Free and Forced Vibrations of Elastically Connected Double-beam Systems*

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The free vibrations and the forced vibrations of a system of two elastically connected parallel upper and lower beams having unequal masses and unequal flexural rigidities are analysed by using a generalized method of finite integral transformation and the Laplace transformation. The natural frequencies and the amplitude ratios for the normal modes of vibration are presented explicitly, and the effects of mass ratio and flexural rigidity ratio of the beams are discussed. It is also shown that a system consisting of a lower beam and coupling springs can be used as a dynamic vibration absorber to reduce the vibrations of the upper beam subjected to a periodic force. An elastically connected multi-beam system is compared with a multi-degree-of-freedom-mass-spring system, and their similarity is pointed out.

Key Words: Vibrations, Vibrations of Beams, Elastically Connected Double-Beam Systems, Dynamic Vibration Absorber

connected Bernoulli-Euler uniform beams having unequal masses and unequal flexural rigidities when spring mass is neglected.

1. Introduction

The vibration problems of two Bernoulli-Euler uniform beams of equal length connected by spring-like materials are of technological interest because the system can be regarded as an ideal model of structural elements practically used, or as an approximate model of sandwich beams, or as a model which could be used in the future.

The free vibrations⁽¹⁾ and the impact problem⁽²⁾ of an elastically connected double-beam system were first studied by Seelig and Hoppmann II, both theoretically and experimentally. Saito and Chonan analysed the free vibrations, the forced vibrations⁽³⁾, and the impact problem⁽⁴⁾ of the same system taking the effects of spring mass into account. However, these studies are limited to a particular case of equal beams; the general case, when the mass and the flexural rigidity of the beams are different, has not been investigated. Since the case of equal beams is very special, it is necessary to analyse the general case in order to apply the theory of double-beam systems more widely to the analysis of the models mentioned above. The purpose of this paper is to reveal the properties of free vibrations and forced vibrations of a system of two elastically

2. Nomenclature

- E_i : Young's modulus
- I_i : moment of inertia of beam cross section
- j : number of j th modes of vibration, $= 1, 2, 3, \dots$
- k_c : stiffness of coupling springs per unit length of beam
- L : length of beam
- M_{aj}, M_{bj} : amplitude ratios of j th mode of vibration
- P_0 : concentrated force of constant magnitude
- $q_i(\xi, t)$: applied force per unit length of beam
- t : time
- v_0 : static deflection at mid-span of simply supported beam under force P_0 at mid-span, $= P_0 L^3 / (48 E_1 I_1)$
- $v_j(\xi)$: j th normal function of beam
- $v_i(\xi, t)$: deflection of beam
- x : length co-ordinate of beam
- α : stiffness ratio of coupling springs to upper beam, $= k_c L / (E_1 I_1 / L^3)$
- β : flexural rigidity ratio of lower beam to upper beam, $= E_2 I_2 / (E_1 I_1)$
- γ : mass ratio of lower beam to upper beam, $= \mu_2 / \mu_1$
- $\chi_i = k_c / \mu_i$
- λ_j : root of frequency equation
- μ_i : mass per unit length of beam
- ξ : dimensionless length co-ordinate of beam, $= x / L$
- ω : forcing circular frequency

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ω_{ij} : natural circular frequency of single beam

ω_{aj}, ω_{bj} : natural circular frequencies of double-beam system

Subscript i denotes quantities concerned with upper beam ($i=1$) and lower beam ($i=2$).

3. Method of analysis

When the effects of rotatory inertia and shear deformation of beams, and the effects of damping are neglected, the differential equations of a system of two Bernoulli-Euler beams connected by innumerable coupling springs, shown in Fig.1, are

$$\begin{aligned} \partial^2 v_i / \partial t^2 + \{E_i I_i / (\mu_i L^4)\} \partial^4 v_i / \partial \xi^4 \\ = (-1)^{i+1} \kappa_i (v_2 - v_1) + q_i / \mu_i \quad (i=1, 2) \end{aligned} \quad (1)$$

Fig.1 Elastically connected double-beam system

Assume that the boundary conditions of all beams are the same, and that one of the following combinations of boundary conditions is given to the beams.

$$\text{S-S, C-C, F-F, C-F, S-C} \quad (2)$$

where the symbols S, C, and F represent simply supported end, clamped end, and free end, respectively. The initial conditions of the beams are assumed to be written as functions of ξ in the form:

$$\begin{cases} v_i(\xi, t) = f_i(\xi) \\ \partial v_i(\xi, t) / \partial t = g_i(\xi) \end{cases} \quad \text{at } t=0 \quad (3)$$

Consider the following finite integral transforms and their inverse transforms, where $v_j(\xi)$ is the j th normal function of a single beam which satisfies given boundary conditions of the double-beam system.

$$V_{ij}^*(s) = V_{aij}^*(s) + V_{bij}^*(s) + V_{cij}^*(s) \quad (i=1, 2; j=1 \sim \infty) \quad (7\cdot a)$$

$$V_{a1j}^*(s) = \{(s^2 + \omega_{2j}^2 + \kappa_2) Q_{1j}^*(s) / \mu_1 + \kappa_1 Q_{2j}^*(s) / \mu_2\} / D(s) \quad (7\cdot b)$$

$$V_{a2j}^*(s) = \{\kappa_2 Q_{1j}^*(s) / \mu_1 + (s^2 + \omega_{1j}^2 + \kappa_1) Q_{2j}^*(s) / \mu_2\} / D(s) \quad (7\cdot c)$$

$$V_{b1j}^*(s) = s \{(s^2 + \omega_{2j}^2 + \kappa_2) F_{1j} + \kappa_1 F_{2j}\} / D(s) \quad (7\cdot d)$$

$$V_{b2j}^*(s) = s \{\kappa_2 F_{1j} + (s^2 + \omega_{1j}^2 + \kappa_1) F_{2j}\} / D(s) \quad (7\cdot e)$$

$$V_{c1j}^*(s) = \{(s^2 + \omega_{2j}^2 + \kappa_2) G_{1j} + \kappa_1 G_{2j}\} / D(s) \quad (7\cdot f)$$

$$V_{c2j}^*(s) = \{\kappa_2 G_{1j} + (s^2 + \omega_{1j}^2 + \kappa_1) G_{2j}\} / D(s) \quad (7\cdot g)$$

$$D(s) = s^4 + (\omega_{1j}^2 + \omega_{2j}^2 + \kappa_1 + \kappa_2) s^2 + \omega_{1j}^2 \omega_{2j}^2 + \kappa_2 \omega_{1j}^2 + \kappa_1 \omega_{2j}^2 = (s^2 + \omega_{aj}^2)(s^2 + \omega_{bj}^2) \quad (8\cdot a)$$

$$\left. \begin{aligned} \omega_{aj}^2 \\ \omega_{bj}^2 \end{aligned} \right\} = \frac{1}{2} [\omega_{1j}^2 + \omega_{2j}^2 + \kappa_1 + \kappa_2 \mp \{(\omega_{1j}^2 - \omega_{2j}^2 + \kappa_1 - \kappa_2)^2 + 4\kappa_1 \kappa_2\}^{1/2}] \quad (8\cdot b)$$

where ω_{aj}, ω_{bj} ($j=1, 2, 3, \dots$) are the natural frequencies of the system.

If the inverse transformation of Eqs.(7.a) can be performed under given applied force and initial conditions, $V_{ij}(t)$ can be obtained. Then, the response of the upper and the lower beams can be calculated from expressions (4.b) by using the normal functions which satisfy given boundary conditions.

4. Free vibrations

$$V_{ij}(t) = \int_0^1 v_i(\xi, t) v_j(\xi) d\xi \quad (4\cdot a)$$

$$v_i(\xi, t) = \sum_{j=1}^{\infty} V_{ij}(t) v_j(\xi) \quad (4\cdot b)$$

$$Q_{ij}(t) = \int_0^1 q_i(\xi, t) v_j(\xi) d\xi \quad (4\cdot c)$$

$$q_i(\xi, t) = \sum_{j=1}^{\infty} Q_{ij}(t) v_j(\xi) \quad (4\cdot d)$$

$$F_{ij} = \int_0^1 f_i(\xi) v_j(\xi) d\xi \quad (4\cdot e)$$

$$f_i(\xi) = \sum_{j=1}^{\infty} F_{ij} v_j(\xi) \quad (4\cdot f)$$

$$G_{ij} = \int_0^1 g_i(\xi) v_j(\xi) d\xi \quad (4\cdot g)$$

$$g_i(\xi) = \sum_{j=1}^{\infty} G_{ij} v_j(\xi) \quad (4\cdot h)$$

in which the functions $v_j(\xi)$ are normalized by the condition $\int_0^1 v_j^2(\xi) d\xi = 1$ ⁽⁵⁾. Next, let the natural frequencies of the separated single upper and lower beams of the system be denoted by ω_{ij} , where

$$\omega_{ij}^2 = (\lambda_j / L)^4 E_i I_i / \mu_i \quad (i=1, 2; j=1 \sim \infty) \quad (5)$$

and where λ_j is the root, corresponding to the j th normal mode of vibration, of the frequency equation of a single beam under given boundary conditions.

By multiplying both sides of Eqs.(1) by $v_j(\xi)$ and integrating them from 0 to 1 with respect to ξ , the following equations are obtained using Eqs.(2), (4) and (5):

$$\begin{aligned} d^2 V_{ij}(t) / dt^2 + \omega_{ij}^2 V_{ij}(t) \\ = (-1)^{i+1} \kappa_i \{V_{2j}(t) - V_{1j}(t)\} \\ + Q_{ij}(t) / \mu_i \quad (i=1, 2; j=1 \sim \infty) \end{aligned} \quad (6)$$

Now let the Laplace transforms of $V_{ij}(t)$ and $Q_{ij}(t)$ with respect to t be denoted by $V_{ij}^*(s)$ and $Q_{ij}^*(s)$, respectively. The Laplace transformation of both sides of Eqs.(6) under initial conditions (3) gives the following fundamental transformed formulae:

4.1 Normal modes of vibration

Substituting the expressions $Q_{ij}^*(s) = F_{ij} = 0$ ($i=1,2; j=1,2,3,\dots$) into Eqs.(7) reduces $V_{ij}^*(s)$ to the following form:

$$\begin{aligned} V_{1j}^*(s) &= \{(s^2 + \omega_{2j}^2 + x_2)G_{1j} + x_1 G_{2j}\} / D(s) \\ V_{2j}^*(s) &= \{x_2 G_{1j} + (s^2 + \omega_{1j}^2 + x_1)G_{2j}\} / D(s) \end{aligned} \quad (9)$$

Then, introduce the following quantities M_{aj} , M_{bj} ($j=1,2,3,\dots$) defined by

$$M_{aj} = (\omega_{bj}^2 - \omega_{2j}^2 - x_2) / x_1 = x_2 / (\omega_{bj}^2 - \omega_{1j}^2 - x_1) \quad (10 \cdot a)$$

$$M_{bj} = -(\omega_{aj}^2 - \omega_{2j}^2 - x_2) / x_1 = -x_2 / (\omega_{aj}^2 - \omega_{1j}^2 - x_1) \quad (10 \cdot b)$$

where the validity of the second signs of equality can be verified by substituting $s = i\omega_{bj}$, $i\omega_{aj}$ ($i = \sqrt{-1}$) into Eq.(8.a). Further, the values of M_{aj} and M_{bj} are positive because the relations

$$D(i\sqrt{\omega_{1j}^2 + x_1}) = D(i\sqrt{\omega_{2j}^2 + x_2}) = -x_1 x_2 \text{ hold.}$$

Now choose the expressions $G_{1j} = \omega_{aj} \delta_{jn}$ and $G_{2j} = \omega_{aj} M_{aj} \delta_{jn}$ as the initial conditions of the beams, where δ_{jn} is Kronecker delta symbol and n is a natural number. By substituting into Eqs.(9) the above initial conditions and the following expressions

$$x_1 M_{aj} = \omega_{bj}^2 - \omega_{2j}^2 - x_2, \quad x_2 = (\omega_{bj}^2 - \omega_{1j}^2 - x_1) M_{aj}$$

obtained from Eqs.(10.a), and using Eqs.(4.b) after performing an inverse transformation of $V_{ij}^*(s)$ with respect to s , the normal modes of vibration of the system can be expressed by

$$\left. \begin{aligned} v_1(\xi, t) &= v_j(\xi) \sin \omega_{aj} t \\ v_2(\xi, t) &= M_{aj} v_j(\xi) \sin \omega_{aj} t \end{aligned} \right\} \quad (j=n) \quad (11)$$

In a similar manner, by using the initial conditions $G_{1j} = \omega_{bj} \delta_{jn}$, $G_{2j} = -\omega_{bj} M_{bj} \delta_{jn}$ together with Eqs.(4.b), (9) and (10.b), the following expressions can be obtained:

$$\left. \begin{aligned} v_1(\xi, t) &= v_j(\xi) \sin \omega_{bj} t \\ v_2(\xi, t) &= -M_{bj} v_j(\xi) \sin \omega_{bj} t \end{aligned} \right\} \quad (j=n) \quad (12)$$

As can be seen from expressions (11) and (12), when the system performs a simple harmonic motion with one of the natural frequencies ω_{aj} , both beams vibrate in phase with each other, and the corresponding M_{aj} represents the amplitude ratio of the lower beam to the upper beam. Again, when the system performs a simple harmonic motion with one of the natural frequencies ω_{bj} , both beams vibrate π radians out of phase with each other, and the corresponding M_{bj} represents the absolute values of the amplitude ratio of the lower beam to the upper beam.

4.2 Natural frequencies and amplitude ratios

Introducing the dimensionless parameters α , β , γ and using the expressions $\omega_{1j}^2 = x_1(\lambda_j^2/\alpha)$ and $(\omega_{2j}/\omega_{1j})^2 = \beta/\gamma$, expressions (8.b) and (10) take the following forms:

$$\left. \begin{aligned} (\omega_{aj}/\omega_{1j})^2 &= \frac{1}{2} \frac{\alpha}{\lambda_j^2} \left[\frac{\lambda_j^2}{\alpha} \left(1 + \frac{\beta}{\gamma} \right) + 1 + \frac{1}{\gamma} \mp \left[\left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\}^2 + \frac{4}{\gamma} \right]^{1/2} \right] \\ (\omega_{bj}/\omega_{1j})^2 &= \frac{1}{2} \frac{\alpha}{\lambda_j^2} \left[\frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \pm \left[\left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\}^2 + \frac{4}{\gamma} \right]^{1/2} \right] \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} M_{aj} &= \frac{1}{2} \left[\left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\}^2 + \frac{4}{\gamma} \right]^{1/2} \pm \left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\} \\ M_{bj} &= \frac{1}{2} \left[\left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\}^2 + \frac{4}{\gamma} \right]^{1/2} \mp \left\{ \frac{\lambda_j^2}{\alpha} \left(1 - \frac{\beta}{\gamma} \right) + 1 - \frac{1}{\gamma} \right\} \end{aligned} \right\} \quad M_{aj} M_{bj} = \frac{1}{\gamma} \quad (14)$$

When the natural frequencies of the single upper beam coincide with those of the single lower beam (i.e., ω_{1j}, ω_{2j} for $j=1,2,3,\dots$), the relation $\beta = \gamma$ holds. In this case, the above expressions are reduced to the following simpler forms:

$$\left\{ \begin{aligned} (\omega_{aj}/\omega_{1j})^2 &= 1 \\ M_{aj} &= 1 \end{aligned} \right\}, \quad \left\{ \begin{aligned} (\omega_{bj}/\omega_{1j})^2 &= 1 + (1 + 1/\gamma) \alpha / \lambda_j^2 \\ M_{bj} &= 1/\gamma \end{aligned} \right\} \quad (15)$$

from which the same results, given by Seelig and Hoppmann II, are obtained when the system consists of two equal beams (i.e., $\beta = \gamma = 1$).

From evaluation of the limits of Eqs.(13) as $j \rightarrow \infty$ or $\alpha \rightarrow 0$, it follows that

$$\omega_{aj}/\omega_{1j} \rightarrow \begin{cases} 1 & (\gamma \leq \beta) \\ (\beta/\gamma)^{1/2} & (\gamma \geq \beta) \end{cases}, \quad \omega_{bj}/\omega_{1j} \rightarrow \begin{cases} (\beta/\gamma)^{1/2} & (\gamma \leq \beta) \\ 1 & (\gamma \geq \beta) \end{cases} \quad (16)$$

That is, as the modal number j becomes higher or the value of the coupling-spring constant becomes smaller, the natural frequencies ω_{aj} and ω_{bj} approach smaller and larger values of both ω_{1j} and ω_{2j} , respectively.

Next, when the material and the shape of the upper beam are kept invariant, consider the effects of the parameters β and γ on the natural frequencies and the amplitude ratios of the system.

It is readily verified by taking the partial derivatives of ω_{aj} , ω_{bj} , M_{aj} and M_{bj} with respect to γ that the following inequalities always hold:

$$\left. \begin{aligned} \partial \omega_{aj} / \partial \gamma &< 0, \quad \partial \omega_{bj} / \partial \gamma < 0 \\ \partial M_{aj} / \partial \gamma &> 0, \quad \partial M_{bj} / \partial \gamma < 0 \end{aligned} \right\} \quad (17)$$

Hence, when β is constant, ω_{aj} , ω_{bj} and M_{bj} are monotonically decreasing functions of γ , and M_{aj} is a monotonically increasing function of γ . Similarly, the partial differentiation of ω_{aj} , ω_{bj} , M_{aj} and M_{bj} with respect to β gives

$$\left. \begin{aligned} \partial \omega_{aj} / \partial \beta &> 0, \quad \partial \omega_{bj} / \partial \beta > 0 \\ \partial M_{aj} / \partial \beta &< 0, \quad \partial M_{bj} / \partial \beta > 0 \end{aligned} \right\} \quad (18)$$

Hence, when γ is constant, ω_{aj} , ω_{bj} and M_{bj} are monotonically increasing functions of β ,

and M_{aj} is a monotonically decreasing function of β . Further, concerning the values of M_{aj} and M_{bj} in Eqs(14), the following inequalities generally hold:

$$\left. \begin{aligned} M_{aj} &> 1 & \text{if } \gamma > \beta \\ M_{aj} &< 1 & \text{if } \gamma < \beta \\ M_{bj} &< 1 & \text{if } \gamma M_{aj} > 1 \\ M_{bj} &> 1 & \text{if } \gamma M_{aj} < 1 \end{aligned} \right\} \quad (19)$$

As an example, the natural frequencies and the amplitude ratios for $j=1,2,3$ of the system having $\alpha=100$ and S-S boundary conditions are illustrated as functions of γ and β in Fig.2 to Fig.5. For the systems with the other boundary conditions, it is clear that the dependence of the natural frequencies and the amplitude ratios on the parameters γ and β shows similar tendencies.

So far, the cases of γ and β varying independently have been discussed. However, the relation $\beta=\gamma^2$ holds when both beams of the same material have either square or circular cross sections. Hence, the natural frequencies and the amplitude ratios for $j=1,2,3$ of the system having $\beta=\gamma^2$, $\alpha=100$ and S-S are shown in Fig.6 and Fig.7 as functions of γ . In this case, both ω_{aj} and ω_{bj} decrease monotonically as the value of γ increases, and then they increase monotonically after attaining their minimum

values. Thereupon, each M_{aj} increases monotonically from unity, and attains its maximum value at a certain value of γ (i.e., in this example, $\gamma=0.403$ for $j=1$, $\gamma=0.239$ for $j=2$ and $\gamma=0.112$ for $j=3$) which gives the minimum value of ω_{aj} , and then decreases monotonically. Meanwhile, each M_{bj} decreases monotonically from infinity, and attains its minimum value at a certain value of γ (i.e., in this example, $\gamma=1.475$ for $j=1$, $\gamma=0.541$ for $j=2$ and $\gamma=0.507$ for $j=3$) which gives the minimum value of ω_{bj} , and then increases monotonically.

In general, the values of γ which give maxima or minima of the functions ω_{aj} , ω_{bj} , M_{aj} and M_{bj} ($\beta=\gamma^2$) can be readily obtained by solving the following necessary condition (20) derived from any one of the equations $d\omega_{aj}/d\gamma=0$, $d\omega_{bj}/d\gamma=0$, $dM_{aj}/d\gamma=0$ and $dM_{bj}/d\gamma=0$.

$$2(\lambda_j^4/\alpha)\gamma^3 - \{1 + (\lambda_j^4/\alpha)\}\gamma^2 - 2\gamma + 1 = 0 \quad (20)$$

The solutions of the above cubic equation can also be obtained by the abscissas of the intersections of the curves expressed by

$$\begin{aligned} y &= 1/\gamma^2 + 1/(2\gamma - 1) & (21.a) \\ y &= \lambda_j^4/\alpha & (21.b) \end{aligned}$$

The curve of Eq.(21.a) is plotted for $\gamma > 0$ in Fig.8. In this example, since $\lambda_1=\pi$ and

Fig.2 ω_{aj}, ω_{bj} ($\beta=1, \alpha=100$)

Fig.3 M_{aj}, M_{bj} ($\beta=1, \alpha=100$)

Fig.4 ω_{aj}, ω_{bj} ($\gamma=1, \alpha=100$)

Fig.5 M_{aj}, M_{bj} ($\gamma=1, \alpha=100$)

Fig.6 ω_{aj}, ω_{bj} ($\beta=\gamma^2, \alpha=100$)

Fig.7 M_{aj}, M_{bj} ($\beta=\gamma^2, \alpha=100$)

Fig.8 Graph of $y=1/\gamma^2+1/(2\gamma-1)$

$\lambda_2=2\pi$, the values of Eq.(21.b) are $y=0.974$ for $j=1$ and $y=15.585$ for $j=2$. Hence, the values of γ corresponding to the above values of y are $\gamma=0.403$, $\gamma=1.475$ for $j=1$, and $\gamma=0.239$, $\gamma=0.541$ for $j=2$. From these results, it may be concluded that there are two positive roots which satisfy Eq.(20) for each j , and that one root satisfying $0 < \gamma < \sqrt{2}-1$ gives the minimum and the maximum values of ω_{aj} and M_{aj} , respectively, and the other root satisfying $0.5 < \gamma$ gives the minimum values of ω_{bj} and M_{bj} .

4.3 Special cases derived from the normal modes of vibration of the double-beam system

Normal modes of vibration of some particular systems can be derived from Eqs. (11), (13) and (14) as special cases of the double-beam system. In this section the normal modes of vibration of the following three cases will be discussed.

Case(I): Coupling springs are rigid. ($k_c \rightarrow \infty$)

In this case, since $\alpha \rightarrow \infty$, it follows from Eqs.(13) that $\lim_{\alpha \rightarrow \infty} \omega_{bj} = \infty$ and

$$\begin{aligned} \theta_{ij} &= \lim_{\alpha \rightarrow \infty} \omega_{aj} = \left(\frac{1+\beta}{1+\gamma} \right)^{1/2} \omega_{1j} \\ &= (\lambda_j/L)^2 \left(\frac{E_1 I_1 + E_2 I_2}{\mu_1 + \mu_2} \right)^{1/2} \end{aligned} \quad (22 \cdot a)$$

Hence, in the limit, the out-of-phase vibration does not occur, and the natural frequencies of the system become θ_{ij} only. Then, since $\lim_{\alpha \rightarrow \infty} M_{aj} = 1$ from Eqs.(14), expressions (11) take the following form:

$$v_1(\xi, t) = v_2(\xi, t) = v_j(\xi) \sin \theta_{ij} t \quad (j=n) \quad (22 \cdot b)$$

Case(II): Lower beam is massless. ($\mu_2 \rightarrow 0$)

In this case, since $\gamma \rightarrow 0$, it follows from Eqs.(13) that $\lim_{\gamma \rightarrow 0} \omega_{bj} = \infty$ and

$$\theta_{ij} = \lim_{\gamma \rightarrow 0} \omega_{aj} = \left\{ \frac{\beta(\lambda_j^2/\alpha) + \beta + 1}{\beta(\lambda_j^2/\alpha) + 1} \right\}^{1/2} \omega_{1j} \quad (23 \cdot a)$$

Then, since $\lim_{\gamma \rightarrow 0} M_{aj} = 1/\{\beta(\lambda_j^2/\alpha) + 1\}$ from Eqs.(14), expressions (11) take the following form:

$$\begin{aligned} v_1(\xi, t) &= v_j(\xi) \sin \theta_{ij} t, \\ v_2(\xi, t) &= [1/\{\beta(\lambda_j^2/\alpha) + 1\}] v_j(\xi) \sin \theta_{ij} t \end{aligned} \quad (j=n) \quad (23 \cdot b)$$

Case(III): Beam on an elastic foundation ($E_2 I_2 \rightarrow \infty$)

In this case, since $\beta \rightarrow \infty$, it follows from Eqs.(13) that $\lim_{\beta \rightarrow \infty} \omega_{bj} = \infty$ and

$$\theta_{ij} = \lim_{\beta \rightarrow \infty} \omega_{aj} = \left\{ \frac{\lambda_j^2/\alpha + 1}{\lambda_j^2/\alpha} \right\}^{1/2} \omega_{1j} \quad (24 \cdot a)$$

Then, since $\lim_{\beta \rightarrow \infty} M_{aj} = 0$ from Eqs.(14), expressions (11) take the following form:

$$\begin{aligned} v_1(\xi, t) &= v_j(\xi) \sin \theta_{ij} t, \\ v_2(\xi, t) &= 0 \end{aligned} \quad (j=n) \quad (24 \cdot b)$$

5. Forced vibrations

5.1 Steady-state solution of the system subjected to a sinusoidal force on the upper beam.

When a concentrated force $P_0 \sin \omega t$ is applied to the upper beam at point $\xi = \xi_0$ ($0 \leq \xi_0 \leq 1$), $q_i(\xi, t)$ can be described by

$$\begin{aligned} q_1(\xi, t) &= (P_0/L) \delta(\xi - \xi_0) \sin \omega t \\ q_2(\xi, t) &= 0 \end{aligned} \quad (25)$$

where $\delta(\xi)$ denotes Dirac delta function. Substituting the expressions $F_{ij} = G_{ij} = 0$ ($i=1,2$; $j=1,2,3,\dots$) together with the above expressions (25) into fundamental formulae (7), and evaluating the sum of residues of $V_{ij}^*(s)e^{st}$ at two simple poles $s = \pm i\omega$ ($i = \sqrt{-1}$), $V_{ij}(t)$ are obtained. The steady-state response is then described according to expressions (4.b) as follows:

$$v_i(\xi, t) = X_i(\xi, \xi_0) \sin \omega t \quad (i=1,2) \quad (26 \cdot a)$$

$$\begin{aligned} X_1(\xi, \xi_0) &= \frac{P_0}{\mu_1 L} \sum_{j=1}^{\infty} v_j(\xi) v_j(\xi_0) \\ &\times \frac{-\omega^2 + \omega_{2j}^2 + \kappa_2}{(\omega_{aj}^2 - \omega^2)(\omega_{bj}^2 - \omega^2)} \end{aligned} \quad (26 \cdot b)$$

$$\begin{aligned} X_2(\xi, \xi_0) &= \frac{P_0}{\mu_1 L} \sum_{j=1}^{\infty} v_j(\xi) v_j(\xi_0) \\ &\times \frac{\kappa_2}{(\omega_{aj}^2 - \omega^2)(\omega_{bj}^2 - \omega^2)} \end{aligned} \quad (26 \cdot c)$$

If the forcing frequency ω satisfies the condition

$$\omega^2 = \omega_{2n}^2 + \kappa_2 \quad (27)$$

it is clear, in view of Eq.(26.b), that the n th term of Eq.(26.b) is always zero whatever the values of ξ and ξ_0 may be. Incidentally, the steady-state solutions for the three special cases discussed in section 4.3 can be readily obtained by evaluating the limits of Eqs.(26) as $\alpha \rightarrow \infty$, $\gamma \rightarrow 0$ and $\beta \rightarrow \infty$, respectively.

By using Eqs.(26.b) and (26.c), the spectra for the steady-state amplitudes at any point of the system subjected to a concentrated force $P_0 \sin \omega t$ at a point ξ_0 of the upper beam can be obtained. Figure 9 shows the spectra for the steady-state amplitudes of the upper and the lower beams in the case of S-C, $\beta = \gamma = 1$, $\alpha = 100$ and $\xi = \xi_0 = 1/2$ within the interval $0 \leq \omega/\omega_{11} \leq 5$.

Fig.9 Spectra for the steady-state amplitudes of the system: (a) Upper Beam, (b) Lower Beam.

5.2 Effect of a lower beam used as a dynamic vibration absorber.

When regarding a single upper beam as a principal system, and a system consisting of a lower beam and coupling springs as an auxiliary system, the amplitudes of the principal system at or near resonance can be suppressed considerably by attaching an appropriate auxiliary system to the principal system. For example, consider the case when the forcing frequency ω of a sinusoidal force applied to the principal system coincides with the first natural frequency ω_{11} of the principal system. Thus, when the principal system is attached on auxiliary system which, with Eq.(27) taken into account, satisfies the condition $\omega^2 = \omega_{11}^2 = \omega_{21}^2 + \kappa_2$, i.e.,

$$\alpha/\lambda_1^4 = \gamma - \beta \quad (\gamma > \beta) \quad (28)$$

not only the resonance of the principal system is avoided, but also the first term of Eq.(26.b) becomes zero. Further considering the case of $\beta = \gamma^2$, expression (28) takes the form:

$$\alpha/\lambda_1^4 = \gamma(1 - \gamma) \quad (\gamma < 1) \quad (29)$$

Figure 10 shows the relationship between γ and the maximum amplitude of the lower beam for the S-S double-beam system so tuned as to satisfy condition (29) and subjected to a sinusoidal force $P_0 \sin \omega t$ at mid-span of the upper beam. In this figure, the symbols $v_{i \max}$ ($i=1,2$) denote the maximum amplitudes of both beams, and are defined by

$$v_{i \max} = \max\{|X_i(\xi, 1/2)|\} \quad (i=1,2; 0 \leq \xi \leq 1)$$

Fig.10 Relationship between γ and $v_{2 \max}$

According to the numerical results, the maximum amplitudes $v_{i \max}$ occur at mid-span of both beams, and $v_{1 \max}$ has almost the same value $0.0146 v_0$ within the interval $0 < \gamma < 1$. Hence, the maximum amplitude of the upper beam becomes very small when the auxiliary system is tuned as stated above. On the contrary, $v_{2 \max}$ varies parabolically as the value of γ increases, and takes its minimum value $3.94 v_0$ at $\gamma = 1/2$. In order to design dynamic vibration absorbers of this kind, first select a value of γ where $v_{2 \max}$ has reasonable magnitude, and then determine the value of α corresponding to the selected value of γ by using Eq.(29).

In this example, considering the auxiliary system tuned so as to make $v_{2 \max}$ minimum as the optimum absorber, $\gamma = 1/2$ should be selected. Then, $\alpha = 24.35$ is calculated by substitution of $\gamma = 1/2$ and $\lambda_1 = \pi$ into Eq.(29). In this case, since the static deflection at mid-span of the lower beam due to P_0 is calculated as $0.438 v_0$ by substitution of $\omega = 0$ into Eq.(26.c), the maximum amplitude of the lower beam becomes about nine times as large as the static deflection.

6. Discussion on the similarity between multi-beam systems and multi-degree-of-freedom-mass-spring systems

Consider a two-degree-of-freedom system shown in Fig.11. It consists of two masses μ_1 and μ_2 connected in series by three springs k_1 , k_c and k_2 . Let the displacements and the applied force be denoted by $V_i(t)$ and $Q_i(t)$, respectively. Introducing the notations $\omega_i^2 = k_i/\mu_i$ and $\chi_i = k_c/\mu_i$ ($i=1,2$), the differential equations of the system is given by

Fig.11 Two-degree-of-freedom-mass-spring system

$$\begin{aligned} d^2 V_i(t)/dt^2 + \omega_i^2 V_i(t) \\ = (-1)^{i+1} x_i \{ V_2(t) - V_1(t) \} + Q_i(t)/\mu_i \\ (i=1, 2) \end{aligned} \quad (30)$$

where ω_i is the natural frequency of an isolated system composed of the mass μ_i and spring k_i . As can be seen by comparing Eqs.(30) with Eqs.(6), equations (6) are identical in form to Eqs.(30) except for the fact that the quantities ω_i , V_i and Q_i in Eqs.(30) do not carry the subscript j . That is, with the aid of the generalized method of finite integral transformation, by separating the double-beam system discussed in this paper into infinite two-degree-of-freedom systems, each of the separated systems is made equivalent to the two-mass system shown in Fig.11. This equivalence arises from the assumption of both beams having the same boundary conditions, and because of this equivalence, natural frequencies (8.b) and amplitude ratio (10) of the double-beam system can be represented in the same forms as those of the two-mass system.

Expanding the above discussion, it is clear that the differential equations of the elastically connected multi-beam system shown in Fig.12, whose beams have the same boundary conditions, are reduced to the differential equations of the multi-degree-of-freedom system shown in Fig.13 under the finite integral transformation. Hence the well-known methods for analysing multi-degree-of-freedom systems can also be applied to the subsequent analysis of the multi-beam system.

The correspondence between the multi-beam system and the multi-mass system stated above is also valid even when the effects of the mass of coupling springs are taken into account. For instance, the double-beam system analysed by Saito and Chonan corresponds to the two-mass system in which $\mu_1 = \mu_2$, $k_1 = k_2$ and the longitudinal vibration of the coupling spring k_c is taken into account.

Fig.12 Elastically connected multi-beam system

Fig.13 Multi-degree-of-freedom-mass-spring system

7. Conclusions

The free vibrations and the forced vibrations of a system of two Bernoulli-Euler uniform beams of equal length connected by innumerable coupling springs and having the same boundary conditions have been analysed taking into consideration the case of unequal masses and unequal flexural rigidities of the beams. The summary of this paper is as follows: (1) The fundamental transformed formulae have been presented by applying a generalized method of finite integral transformation with respect to the length co-ordinate along the beams and the Laplace transformation with respect to time. (2) When the material and the shape of the upper beam are kept invariant, the effects of mass ratio and flexural rigidity ratio of the beams on the natural frequencies and the amplitude ratios of the double-beam system are clear. (3) It has been shown that the normal modes of vibration for the three cases of coupling springs being rigid, the lower beam being massless and a beam being on an elastic foundation can be derived as special cases of the double-beam system. (4) The steady-state solution of the double-beam system subjected to a sinusoidal force on the upper beam has been presented. Further, it has been shown that the system of a lower beam with coupling springs can be used, by tuning it appropriately, as a dynamic vibration absorber to reduce the vibrations of the upper beam. (5) It has been pointed out that the analysis of elastically connected multi-beam systems, whose beams have the same boundary conditions, is reduced to that of elastically connected multi-mass systems under the finite integral transformation.

It is clear that the method used in this paper may also be applied to the transient analysis of multi-beam system. In addition, this method enables one to analyse visco-elastically connected multi-beam systems, in which innumerable dashpots are inserted parallel to the coupling springs. Considering the vibrations of a visco-elastically connected double-beam system, a system consisting of a lower beam, coupling springs and dashpots would act as a dynamic vibration absorber to reduce the vibrations of the upper beam caused by a machine operating at variable speeds.

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