Vibration Analysis for Periodically Symmetric Structures with Solid Centers*

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"Wave Propagation Technique" and "Cyclic Symmetry Method" are very useful for analyzing vibration of periodically symmetric structures consisting of columns with hollow centers. However, these two methods are not applicable to periodically symmetric structures consisting of columns with solid centers, because the solid center connects all the substructures to form a total system.

As a result of improving these two methods, a new analytical method is proposed in this paper. Using the proposed method, the vibration characteristics of a total system can be predicted simply by analyzing one fundamental substructure which is separated from the total system regardless of whether the center of a periodically symmetric structure is solid or hollow. The application of this method to specific structures, and the accuracy and efficiency of the method are discussed in detail.

Key Words: Vibration of Continuous System, Vibration Analysis, Finite Element Method, Periodically Symmetric Structure, Natural Frequency, Natural Mode, Cyclic Symmetry Method.

1. Introduction

For the vibration analyses of periodically symmetric structures with hollow centers, two useful methods namely (i) "Wave Propagation Technique" and (ii) "Cyclic Symmetry Method" were developed. Such a structure consists of a finite number of identical substructures coupled together in a specific way to form a closed ring-type structure, therefore the dynamic behavior of a whole system can be analyzed simply by analyzing one fundamental substructure which is separated from the total system. These methods may be summarized as follows; (i) Natural frequencies and natural modes of such a structure are predicted for each nodal line appearing on the system, assuming that displacements and forces of both sides of a single substructure are approximately variable sinusoidally and (ii) a large eigenvalue problem can be mathematically reduced to a number of small eigenvalue problems, considering that the stiffness matrix, the mass matrix, etc. of such a structure are real, symmetric and block-cyclic. However, if the center of a structure is solid, these two methods cannot be applied because the solid center interconnects all the substructures.

Recently the coupled vibration between a flexible disk and a rotating flexible shaft was studied. If a disk and a shaft are divided into finite elements for application of the Finite Element Method (F.E.M.), there may be some nodes in the contacting region of the disk when the disk is hollow. However, there is only one node in the contacting region of the shaft and therefore some specific assumptions are necessary to combine these nodes. On the other hand, when the disk is solid, it is possible to combine two nodes without any assumptions because there is only one node in the contacting region of the disk and the shaft.

This paper is mainly devoted to solution of this problem. Vibration analysis for periodically symmetric structures with solid centers is explained. Applying this new method to such structures, the dynamic behavior of a total system can be predicted simply by analyzing one fundamental substructure which is separated from the total system. The accuracy, efficiency and application of this method are discussed by calculating the natural frequencies and the natural modes of some structures.

2. Method of Analysis

2.1 Characteristic matrices of periodically symmetric structures with solid centers

A periodically symmetric structure consists of identical substructures linked together in a specific way as shown in Fig. 1(a). Representing the characteristics of one fundamental substructure by the F.E.M. or other methods, the relation between displacements and forces is written as follows:

\[
\begin{bmatrix}
H_{11} & H_{12} & H_{13} & H_{14} \\
H_{21} & H_{22} & H_{23} & H_{24} \\
H_{31} & H_{32} & H_{33} & H_{34} \\
H_{41} & H_{42} & H_{43} & H_{44}
\end{bmatrix}
\begin{bmatrix}
u^\prime \\
u^\prime \\
u^\prime \\
u^\prime \\
\end{bmatrix} =
\begin{bmatrix}
f^\prime \\
f^\prime \\
f^\prime \\
f^\prime \\
\end{bmatrix}
\]

...(1)

where the symbols \(t, g, v^\prime\) and \(a\) denote the regions shown in Fig. 1(b), respectively. The superscript \(\prime\) means that the displacement and the forces are expressed in a fixed
coordinate system \(O-x'y'z'\), located at the 
\(k\)-th substructure. \(\mathbf{B}^K\) is the stiffness 
matrix \((6\times 6)\) of the region (point) \(s\). 
Let the displacements and the forces of the regions \(t, g, t'\) be represented in the 
cylindrical coordinate system \((\rho, \phi)\) denoted by 
the superscript \(p\) and those of the region 
\(s\) be represented in the orthogonal coordinate 
system \(O-xyz\) denoted by the superscript \(c\), i.e.,

\[
\begin{bmatrix}
  u_t^p \\
  u_q^p \\
  u_t'^p \\
  f_t^p \\
  f_q^p \\
  f_t'^p
\end{bmatrix} =
\begin{bmatrix}
  R_t & 0 & 0 \\
  R_q & 0 & 0 \\
  R_t' & 0 & 0 \\
  0 & R_\rho & 0 \\
  0 & 0 & R_\rho \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  u_t^c \\
  u_q^c \\
  u_t'^c \\
  f_t^c \\
  f_q^c \\
  f_t'^c
\end{bmatrix}
\]

\[\cdots (2a)\]

\[
\begin{bmatrix}
  f_t^p \\
  f_q^p \\
  f_t'^p
\end{bmatrix} =
\begin{bmatrix}
  R_t & 0 & 0 \\
  R_q & 0 & 0 \\
  R_t' & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  f_t^c \\
  f_q^c \\
  f_t'^c
\end{bmatrix}
\]

\[\cdots (2b)\]

in which \(R_\rho, R_\phi, R_\theta\) are the orthogonal matrices. Particularly \(R_\rho\) is expressed as follows;

\[
R_\rho =
\begin{bmatrix}
  \cos \alpha_\rho & \sin \alpha_\rho & 0 \\
  -\sin \alpha_\rho & \cos \alpha_\rho & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

\[\cdots (3)\]

Substituting Eqs. (2-a) and (2-b) into Eq. (1) results in the following equation;

\[
\begin{bmatrix}
  K_{tt} & K_{tq} & K_{tv} & K_{tR} & R^T R & \vdots \\
  K_{qt} & K_{qq} & K_{qv} & K_{qR} & R^T R & \vdots \\
  K_{vt} & K_{vt} & K_{vq} & K_{vR} & R^T R & \vdots \\
  R^T R & R^T R & R^T R & R^T R & R^T R & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
  u_t^p \\
  u_q^p \\
  u_t'^p \\
  f_t^p \\
  f_q^p \\
  f_t'^p
\end{bmatrix} =
\begin{bmatrix}
  f_t^c \\
  f_q^c \\
  f_t'^c
\end{bmatrix}
\]

\[\cdots (4)\]

Since \(K_{tt}, K_{tq}, \ldots, K_{tt}, K_{qR}, \ldots, K_{tt}, K_{tt}, \ldots, K_{tt}\) and \(R^T R\) are common to all other substructures, the relation between the displacements and the forces of the total system is of the form

\[
\begin{bmatrix}
  K_0 & 0 & \cdots & 0 & K_{n-1} & K_n \\
  0 & K_0 & \cdots & 0 & K_{n-1} & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  K_{n-1} & 0 & \cdots & 0 & K_0 & K_n R^{n-1} \\
  K_n R^{n-1} & R^T K_n & R^T R & \cdots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots
\end{bmatrix}
\begin{bmatrix}
  u_t^c \\
  u_q^c \\
  u_t'^c \\
  f_t^c \\
  f_q^c \\
  f_t'^c
\end{bmatrix} =
\begin{bmatrix}
  f_t^c \\
  f_q^c \\
  f_t'^c
\end{bmatrix}
\]

\[\cdots (5a)\]

where

\[
K_0 =
\begin{bmatrix}
  K_{tt} + K_{tv} & K_{tv} \\
  K_{vt} & K_{tt}
\end{bmatrix}
\]

\[
K_{n-1} =
\begin{bmatrix}
  K_{vt} & K_{tv} \\
  0 & 0
\end{bmatrix}
\]

\[
K_n =
\begin{bmatrix}
  K_{tt} + K_{tv} R^T \\
  K_{vt}
\end{bmatrix}
\]

\[\cdots (5b)\]

\[
K_n = R^T H_n R + R^T H_{n-1} R + \cdots + R^T H_1 R + \sum_{k=0}^{n-1} R^T H_k R^T
\]

\[\cdots (5c)\]

Deriving the equations above, the relationships of \(R_n, R_0, R^n, R^T\) is a unit matrix \((6 \times 6)\) and

\[
R_n R_0 = R
\]

\[\cdots (5d)\]

is taken. \(K_n\) is a real symmetric matrix \((6 \times 6)\) and \(K_n\) is a matrix of which the column number is 6. In order to simplify the total stiffness matrix, Eq. (5-a) is rearranged as follows;
\[
\begin{bmatrix}
A & E \\
E^T & K_{ss}
\end{bmatrix}
\]
\[
E^T = \begin{bmatrix}
K_1 & R^T K_1 & R^T K_2 & \ldots & R^T K_{n-1}
\end{bmatrix}
\]

in which \( A \) is real, symmetric and block-cyclic and \( E \) is a matrix of which the column number is 6.

2.2 Eigenvalue problems for periodically symmetric structures with solid centers

The eigenvalue problem for the real and symmetric matrix given in Eq. (6-a) is defined as follows:

\[
\begin{bmatrix}
A & E \\
E^T & K_{ss}
\end{bmatrix}
\begin{bmatrix}
u \\
u_s
\end{bmatrix} = \lambda \begin{bmatrix}
u \\
u_s
\end{bmatrix}
\]

in which \( \lambda \) is an eigenvalue (a real number) and \( \begin{bmatrix} u^T & u_s^T \end{bmatrix} \) is an eigenvector \( u_s \) is a six by one column vector.

Now we consider a new column vector \( \begin{bmatrix} v^T & v_s^T \end{bmatrix} \) defined as follows:

\[
\begin{bmatrix}
u \\
u_s
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{n}} B & 0 \\
0 & X
\end{bmatrix}
\begin{bmatrix}
v \\
v_s
\end{bmatrix}
\]

where

\[
B = \begin{bmatrix}
W_0 & W_i & W_i & \ldots & W_{2^{-1}} \\
W_i & W_0 & W_i & \ldots & W_{2^{-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
W_{2^{-1}} & W_{2^{-1}} & W_0 & \ldots & W_{2^{-1}}
\end{bmatrix}
\]

\[
W_0^s = \epsilon^s I \quad (I: \text{unit matrix})
\]

\[
\epsilon_s = e^{i \alpha_s} \quad (i = \sqrt{-1})
\]

\[
a_s = \frac{2\pi}{n}
\]

and \( X \) is the eigenvectors matrix of \( R(=R_s) \). For example it is written as follows:

\[
X = \begin{bmatrix}
\beta & \beta & 0 \\
\beta e^{-i\beta} & -i \beta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Defining \( \Gamma \) as an eigenvalue matrix (diagonal) of \( R \), i.e.,

\[
\Gamma = \begin{bmatrix}
e^{i\alpha} & 0 & e^{-i\alpha} & 0 \\
0 & 1 & 0 & e^{i\alpha} \\
e^{-i\alpha} & 0 & e^{i\alpha} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\beta = 1/\sqrt{2}
\]

\[
\alpha = \frac{2\pi}{n}
\]

\( R \) is expressed as follows using \( X \) and \( \Gamma \)

\[
R = X \Gamma X^T
\]

Considering the fact that \( 1/\sqrt{n} B \) and \( X \) are unitary matrices, i.e.,

\[
B^T B = BB^T = nI
\]

\[
X^T X = XX^T = I
\]

and substituting Eq. (8-a) into Eq. (7), Eq. (7) is rearranged as

\[
\begin{bmatrix}
\tilde{A} & \tilde{E} \\
\tilde{E}^T & K_{ss}
\end{bmatrix}
\begin{bmatrix}
v \\
v_s
\end{bmatrix} = \lambda \begin{bmatrix}
v \\
v_s
\end{bmatrix}
\]
in which
\[ \hat{A} = \frac{1}{n} \hat{B}^T \hat{A} \hat{B} = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{n-1} \\ Q_1 & \cdots & Q_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-1} & \cdots & Q_1 & Q_0 \end{bmatrix} \] .............................................. (11•b)

\[ Q_j = K_j e^{i \theta_j} + K_j^* e^{-i \theta_j} \] .............................................. (11•c)

\[ Q_j^* = Q_j (j = 0, 1, \ldots, n-1) \] .............................................. (11•d)

\[ \hat{K}_{ss} = \hat{X}^T \hat{K}_{ss} \hat{X} = \hat{X}^T \left( \sum_{k=0}^{n} \hat{r}_k \hat{R}_k \hat{R}_k^* \right) \hat{X} = \hat{X}^T \left( \sum_{k=0}^{n} \hat{X}^T \hat{r}_k \hat{R}_k \hat{R}_k^* \hat{X} \right) \hat{X} \] .............................................. (11•e)

\[ \hat{E}_j^* = \frac{1}{\sqrt{n}} \hat{X}^T \hat{E} \hat{B} = \frac{1}{\sqrt{n}} \left[ \hat{f}_0 \hat{X} \hat{E} \hat{K}_0 \hat{X} \hat{E} \hat{K}_0^* \cdots \hat{f}_n \hat{X} \hat{E} \hat{K}_n \hat{X} \hat{E} \hat{K}_n^* \right] \] .............................................. (11•f)

Derivation of Eqs. (11•b) and (11•c) is shown in reference (3). Detail calculation of \( \hat{K}_{ss} \) and \( \hat{E} \) in Eqs. (11•e) and (11•f) gives the following equations

\[ \hat{K}_{ss} = \begin{bmatrix} \beta_0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_2 & 0 & \beta_6 \\ 0 & 0 & \beta_4 & 0 & \beta_5 & 0 \\ \beta_3 & 0 & 0 & \beta_2 & 0 & \beta_6 \\ 0 & \beta_5 & 0 & \beta_2 & 0 & \beta_6 \\ 0 & 0 & \beta_4 & 0 & \beta_5 & 0 \end{bmatrix} \] .............................................. (12•a)

\[ H_{ss} = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} & \eta_{15} & \eta_{16} \\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} & \eta_{25} & \eta_{26} \\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} & \eta_{35} & \eta_{36} \\ \text{sym.} & \eta_{44} & \eta_{45} & \eta_{46} & \eta_{46} & \eta_{46} \\ \eta_{55} & \eta_{56} & \eta_{56} & \eta_{56} & \eta_{56} & \eta_{56} \\ \eta_{66} & \eta_{66} & \eta_{66} & \eta_{66} & \eta_{66} & \eta_{66} \end{bmatrix} \] .............................................. (12•b)

\[ \beta_0 = n (\eta_{11} + \eta_{22}) / 2 \] \[ \beta_1 = n (\eta_{44} + \eta_{55}) / 2 \] \[ \beta_2 = n (\eta_{44} + \eta_{55}) / 2 \] \[ \beta_3 = n (\eta_{11} + \eta_{22}) / 2 \] \[ \beta_4 = n \eta_{33} \] \[ \beta_5 = n \eta_{33} \] \[ \beta_6 = n \eta_{33} \] .............................................. (12•c)

\[ E_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \] \[ E_1 = \begin{bmatrix} e_1 & 0 & 0 & 0 \end{bmatrix} \] \[ E_{n-1} = \begin{bmatrix} 0 & e_0 & 0 & 0 \end{bmatrix} \] .............................................. (12•d)

\[ K_0 = \begin{bmatrix} k_{s1} & k_{s2} & k_{s3} & k_{s4} & k_{s5} & k_{s6} \end{bmatrix} \] .............................................. (12•f)

\[ e_1 = \sqrt{n/2} (k_{s1} + ik_{s2}) \] \[ e_2 = \sqrt{n/2} (k_{s1} - ik_{s2}) = \bar{e}_1 \] \[ e_3 = \sqrt{n/2} (k_{s3} + ik_{s4}) \] \[ e_4 = \sqrt{n/2} (k_{s3} - ik_{s4}) = \bar{e}_3 \] \[ e_5 = \sqrt{n/2} (k_{s5} + ik_{s6}) \] \[ e_6 = \sqrt{n/2} (k_{s5} - ik_{s6}) = \bar{e}_5 \] .............................................. (12•g)

in which \( \eta_{ji}, \eta_{ji}^*, \eta_{ji}^{**}, \) etc. are the components of \( H_{ss} \) (a six by six matrix), \( \beta_j, \) \( \beta_j^* \) are defined from \( H_{ss} \), \( \bar{e}_j \) are real numbers. \( k_{s1}, k_{s2}, k_{s5}, k_{s6}, k_{s5}, k_{s6} \) are the components of \( B_{ss} \). Details of \( k_{s1}, k_{s2}, k_{s3}, k_{s4}, k_{s5}, k_{s6} \) are the column vectors used in the calculation. Although the detail derivation of Eqs. (12•a) to (12•g), and (11•f) to Eqs. (11•e) and (11•f) is also shown, the following relation is generally taken

\[ \sum_{j=0}^{n} e_j^{2} = \frac{n}{2} \sum_{k} \eta_{kk} = \sum_{j=0}^{n} e_j^{2} \] .............................................. (12•h)

Consequently, it is possible to break up the large eigenvalue problem defined in Eq. (11•a) into a number of small eigenvalue problems which are shown as follows;
Equations (13-f) ~ (13-i) explain the properties of the eigenvectors \( [u_{1}, u_{2}, \ldots, u_{n}] \) obtained for each index \( j \). First, the eigenvector \( u_{k} \) of each substructure except the center \( s \) has the following characteristics. The eigenvectors obtained at \( j=0 \) are identical for every substructure (see Eq. (13-f)). For \( j=1 \) to \( n_{r} \), they are variable sinusoidally (the period is \( n/2 \)). See Eqs. (13-g) and (13-h)). For \( j=n/2 \) (if \( n \) is even), every other substructure is identical (see Eq. (13-i)). Second, the eigenvector \( u_{0} \) of the center \( s \) has the following characteristics. For \( j=0 \), the displacement and the angular displacement of \( z \) direction \( (u_{x}^{z} \) and \( u_{y}^{z} ) \) take certain values, and those of \( x \) and \( y \) directions \( (u_{x}^{y}, u_{y}^{x}, u_{x}^{y} \) and \( u_{y}^{x} ) \) are always zero, i.e., the natural modes of the center \( s \) do not have the components of \( x \) and \( y \) directions when \( j=0 \) (see Eq. (13-f)). For \( j=1 \), the displacement and the angular displacement of \( z \) direction are always zero and those of \( x \) and \( y \) directions take certain values. For \( j=2 \), the displacements and the angular displacements of all directions are always zero (see Eqs. (13-h) and (13-i)), i.e., the center is always at rest or fixed.

3. Discussion of Results

Applying this method developed in this paper to circular plates, square plates and pentagonal plates shown in Fig. 2, the calculated natural pairs are compared with the exact solutions. These plates are divided into triangular elements (one element has three nodes and one node has six degrees of freedom). For the square plate and the pentagonal plate, 1/4 and 1/5 of a whole plate respectively are considered one fundamental substructure. From a mathematical point of view, the circular plate may be divided into any number of substructures, but, 1/8 of a whole plate is considered a fundamental substructure in this paper because of the shape of the triangular elements.

The natural pairs of the circular plate are shown in Table 1 for each index \( j(0/n/2) \). According to this table, the calculated values are in agreement with the true values and the number of nodal diameters \( n_{r} \), and the index \( j \) are in the following relation:

\[ n_{r} = \lfloor j + \frac{1}{2} \rfloor \quad (I: \text{integer}) \]  

The natural pairs of square plates are shown in Fig. 3 for each index \( j(0/n/2) \). According to these figures, the natural modes of each index \( j \) have the following characteristics. When \( j=0 \), the natural modes of each substructure are identical (see Eq.
Fig. 2(a) The circular plate of which periphery is fixed [thickness; 1 mm, material; steel (elastic modulus; 2.058 x 10^11 N/m², density; 7.86 x 10^3 kg/m³, Poisson's ratio; 0.3)]

Fig. 2(b) The square plate of which periphery is fixed [thickness; 1 mm, material; steel]

Fig. 2(c) The pentagonal plate of which periphery is fixed [thickness; 1 mm, material; steel]

Table 1. The natural frequencies (Hz) and the natural modes of the circular plate

<table>
<thead>
<tr>
<th>j=0</th>
<th>j=1</th>
<th>j=2</th>
<th>j=3</th>
<th>j=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-0</td>
<td>1-0</td>
<td>2-0</td>
<td>3-0</td>
<td>4-0</td>
</tr>
<tr>
<td>983 [980]</td>
<td>1503* [1499]</td>
<td>2077* [2085]</td>
<td>2224* [2236]</td>
<td>3536* [3454]</td>
</tr>
<tr>
<td>0-1</td>
<td>1-1</td>
<td>2-1</td>
<td>5-0</td>
<td>4-1</td>
</tr>
<tr>
<td>2195 [2196]</td>
<td>3005* [2959]</td>
<td>2750* [2814]</td>
<td>2719* [2736]</td>
<td>5694* [5656]</td>
</tr>
<tr>
<td>0-2</td>
<td>1-2</td>
<td>6-0</td>
<td>3-1</td>
<td>4-2</td>
</tr>
</tbody>
</table>

Fig. 3 The natural frequencies (Hz) and the natural modes of the circular plate
(The values bracketed are the true values of the natural frequencies. M10, etc. are the names of the natural modes.)
modes marked by the symbol ' at j=0,2 (e.g., M13', M15', etc.), because of the relation shown in Fig. 4 (e.g., M13', M13"=M13|M51). Comparing these natural frequencies, they are found nearly equal (for example the natural frequency of M13' is 2920 Hz and that of M51' is 2919 Hz). Therefore, these frequencies must be degenerated if the square plate is analyzed as one system. Further the natural modes shown in Fig. 4 (M02,M02', M51,M15,M04, M40) cannot be obtained by the method developed in this paper, i.e., these modes cannot belong to the natural modes obtained by this method at each index j.

The natural pairs of pentagonal plates are shown in Fig. 5. Since the number of substructures n=5 is odd, the natural frequencies obtained at j=1, 2 are inevitably degenerated and the two orthogonal natural modes are described. According to these figures, the natural modes of each substructure against each index j are variable sinusoidally (see Eq. (15-f))-(13-i)) and the number of nodal lines drawn through the center is zero or five at j=0, one at j=1, two at j=2.

4. Conclusions

(1) A new general method of vibration analysis for periodically symmetric structures with solid centers is proposed. Applying this method to such structures, the dynamic behavior of them can be predicted simply by analyzing one fundamental substructure in the same manner as for hollow center structures.

(2) This method can reduce a large eigenvalue problem a number of small eigenvalue problems of each index j without making approximations. Therefore, this method seems to be useful in view of the capacity and the operating time of computers and the accuracy of the solutions.

(3) Natural modes of a whole system except its center are formed, because natural modes of one fundamental substructure propagate sinusoidally (the period is n/|j|). Natural modes of its center and indexes j have the following relations. When j=0, displacements and angular displacements of x and y directions are always zero. When j=1, those of z direction are always zero. When j=2, those of all directions are always zero, i.e., the center is fixed.

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References