Vibration of Rotating Bladed Disc
Excited by Stationary Distributed Forces*

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This paper presents the results of theoretical and experimental studies which have been carried out to investigate the diametral vibration modes of rotating bladed discs. The theoretical model has been considered as a linear system and only the diameter modes have been discussed. The vibration of the bladed disc has been obtained by superposing vibrations of discs assumed to have a single blade respectively, so that the exciting condition has been derived. This condition depends upon the relations among the number of rotating blades, the number of exciting sources on the stationary side and the number of diametral nodal lines. These relations have been verified by experimental results. It is confirmed that the disc does not resonate when the exciting condition is not satisfied, even if the exciting frequencies coincide with natural frequencies of the disc.

1. Introduction

This paper describes on the bladed disc vibrations which often occur in the operation of such fluid machinery as Francis hydraulic turbines, pump-turbines, steam turbines and centrifugal compressors. The present studies have clarified what kind of diametral modes are excited on a rotating bladed disc when the disc is rotating in the exciting sources, such as stationary vanes and nozzles (in this paper, stationary vanes, nozzles, etc. are collectively designated “exciting sources”) regularly distributed on the stationary side.

As regards the diametral modes of the rotating bladed discs, W. Campbell(1) has performed well-known studies on steam turbines, discussing mainly the fixed waves at the stationary side taking place on the disc, in case of a single exciting source. In the problems of fluid machinery which have plural exciting sources, however, it is important to consider the interaction between the rotating blades and the stationary exciting sources. Concerning this interaction, F. Kushner(2) has made a study on some combinations of the number of blades and the number of exciting sources. As a general consideration, however, his study seems to be inadequate.

This paper reveals theoretically the general relations among the number of rotating blades, the number of exciting sources distributed on the stationary side, and the number of diametral nodal lines of vibration mode excited on the disc. These relations are verified by results of experimental studies.

2. Theory

The theoretical model is shown in Fig. 1. $\theta$ represents co-ordinate fixed on stationary side and $\phi$ represents coordinate fixed on the disc. On the stationary side there are the $Z$s exciting sources, from $S(1)$ to $S(Z s)$, which are equally spaced. While the $Z_r$ blades, from $R(1)$ to $R(Z_r)$, are attached to the disc at regular intervals. The disc is rotating at angular velocity $\Omega$. On the stationary side there are distributed forces of $2\pi/Z_r$ period caused by the $Z_r$ exciting sources, such as the wakes of the stationary vanes and the fluid injection from the nozzles. The maximum excit-
ing force is exerted on the disc when the blades pass the exciting sources during the disc rotation. The disc receives the exciting force only at the sections to where the blades are attached. Further, in this paper, the circular modes of the disc are not taken into account, only the diametral modes are considered. Therefore, only the vibrations in the same radial positions of the disc are discussed. It is assumed that all phenomena in this model are regarded as linear systems.

In order to obtain the vibration excited on the disc which has the 2r blades, it is only necessary to superpose the vibration taking place on the disc assumed to have a single blade, because the model is linear. When either the forward waves (traveling in the direction of disc rotation) or the backward waves (traveling in the opposite direction) are observed at an arbitrary point P in Fig. 1, the phases of waves occurring at respective blade positions are different. These differences in phases are caused by the difference in distance between each blade and the point P and by the time lags which arise when each blade passes the exciting sources. Assuming that \( \phi_0 \) is the phase difference at the point P between the wave occurring at the blade R(1) and the wave occurring at the blade R(2), the phase difference at the point P between the waves occurring at the blades R(2) and R(3) should also be \( \phi_0 \). That is, the waves occurring at the adjacent blades all have the same phase difference, \( \phi_0 \). Further, since the number of blades is 2r, the phase difference which has 2r times as large as \( \phi_0 \) must vanish. Therefore, 2r times \( \phi_0 \) will be 2\( \pi \) (j is an integer). For example, when 2r = 6 and respective waves observed at the point P are expressed by vectors, phase difference of respective vectors is 2\( \pi \)/6. Accordingly, the superposition of vectors will be as shown in Fig. 2(a) when \( j=1, 5, \ldots \), Fig. 2(b) when \( j=2,4,8,\ldots \), Fig. 2(c) when \( j=3,9,15,\ldots \), and Fig. 2(d) when \( j=6,12,18,\ldots \). In Figs. 2(a), 2(b) and 2(c), respective vectors at the point P cancel one another, so that the total amplitude of vector at the point P becomes zero. In Fig. 2(d), the vectors of respective waves are aligned in the same direction, so that the total amplitude of vector at the point P becomes six times as large as that of one wave. The waves sometimes cancel one another and sometimes overlap one another, depending upon a value of the phase difference. Under what condition these phenomena take place will be theoretically discussed.

Taking into consideration higher harmonics, when the force distribution on the stationary side caused by the exciting sources is not a sinusoidal form, the angular frequency of the exciting force, \( \omega \), being exerted on the blade sections of the disc is

\[
\omega = \omega_0 h \pi
\]

where \( h \) is a positive integer representing the higher harmonic order. The disc is forced to oscillate at this angular frequency, \( \omega \). When the vibration of the n-diameter mode, excited on the disc at angular frequency, \( \omega \), is \( \xi_n \), the disc vibration at \( \omega \), \( \xi_n \), can be expressed by

\[
\xi_n = \sum_{n=1}^{\infty} \xi_n
\]

Assuming that an observation is started when the blade R(1) has passed the exciting source S(i), as shown in Fig. 1, and that the disc has only one blade, R(1), the vibration of the n-diameter mode, \( \xi_{n,1} \), can be written on the rotating coordinate, with the amplitude \( \Lambda \), as

\[
\xi_{n,1} = A \cos n \phi \sin \omega t
\]

In a similar manner, the vibration occurring at the blade R(k), located in the kth position from R(1), \( \xi_{n,k} \), can be given by the following equation, considering the time length which it takes the blade R(k) to reach the position of the exciting source S(i) located in the ith position from s(1) as shown in Fig. 1.

\[
\xi_{n,k} = A \cos (\phi - \phi_k) \sin \omega (t - \frac{1}{\Omega} (\theta_i - \phi_k))
\]

where \( \phi_k \) represents the position of the blade R(k) on the rotating co-ordinate,

\[
\phi_k = \frac{2\pi}{2r} (k-1)
\]

\( \theta_i \) represents the position of the exciting source S(i) on the stationary co-ordinate,

\[
\theta_i = \frac{2\pi}{2r} (i-1)
\]

Substituting Eqs. (1) and (6) into Eq. (4) gives the following equation.

\[
\xi_{n,k} = A \cos (\phi - \phi_k) \sin \omega (t + \frac{\phi_k}{\Omega})
\]

Hence the vibration of the n-diameter mode on the rotating disc which has the 2r blades, \( \xi_n \), can be expressed by

\[
\xi_n = \sum_{k=1}^{2r} A \cos (\phi - \phi_k) \sin \omega (t + \frac{\phi_k}{\Omega})
\]

\[Fig. 2 \; \text{Superposition of Waves}\]
By substituting Eq. (5), Eq. (8) can be written as

\[ \xi_n = \frac{A}{2} \left( \frac{\sin(\omega t + \phi)}{2\pi} + \frac{2\pi}{2\pi}(h_2s + n)(k-1) \right) \]

\[ + \sin(\omega t + \phi) + \frac{2\pi}{2\pi}(h_2s - n)(k-1) \]  

(9)

Equation (9) expresses the superposition of waves with different phases, as shown in Fig. 2. By putting the summation of Eq. (9) into practice, the following equation can be obtained:

\[ \xi_n = \frac{A}{2} \left( \frac{\sin \frac{\pi}{2\pi}(h_2s + n)}{2\pi} \sin \frac{\pi}{2\pi}(h_2s + n) \right) \]

\[ + \frac{\pi}{2\pi}\left( \frac{n}{2\pi}(h_2s + n)(2\pi - 1) \right) + \frac{\pi}{2\pi}\left( \frac{n}{2\pi}(h_2s - n)(2\pi - 1) \right) \]  

(10)

The first term in the right-hand side of Eq. (10) indicates the forward wave excited on the rotating disc. The second term expresses the backward wave. In Eq. (10), \((h_2s/nm)\) in the amplitude terms for the forward and backward waves are integers, so that the numerators of amplitude terms are always \( \sin(\pi h_2s/nm) = 0 \). Therefore, when the denominators of amplitude terms are \( \sin(\pi h_2s/nm) = 0 \), \( \xi_n = 0 \) from Eq. (10).

Accordingly, no vibration of the n-diameter mode is excited on the rotating disc.

Vibration is excited on the disc, when the denominator of amplitude term becomes zero, namely

\[ h_2s \pm n = m\pi \]  

(11)

where \( m \) is an arbitrary integer. When Eq. (11) is satisfied, the amplitude term in Eq. (10) becomes an indeterminate form of \( \sin(m\pi r)/\sin(m\pi) \). So, assuming that \( m\pi r \) is \( (m\pi r^2) \), the limit of \( e^0 = 0 \) is obtained as

\[ \lim_{\pi \to 0} \frac{\sin(m\pi r^2)}{\sin(m\pi)} = (-1)^m(2\pi - 1) \]  

(12)

Therefore, when Eq. (11) is satisfied, Eq. (10) will be expressed as

\[ \xi_n = \frac{A}{2} \left( \frac{\sin(\omega t + \phi) + m(2\pi - 1)}{2\pi} \right) \]  

(13)

Observing Eq. (13) on the stationary coordinate, since \( \phi = 0 \)-\( \pi \),

\[ \xi_n = \frac{A}{2} \sin(\omega t + \phi) + n\phi \]  

(14)

cited. In this case, two kinds of angular frequencies, \((\omega - m\pi)\) and \((\omega + m\pi)\), will be observed on the stationary side. When the arbitrary integer \( m \), which satisfies Eq. (11), does not exist, no vibration of the n-diameter mode will be excited on the rotating disc. The above mentioned holds good, irrespective of whether or not the disc is resonating.

The rotating disc resonates under the conditions that \( m \) which satisfies Eq. (11) exists and that \( \omega = \omega_n \) when the disc natural angular frequency of the n-diameter mode is \( \omega_n \). Therefore, even if \( \omega = \omega_n \), when Eq. (11) is not satisfied, the disc will not resonate.

3. Experiments

To verify that the diametral modes are selectively excited on the rotating disc in accordance with this theory, experiments were carried out on several combinations of the number of blades and the number of exciting sources.

3.1 Experimental apparatus

The outline of the experimental apparatus is shown in Fig. 3. The disc is 300 mm in diameter, 3 mm in thickness and made of steel. The 2r small steel pieces are attached to the disc at equal intervals and their outer ends project 5 mm outside the disc periphery. This disc is directly coupled with a DC motor through a rubber coupling. Its rotation speed can be continuously varied from 0 to 2000 r.p.m. On the other hand, the Zs nozzles are installed as the exciting sources at equal intervals on the fixed side. Water under 0.4 kg/cm² (on gage) pressure is jetted out of these nozzles, striking only the projecting pieces attached to the disc. Vibrations are detected by using a strain gage affixed on the disc and a gap sensor fixed on the stationary side.

Experiments were conducted concerning six combinations of Zs and 2r, as shown in the Table 1. On two cases: when the water jet is stopped and when the water jet is
injected from the nozzles, the rotating speed of the disc $N$ is changed continuously from 0 to 2000 r.p.m. The data detected by the strain gage were recorded on an X-Y plotter, the rotating speed $N$ as the horizontal axis and the r.m.s. value of the overall stress amplitude as the vertical axis. Frequency analyses were made on vibrations detected by the gap sensor and the strain gage at the speeds at which extremely great stress appeared in the disc when the water jet was injected from the nozzles.

3.2 Experimental results

The natural frequencies of the disc obtained by the impulse tests are shown in Table 2. The natural frequencies of $Z_r = 7$ are slightly lower than those of $Z_r = 6$. The zero-diameter and the one-diameter mode were not observed.

According to the results of experiment Case No. 1, Fig. 4 shows stress occurring on the disc when water jet from the nozzles is stopped. This stress is caused by unbalance and centrifugal force. Fig. 5 shows stress which occurs when the water under pressure is being jetted out from the nozzles; great stress appeared at $N = 270$ r.p.m., 542 r.p.m. and 1049 r.p.m. respectively. The vertical axes in Figs. 4 and 5, and Figs. 9 through 13 show non-dimensional value of disc stress which is normalized by value at $N = 270$ r.p.m. in Fig. 5. Fig. 6(a) shows the frequency analysis result of disc stress at $N = 270$ r.p.m. in Fig. 5, showing that 180 Hz is predominant. From this, it is seen that the disc is resonating in the two-diameter mode with at the second higher harmonic ($h = 2$) of the exciting frequency. Meanwhile, Fig. 6(b) shows the result of frequency analysis for vibration detected by the gap sensor at

<table>
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<th>Case</th>
<th>No.1</th>
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<th>No.4</th>
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<td>20</td>
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<td>18</td>
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</tr>
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<td>6</td>
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Table 1: Combinations of $Z_s$ and $Z_r$

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<tr>
<th>Mode</th>
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<tr>
<td>$n = 2$</td>
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<td>$n = 3$</td>
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<td>$n = 5$</td>
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Table 2: Disc Natural Frequencies (Hz)

Fig. 4 Disc Stress without Jet

Fig. 5 Disc Stress with Jet

Fig. 6(a) Disc Stress ($N = 270$)

Fig. 6(b) Vibration Detected by Gap Sensor ($N = 270$)

Fig. 7(a) Disc Stress ($N = 542$)

Fig. 7(b) Vibration Detected by Gap Sensor ($N = 542$)

Fig. 8(a) Disc Stress ($N = 1049$)

Fig. 8(b) Vibration Detected by Gap Sensor ($N = 1049$)
$N = 270$ r.p.m. in Fig. 5. As shown, these are two predominant frequencies: 172 Hz and 190 Hz. These are equivalent to $(2m+1)$ in Eq. (14). Since the amplitude of 190 Hz is larger than that of 172 Hz, it is understood that the forward wave is dominant. Fig. 7(a) and 7(b) show the frequency analysis results of disc stress and vibration observed at the stationary side at $N = 542$ r.p.m. in Fig. 5. From these, also, it is seen that the disc is resonating in the two-diameter mode at the fundamental exciting frequency ($h = 1$) and that the backward wave is dominant. Figs. 8(a) and 8(b) show the frequency analysis results of disc stress and vibration observed at the stationary side at $N = 1049$ r.p.m. in Fig. 5. It is seen from these results that the disc is resonating in the four-diameter mode at the second higher harmonic ($h = 2$) of the exciting frequency and that the backward wave is dominant. The frequency analyzer used in the experiments has a resolution of 1 Hz in Figs. 6 and 7 and 2 Hz in Fig. 8.

Figs. 9 through 13 show stress occurring in the disc when the water jet is injected from the nozzles, concerning Cases No. 2 through No. 6. The frequency analysis results of disc stress and vibration observed at the stationary side in Cases No. 2 through No. 6 are omitted here. The order of higher harmonic of the exciting frequency, $h$, the number of diametral nodal lines, $m$, the arbitrary integer satisfying Eq. (11), and a distinction between the forward wave and the backward wave, which are all seen from these results, are shown in Figs. 9 through 13. In these figures, symbol $\bigcirc$ signifies a forward wave while symbol $\blacklozenge$ signifies a backward wave.

### 3.3 Discussion

In exciting condition, Eq. (11), $h$ represents the order of higher harmonic of the exciting force. In general, it is considered that the greater the higher harmonic order, $h$, the weaker the exciting force. However, the state of its attenuation naturally differs with the condition of the force distribution caused by the exciting sources. In these experiments, since no resonance of the disc appears at $h = 3$ or higher, it is sufficient to consider that $h = 1$ or $h = 2$. Particularly, since greater disc stress appears at the resonance at $h = 1$ than at the resonance at $h = 2$, it can be seen that the fundamental exciting frequency ($h = 1$) is important. Tables 3(a) and 3(b) show the combinations satisfying Eq. (11) in either $h = 1$ or $h = 2$ cases. In these tables, symbol $\bigcirc$ indicates the forward wave and symbol $\blacklozenge$ indicates the backward wave. Symbol (*) represents the combinations which satisfy Eq. (11) but will not resonate at speeds lower than $N = 2000$ r.p.m.. Comparison between Table 3, obtained by Eq. (11), and Fig. 5 and Figs. 9 through 13, showing experimental results, clarifies that the experimental results completely coincide with theoretical results in Table 3. In Figs. 6(b), 7(b) and 8(b), however, forward or backward waves with smaller amplitude, which does not satisfy the exciting condition, are observed at the stationary side. These waves seem to have been caused by other factors, such as nonuniformity in the nozzles. The experiments also verified that the disc will not resonate if the exciting condition, Eq. (11), are not satis-
Table 3  Combinations Satisfying Eq. (11)

(a) Zr = 6

<table>
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<tr>
<th>n</th>
<th>h</th>
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</tr>
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(b) Zr = 7

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4. Conclusions

Concerning the bladed disc vibrations, the theoretical studies have been carried out to investigate what kind of diametral modes are excited on the rotating disc by the interaction between the number of blades attached to the disc and the number of exciting sounds on the stationary side. The results of the theoretical studies have also been verified by the experimental investigations. The results of the present studies may be summarized as follows:

(1) Vibration of the n-diameter mode is excited on the disc, only when an arbitrary integer m which satisfies the exciting condition, Eq. (11), exists. When (+) in Eq. (11) is satisfied, the exciting frequency coincides with the natural frequency of the disc.

(2) When the disc resonates only when the exciting frequency and the natural frequency of the disc coincide with each other and simultaneously when Eq. (11) is satisfied. The disc will never resonate when the exciting condition, Eq. (11), is not satisfied, even if the exciting frequency coincides with the natural frequency of the disc.

References