quad u_t^{(1)} = u_t^{(2)} + r_t \dots\dots\dots (41)$$

In addition, the subsidiary conditions of velocities, accelerations, and virtual displacements between the two bodies are

$$\left. \begin{aligned} \dot{u}_n^{(1)} &= \dot{u}_n^{(2)}, & \dot{u}_t^{(1)} &= \dot{u}_t^{(2)} + \dot{r}_t \\ \ddot{u}_n^{(1)} &= \ddot{u}_n^{(2)}, & \ddot{u}_t^{(1)} &= \ddot{u}_t^{(2)} + \ddot{r}_t \\ \delta u_n^{(1)} &= \delta u_n^{(2)}, & \delta u_t^{(1)} &= \delta u_t^{(2)} + \delta r_t \end{aligned} \right\} \dots\dots (42)$$

Then, the virtual work principle is expressed as Eq. (37) and its subsidiary conditions are given by Eqs. (32), (38), (41) and (42).

3.6 Summary of Virtual Work Principles and Their Subsidiary Conditions

Table 1 shows the virtual work principles and their subsidiary conditions for two bodies in various two-dimensional contact and separate states.

4. Virtual Work Principles of Three-dimensional Contact Structures

The virtual work principles and their subsidiary conditions for two elastoid impact bodies are three-dimensionally established on the basis of the

two-dimensional principles and their subsidiary condition. (10), (11) Rectangular cartesian coordinates on a contact area S_c are composed of the n -direction normal to the area and both the t_a - and t_b -directions tangential to the area, as shown in Fig. 4. Vectors used in various contact and separate states are three dimensionally calculated by using Eqs. (1) to (4), and (15).

4.1 Stick State of Two Bodies

We consider the motion of two bodies not slipping on a contact area. Let R_n , R_{ta} , and R_{tb} denote, respectively, the contact force vectors in the n -, t_a -, t_b -directions. Sticking conditions exist if

$$|R_{ta}| < |\mu_a R_n|, \quad |R_{tb}| < |\mu_b R_n| \quad \dots\dots\dots (43)$$

where μ_a and μ_b are, respectively, the frictional coefficients in both t_a - and t_b -directions.

The subsidiary condition between $u_c^{(1)}$ and $u_c^{(2)}$ is

$$u_c^{(1)} = u_c^{(2)}, \quad u_c^{(\alpha)} = \{u_n^{(\alpha)}, u_{ta}^{(\alpha)}, u_{tb}^{(\alpha)}\} \quad (\alpha=1, 2) \quad \dots\dots\dots (44)$$

The subsidiary conditions of velocities, accelerations, and virtual displacements on the contact area are expressed as

$$\dot{u}_c^{(1)} = \dot{u}_c^{(2)}, \quad \ddot{u}_c^{(1)} = \ddot{u}_c^{(2)}, \quad \delta u_c^{(1)} = \delta u_c^{(2)} \quad \dots (45)$$

If the subsidiary conditions are provided by Eqs. (44) and (45), the virtual work principle in the stick state of the two bodies is

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) \right\} dt = 0 \quad \dots\dots\dots (46)$$

Fig. 3 Two-dimensional element division near the contact area.

4.2 Slip State of Two Bodies (Case 1)

Let us consider the motion of two bodies slipping on a contact area. Slipping conditions in both the t_a - and t_b -directions are given as

$$|R_{ta}| = |\mu_a R_n|, \quad |R_{tb}| = |\mu_b R_n| \quad \dots\dots\dots (47)$$

Since the two bodies are interconnected in the n -direction, the subsidiary conditions are

$$\left. \begin{aligned} u_n^{(1)} &= u_n^{(2)}, & \dot{u}_n^{(1)} &= \dot{u}_n^{(2)} \\ \ddot{u}_n^{(1)} &= \ddot{u}_n^{(2)}, & \delta u_n^{(1)} &= \delta u_n^{(2)} \end{aligned} \right\} \quad \dots\dots\dots (48)$$

The virtual frictional work δQ_t is

$$\begin{aligned} \delta Q_t &= \int_{S_c} (\partial u_{ta}^{(1)} - \partial u_{ta}^{(2)}) R_{ta} dS \\ &+ \int_{S_c} (\partial u_{tb}^{(1)} - \partial u_{tb}^{(2)}) R_{tb} dS \quad \dots\dots\dots (49) \end{aligned}$$

Then, the virtual work principle in the slip state of the two bodies is

$$\int_{\tau_1}^{\tau_2} \left\{ \sum_{\alpha=1}^2 (\delta E^{(\alpha)} + \delta T^{(\alpha)} - \delta W^{(\alpha)}) + \delta Q_t \right\} dt = 0 \quad \dots (50)$$

The contact force vector R_n is obtained as

$$\left. \begin{aligned} \int_{\tau_1}^{\tau_2} \left(\delta^n E^{(1)} + \delta^n T^{(1)} - \delta^n W^{(1)} \right. \\ \left. + \int_{S_c} \delta^n u_n^{(1)} R_n dS \right) dt = 0 \\ \text{or} \\ \int_{\tau_1}^{\tau_2} \left(\delta^n E^{(2)} + \delta^n T^{(2)} - \delta^n W^{(2)} \right. \\ \left. - \int_{S_c} \delta^n u_n^{(2)} R_n dS \right) dt = 0 \end{aligned} \right\} \quad \dots\dots\dots (51)$$

Fig. 4 Rectangular cartesian coordinates on a contact area.

Thus, the subsidiary conditions are given by Eqs. (47) to (49) and (51).

4.3 Slip State of Two Bodies (Case 2)

By applying the relative movements r_{ta} and r_{tb} to the t_a - and t_b -directions, the subsidiary contact conditions are

$$\left. \begin{aligned} u_{ta}^{(1)} &= u_{ta}^{(2)} + r_{ta}, & u_{tb}^{(1)} &= u_{tb}^{(2)} + r_{tb} \\ \dot{u}_{ta}^{(1)} &= \dot{u}_{ta}^{(2)} + \dot{r}_{ta}, & \dot{u}_{tb}^{(1)} &= \dot{u}_{tb}^{(2)} + \dot{r}_{tb} \\ \ddot{u}_{ta}^{(1)} &= \ddot{u}_{ta}^{(2)} + \ddot{r}_{ta}, & \ddot{u}_{tb}^{(1)} &= \ddot{u}_{tb}^{(2)} + \ddot{r}_{tb} \\ \delta u_{ta}^{(1)} &= \delta u_{ta}^{(2)} + \delta r_{ta}, & \delta u_{tb}^{(1)} &= \delta u_{tb}^{(2)} + \delta r_{tb} \end{aligned} \right\} \dots (52)$$

Thus, Eq. (52) is supplied in the subsidiary conditions presented in section 4.2.

4.4 Separate State of Two Bodies (Case 1)

As discussed in section 3.3, the contact force vectors in a separate state lead to zero, and the two bodies are independent. Hence, the virtual work principle is given by Eq. (11) or (46), and there exist no subsidiary contact conditions.

4.5 Separate State of Two Bodies (Case 2)

Using the relative movement vector r_c corresponding to the clearance between two bodies, the subsidiary conditions are

$$\left. \begin{aligned} u_c^{(1)} - u_c^{(2)} &= r_c, & \dot{u}_c^{(1)} - \dot{u}_c^{(2)} &= \dot{r}_c \\ \ddot{u}_c^{(1)} - \ddot{u}_c^{(2)} &= \ddot{r}_c, & \delta u_c^{(1)} - \delta u_c^{(2)} &= \delta r_c \end{aligned} \right\} \dots (53)$$

Then, the virtual work principle for the two bodies is given by Eq. (46).

4.6 Mixed State of Two Bodies (Case 1)

Let us consider the mixed motion of two bodies with both a stick state and a slip state on a contact area, for example, the lateral slipping behavior of a tire on a road. The mixed state is treated as a slip state in the t_a -direction and as a stick state in the t_b -direction. Mixed conditions in both the t_a - and t_b -directions are represented as

$$|R_{ta}| = |\mu_a R_n|, \quad |R_{tb}| < |\mu_b R_n| \dots (54)$$

Table 1. Virtual work principles and their subsidiary conditions for two bodies in various two-dimensional contact and separate states

States		Principles and subsidiary conditions	Virtual work principles	Subsidiary conditions		
				Contact force vector	Various vectors of displacement system	
					n-direction	t-direction
Stick state			Eq. (31)		Eqs. (28) - (30)	
Slip state	Case 1		Eq. (37)	Eqs. (32), (36), (38)	Eqs. (33) - (35)
	Case 2				Eqs. (41), (42)	
Separate state	Case 1		Eq. (11) or (31)
	Case 2		Eq. (31)		Eqs. (39), (40)	

Table 2. Virtual work principles and their subsidiary conditions for two bodies in various three-dimensional contact and separate states

States		Principles and subsidiary conditions	Virtual work principles	Subsidiary conditions			
				Contact force vector	Various vectors of displacement system		
					n-direction	t_a -direction	t_b -direction
Stick state			Eq. (46)		Eqs. (44), (45)		
Slip state	Case 1		Eq. (50)	Eqs. (47), (49), (51)	Eq. (48)
	Case 2					Eq. (52)	
Separate state	Case 1		Eq. (11) or (46)
	Case 2		Eq. (46)		Eq. (53)		
Mixed state	Case 1		Eq. (50)	Eq. (51), (54), (55)	Eq. (56)		Eq. (56)
	Case 2					Eq. (57)	

The virtual frictional work δQ_f is expressed as

$$\delta Q_f = \int_{S_c} (\delta \mathbf{u}_{ta}^{(1)} - \delta \mathbf{u}_{tb}^{(2)}) \mathbf{R}_{ta} dS \quad \dots\dots\dots (55)$$

The following subsidiary conditions on the contact area are obtained from the continuity of displacements.

$$\left. \begin{aligned} \mathbf{u}_n^{(1)} &= \mathbf{u}_n^{(2)}, & \mathbf{u}_{tb}^{(1)} &= \mathbf{u}_{tb}^{(2)} \\ \dot{\mathbf{u}}_n^{(1)} &= \dot{\mathbf{u}}_n^{(2)}, & \dot{\mathbf{u}}_{tb}^{(1)} &= \dot{\mathbf{u}}_{tb}^{(2)} \\ \ddot{\mathbf{u}}_n^{(1)} &= \ddot{\mathbf{u}}_n^{(2)}, & \ddot{\mathbf{u}}_{tb}^{(1)} &= \ddot{\mathbf{u}}_{tb}^{(2)} \\ \delta \mathbf{u}_n^{(1)} &= \delta \mathbf{u}_n^{(2)}, & \delta \mathbf{u}_{tb}^{(1)} &= \delta \mathbf{u}_{tb}^{(2)} \end{aligned} \right\} \dots\dots\dots (56)$$

If the subsidiary conditions are given by Eqs. (51) and (54) to (56), the virtual work principle for two bodies is represented by Eq. (50).

4.7 Mixed State of Two Bodies (Case 2)

Let us introduce the relative movement H_{ta} in the t_a -direction to an example in section 4.6. The subsidiary conditions are

$$\left. \begin{aligned} \mathbf{u}_{ta}^{(1)} &= \mathbf{u}_{ta}^{(2)} + \mathbf{r}_{ta}, & \dot{\mathbf{u}}_{ta}^{(1)} &= \dot{\mathbf{u}}_{ta}^{(2)} + \dot{\mathbf{r}}_{ta} \\ \ddot{\mathbf{u}}_{ta}^{(1)} &= \ddot{\mathbf{u}}_{ta}^{(2)} + \ddot{\mathbf{r}}_{ta}, & \delta \mathbf{u}_{ta}^{(1)} &= \delta \mathbf{u}_{ta}^{(2)} + \delta \mathbf{r}_{ta} \end{aligned} \right\} \dots\dots\dots (57)$$

If the subsidiary conditions are given by Eqs. (51), and (54) to (57), the virtual work principle for two bodies is expressed as Eq. (50).

4.8 Summary of Virtual Work Principles and Their Subsidiary Conditions

Table 2 shows the virtual work principles and their subsidiary conditions for two bodies in various three-dimensional contact and separate states.

5. Conclusions

In this paper, we formulated the virtual work principles and their subsidiary contact conditions in various contact and separate states of two elastostatic contact bodies.

The followings are concluded:

- (1) It is possible for a virtual work principle to adopt or reject the various virtual works according to the problems to be analyzed: whether or not the problems are conservative, linear, steady, static and so on. Moreover, the principle can be easily transformed into a hybrid type and/or an incremental representation, so that the use of the virtual work principle is more useful than the use of other variational and/or energy principles.
- (2) Using Lagrange multipliers, subsidiary contact conditions of two bodies can be included in a virtual work principle. However, the physical properties of the

multipliers are undefined, and the relation among the multipliers remains unexplained. Therefore, it is very difficult to formulate a hybrid type of the virtual work principle which relaxes subsidiary contact conditions.

(3) The various two-dimensional contact and separate states of two bodies are investigated in detail, to derive the virtual work principles and their subsidiary contact conditions for the two bodies from the conventional principle for a body. In the stick and slip states, the two bodies are regarded as a connective body through a contact area. In a separate state, the conventional principle is applicable to each of the two bodies.

(4) In a separate state, the virtual work principle and its subsidiary conditions for two bodies are obtained by introducing a relative movement corresponding to clearance and regarding the two bodies as a connective body. Moreover, the relative movement in the slip direction can be applied to the subsidiary conditions of a slip state in the same way as those of the separate state. Hence, the concept of the relative movement is effective to discriminate the alternating behavior between contact and separation, between stick and slip, etc. in contact structures.

(5) Finally, the virtual work principles and their subsidiary conditions in various three-dimensional contact and separate states of two bodies can be formulated by applying the basic two-dimensional principles and their subsidiary conditions to two three-dimensional elastic impact bodies. The mixed states are, also, discussed, where the one in the two tangential directions is in a stick state, and the other is in a slip state.

References

- (1) Ohte, S., Bulletin of the JSME, vol. 16, No.95, (1973-5), pp.797-804.
- (2) Tsuta, T. and Yamaji, S., Journal of Japan Soc. Mech. Engrs. (in Japanese), vol. 76, No. 651, (1973-3), pp.348-358.
- (3) Okamoto, N. and Nakazawa, M., Int. J. Num. Mech. Engrs., vol. 14, (1979), pp.337-357.
- (4) Yagawa, G., Hirayama, H. and Andoh, Y., Transaction of Japan Soc. Mech. Engrs. (in Japanese), vol. 46, No. 411, Ser. A., (1980-11), pp. 1220-1229.
- (5) Timoshenko, S., Vibration Problems in Engineering (3rd Ed.), p. 413, D. VAN NOSTRAND COMPANY INC. (1955).
- (6) Doi, O. and Asano, N., Bulletin of the JSME, vol. 24, No. 189, (1981-3), pp. 528-533.
- (7) Asano, N., Proceedings of Japan Soc. Mech. Engrs. (in Japanese), No. 788-3, (1978-10), pp. 97-104.
- (8) Asano, N., Bulletin of the JSME, vol. 23, No. 183, (1980-9), pp. 1421-1427.
- (9) Doi, O., and Asano, N., Proceedings of Japan Soc. Mech. Engrs. (in Japanese), No. 760-17, (1976-10), pp. 141-143, *ibid*, pp. 144-146.
- (10) Asano, N., Proceedings of Japan Soc. Mech. Engrs. (in Japanese), No. 792-1, (1979-10), pp. 1-8.
- (11) Asano, N., Bulletin of the JSME, vol. 24, No. 193, (1981-7), pp. 1123-1129.
- (12) Asano, N., Bulletin of the JSME, vol. 25, No. 202, (1982-4), pp. 501-507.
- (13) Asano, N., Proceedings of Japan Soc. Mech. Engrs. (in Japanese), No. 800-9, pp. 95-100.
- (14) Kamegaya, H. and Asano, N., Bulletin of the JSME, vol. 25, No. 203, (1982-3), pp. 774-781.
- (15) Nemat-Nasser, S. and Lee, K.N., Computer Methods in Appl. Mech. and Engng., vol. 2, No. 1, (1973-2), pp. 33-41.
- (16) Washizu, K., Variational Principles in Elasticity, (in Japanese), (1972), p 26, Baifu-kan Co. Ltd.
- (17) Hughes, T.R., Taylor, R.L., and Kanoknukulchai, W., U.S.-Germany Symp. on FEM, (1976), pp. 468-495.