A Finite Element Approach to Dynamic Characteristics of Helical Spring* (Free Vibration)

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The free vibration of a helical spring is analyzed by a finite element method. The mass and stiffness matrices including the effects of the pitch angle and the number of active coils are derived by using the exact solution of the equations of the helical spring in static equilibrium, and the natural frequencies and associated modes are estimated numerically. The effects of the pitch angle and the number of active coils on the natural frequencies are studied closely, and it is pointed out that the effects of the pitch angle, even if minor, should be taken into consideration. The obtained results are corroborated with the experimental results.

Key Words: Helical Coil Spring, Pitch Angle, Number of Active Coils, Finite Element Approach, Free Vibration, Natural Frequencies

1. Introduction

The helical spring has provided many interesting dynamic problems, such as evaluation of its natural frequencies and associated modes, estimation of stress on surging, finding of conditions for its lateral instability, that is, for its parametric instability, etc., and so considerable research, both theoretical and experimental, has been done on these problems^{(1)~(6)}. However, since few papers have taken the effects of the pitch angle and the wire elongation into account, even the natural frequencies and associated modes, which are the basic data for dynamic analysis of the helical spring, have differed from actual ones with an increase in order of natural frequencies. An attempt⁽¹⁾ to take the effects of the pitch angle and the wire elongation into account has failed to estimate satisfactorily the natural frequencies and associated modes except the lowest translational natural frequency, because of cumbersome calculation involved.

The natural vibration modes of the helical spring can be roughly classified into three groups according to the directions of main motion of all its parts, namely the translational mode along the axis of spring, the torsional mode about the axis of spring and the transverse (bending) mode perpendicular

to the axis of spring. In some industrial applications it may be necessary to estimate the lowest natural mode in each group, but this mode is not always one associated with the lowest natural frequency in increasing order. It serves as an example to indicate that in valve springs used in an internal combustion engine the third natural mode is the lowest transverse mode. Within authors' knowledge, however, there is no research that gives right information about the transverse natural frequencies and associated modes. Taking account of its wide and severe use as machine elements, it would be worth-while to obtain exact natural frequencies and associated modes of the helical spring up to higher order.

The purpose of the present report is to estimate the natural frequencies and associated modes as exactly as possible up to higher order. We, therefore, begin with deriving the governing equations of motion including the effects of the pitch angle and the wire elongation, and analyze the undamped free vibration using the finite element approach based on analytical solution of the equations of the helical spring in static equilibrium. In order to show validity of such a finite element approach, results are corroborated with experiments, and also compared with those obtained by other authors. Moreover, we clarify the effects of the pitch angle and the number of active coils on natural frequencies, and point out that if these quantities are greater than some values, the lowest transverse natural frequency, which is important in analysis of the dynamic instability, is lower than the lowest translational natural frequency, which is important in analysis of the surging phenomenon.

* Received 30th March, 1981.

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2. Nomenclature

The following symbols are used in the present report :

- x, y, z : co-ordinate axes along the tangent, the principal normal, and the binormal at any point on wire axis
 u, v, w : x, y , and z components of displacement at any point on the wire axis
 β, δ, ψ : x, y , and z components of rotation angle at any point on the wire axis
 X, Y, Z : x, y , and z components of section forces at any point on the wire axis
 T, H, M : x, y , and z components of section moment at any point on the wire axis
 κ, τ : curvature and torsion of the wire axis
 θ : angular co-ordinate about the axis of spring
 s : length along the wire axis
 Φ : angle of spring element about the axis of spring
 I : cross-sectional area moment of inertia about wire diameter
 A : area of wire cross-section
 r : mean coil diameter
 α : pitch angle
 N : number of active coils
 E : modulus of longitudinal elasticity
 G : modulus of transverse elasticity
 ρ : mass density of wire material
 t : time
 ω : circular frequency

3. Theoretical Analysis

3.1 Basic equations

Love⁽⁷⁾, Takata and others⁽⁸⁾⁽⁹⁾ referred in detail to methods of derivation of the basic equations, available for analysis of the helical spring in static equilibrium. Extending their methods to deriving the governing equations of undamped free vibration of the helical spring, we

(a) Co-ordinate axes (b) Spring element
Fig. 1

can obtain the following equations

$$\left. \begin{aligned} \rho A \frac{\partial^2 u}{\partial t^2} &= \frac{\partial X}{\partial s} - \kappa Y \\ \rho A \frac{\partial^2 v}{\partial t^2} &= \frac{\partial Y}{\partial s} + \kappa X - \tau Z \\ \rho A \frac{\partial^2 w}{\partial t^2} &= \frac{\partial Z}{\partial s} + \tau Y \\ 2\rho I \frac{\partial^2 \beta}{\partial t^2} &= \frac{\partial T}{\partial s} - \kappa H \\ \rho I \frac{\partial^2 \delta}{\partial t^2} &= \frac{\partial H}{\partial s} + \kappa T - \tau M - Z \\ \rho I \frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial M}{\partial s} + \tau H + Y \end{aligned} \right\} \dots\dots\dots (1)$$

The relations among section force x , section moments T, H, M , displacements u, v, w and rotation angle β are written in the form

$$\left. \begin{aligned} X &= EA \left(\frac{\partial u}{\partial s} - \kappa v \right) \\ T &= 2GI \left(\frac{\partial \beta}{\partial s} + \kappa \frac{\partial w}{\partial s} + \tau v \right) \\ H &= -EI \left(\frac{\partial^2 w}{\partial s^2} + 2\tau \frac{\partial v}{\partial s} - \kappa \beta - \tau^2 w + \kappa \tau u \right) \\ M &= EI \left(\frac{\partial^2 v}{\partial s^2} + \kappa \frac{\partial u}{\partial s} - 2\tau \frac{\partial w}{\partial s} - \tau^2 v \right) \end{aligned} \right\} \dots\dots\dots (2)$$

where

$$\left. \begin{aligned} \kappa &= - \left(\frac{\partial w}{\partial s} + \tau v \right) \\ \phi &= \frac{\partial v}{\partial s} + \kappa u - \tau w \end{aligned} \right\} \dots\dots\dots (3)$$

$$\kappa = \frac{\cos^2 \alpha}{r}, \quad \tau = \frac{\sin 2\alpha}{2r}, \quad \frac{\partial s}{\partial \theta} = \frac{r}{\cos \alpha} \dots\dots\dots (4)$$

If the pitch angle is set equal to zero in Eqs.(1) and (2), the above equations are separated into two sets of equations : one is with respect to u - v motion and the other with respect to w - β motion, and it is possible to obtain analytically the exact natural frequencies and associated modes for each set of equations⁽¹⁾. But as u, v, w and β are coupled with each other in the case of $\alpha \neq 0^\circ$, it is very difficult to obtain the exact natural frequencies and associated modes by analytical method. We, therefore, will estimate those numerically by using the finite element method.

3.2 Analytical solution in static equilibrium

In order to obtain the natural frequencies and associated modes of the helical spring by the finite element method, we need to adopt adequate expressions describing variation of displacement between end points of a spring element as shown in Fig. 1 (b). However, it seems that an adequate expression apparently has not been available for such a space-curve-type structural element with curvature and torsion, as far as the authors know. We will, therefore, adopt analytically obtained displacements u, v, w and rotation

angles β , δ , ψ to describe variation of displacements and rotation angles of the spring element in motion between end points, noticing that Eqs. (1) and (2) may be solved analytically in static equilibrium.

We will describe a procedure for obtaining the solutions of the equations of the helical spring in static equilibrium in the following. We obtain the following equations by ignoring the inertial terms in Eq. (1) (setting the left side in Eq. (1) to zero) and eliminating variables X and Z from the first, the second and third relations in Eq. (1) as well as variables T , M , and Z from the fourth, the fifth, and sixth relations

$$\frac{d^2 Y}{d\theta^2} + Y = 0 \quad \dots\dots\dots (5)$$

$$\frac{d^2 H}{d\theta^2} + H = -(2r \tan \alpha) Y \quad \dots\dots\dots (6)$$

Moreover, we can derive the following relation among displacement v , section force Y , and section moment H from Eq. (2)

$$\frac{d^5 v}{d\theta^5} + 2 \frac{d^3 v}{d\theta^3} + \frac{dv}{d\theta} = \frac{H}{C_3} + \frac{Y}{C_4} r \tan \alpha \quad \dots\dots\dots (7)$$

where

$$\left. \begin{aligned} C_1 &= (2GI \cos^3 \alpha) / r^2, \\ C_2 &= (EI \cos^2 \alpha) / r^2 \\ 1/C_3 &= 2 \sin \alpha \{ \cos^3 \alpha / C_1 + (1 + \sin^2 \alpha) / C_2 \}, \\ 1/C_4 &= 8 \sin \alpha / C_2 \end{aligned} \right\} \quad \dots\dots\dots (8)$$

Note that Eq. (4) is used in derivation of Eqs. (5), (6), and (7). Y , H , and v are obtained successively from Eqs. (5), (6), and (7). Using nine undetermined constants $\beta_1, \beta_2, \dots, \beta_9$ and matrix form, we may write the obtained results as

$$\begin{pmatrix} v \\ Y \\ H \end{pmatrix} = \begin{bmatrix} 1 & \cos \theta & \sin \theta & \theta \cos \theta & \theta \sin \theta & \theta^2 \cos \theta & \theta^2 \sin \theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8C_3 \sin \theta & -8C_3 \cos \theta \\ & & & \theta^3 \cos \theta & \theta^3 \sin \theta & & \\ & & & -C_5 \cos \theta & -C_5 \sin \theta & & \\ & & & 24C_3(\theta \sin \theta - C_5 \cos \theta) & -24C_3(\theta \cos \theta + C_5 \sin \theta) & & \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{pmatrix} \quad \dots\dots\dots (9)$$

where

$$C_5 = \frac{24C_3}{r \tan \alpha}, \quad C_6 = 2 - \frac{C_3}{C_4} \quad \dots\dots\dots (10)$$

Substituting Y and H in Eq. (9) into Eq. (1) (which is set to zero on left side), we obtain the remaining section forces X and Z from the first and second relations, and the remaining section moments T and M from the fourth and sixth relations. Furthermore, we obtain the remaining displacements u and w as well as the remaining rotation angle β by inserting Eq. (9), section forces X and Z , and section moments T and M into Eq. (2). The remaining rotation angles δ and ψ may be determined by inserting displacements u , v , w , and rotation angle β obtained just now into Eq. (3). The relations remaining unused in the above calculation process of X , Z , T , M , u , v , and β are available for rearranging the integral constants. Rewriting the obtained results for notational convenience using displacement vector $\mathbf{u}$, rotation angle vector $\boldsymbol{\theta}$, section force vector $\mathbf{X}$, section moment vector $\mathbf{T}$, and parameter vector $\mathbf{a}$ with its elements $a_1, a_2, \dots, a_{12}$ which are defined as follows

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \beta \\ \delta \\ \psi \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} T \\ H \\ M \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{pmatrix} \quad \dots\dots\dots (11)$$

, we can summarize the results in the following matrix form

$$\mathbf{u} = \mathbf{A}\mathbf{a}, \quad \boldsymbol{\theta} = \mathbf{B}\mathbf{a} \quad \dots\dots\dots (12)$$

$$\mathbf{X} = \mathbf{C}\mathbf{a}, \quad \mathbf{T} = \mathbf{D}\mathbf{a} \quad \dots\dots\dots (13)$$

where each element of 3×12 matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ is a function of the pitch angle α and angular co-ordinate θ . We, furthermore, note that there are relations $\beta_i = a_{i+3}$ ($i=1, 2, \dots, 9$)

between a set of undetermined constants $a_1, a_2, \dots, a_{12}$ and a set of undetermined constants $\beta_1, \beta_2, \dots, \beta_9$.

Undetermined constants $a_1, a_2, \dots, a_{12}$ are related to nodal displacements and rotation angle of the spring element in the following manner. Appending subscripts 1 and 2 to the displacement vector u and rotation angle vector θ at the nodal point ① ($\theta=0$) and the nodal point ② ($\theta=\phi$), respectively, as well as subscripts $\theta=0$ and $\theta=\phi$ to matrices A and B for which values 0 and ϕ are substituted into parameter θ in their elements, respectively, we obtain the following relations from Eq. (12)

$$u_1 = A_{\theta=0}a, \quad \theta_1 = B_{\theta=0}a, \quad u_2 = A_{\theta=\phi}a, \quad \theta_2 = B_{\theta=\phi}a \quad (14)$$

Eq. (14) relates nodal displacements and rotation angles at nodal points ① and ② of the spring element to the undetermined constants $a_1, a_2, \dots, a_{12}$. Using the nodal displacement vector U_e and the 12×12 matrix P defined by

$$U_e = \begin{pmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{pmatrix}, \quad P = \begin{bmatrix} A_{\theta=0} \\ B_{\theta=0} \\ A_{\theta=\phi} \\ B_{\theta=\phi} \end{bmatrix} \quad (15)$$

we can rewrite Eq. (14) as

$$U_e = Pa \quad (16)$$

3.3 Element matrices

We derive the mass and stiffness matrices for the spring element using the kinetic and potential energies. Ignoring the effects of rotary inertia and shear deformation, these energies can be written in the form

$$\begin{aligned} K.E. &= \frac{\rho A r}{2 \cos \alpha} \int_0^{\phi} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right] d\theta + \frac{\rho I r}{\cos \alpha} \int_0^{\phi} \left(\frac{\partial \beta}{\partial t} \right)^2 d\theta \\ &= \frac{\rho A r}{2 \cos \alpha} \int_0^{\phi} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 + \left(\delta_1 r \frac{\partial \beta}{\partial t} \right)^2 \right] d\theta \quad (17) \end{aligned}$$

$$\begin{aligned} P.E. &= \frac{r}{2EA \cos \alpha} \int_0^{\phi} X^2 d\theta + \frac{r}{4GI \cos \alpha} \int_0^{\phi} T^2 d\theta + \frac{r}{2EI \cos \alpha} \int_0^{\phi} (H^2 + M^2) d\theta \\ &= \frac{EI \cos \alpha}{2r^3} \int_0^{\phi} \left[\left(\frac{\delta_2}{C_0} X \right)^2 + \left(\frac{\delta_3 \cos^2 \alpha}{C_1} T \right)^2 + \left(\frac{\cos \alpha}{C_2} H \right)^2 + \left(\frac{\cos \alpha}{C_2} M \right)^2 \right] d\theta \quad (18) \end{aligned}$$

where

$$\delta_1 = \sqrt{\frac{2I}{Ar^2}}, \quad \delta_2 = \frac{\sqrt{2}}{\delta_1}, \quad \delta_3 = \sqrt{\frac{2G}{E}}, \quad C_0 = \frac{EA}{r} \cos \alpha \quad (19)$$

Observing that from Eqs. (12), (13), and (16), we can write

$$\begin{pmatrix} u \\ v \\ w \\ \delta_1 r \beta \end{pmatrix} = Ga = GP^{-1} U_e, \quad \begin{pmatrix} \delta_2 X / C_0 \\ \delta_3 T \cos^2 \alpha / C_1 \\ H \cos \alpha / C_2 \\ M \cos \alpha / C_2 \end{pmatrix} = Qa = QP^{-1} U_e \quad (20)$$

, we can rewrite Eqs. (17) and (18) as

$$K.E. = \frac{1}{2} \dot{U}_e^T \left(\frac{\rho A r}{\cos \alpha} M_e \right) \dot{U}_e, \quad P.E. = \frac{1}{2} U_e^T \left(\frac{EI \cos \alpha}{r^3} K_e \right) U_e \quad (21)$$

$$M_e = (P^{-1})^T \left[\int_0^{\phi} G^T G d\theta \right] P^{-1}, \quad K_e = (P^{-1})^T \left[\int_0^{\phi} Q^T Q d\theta \right] P^{-1} \quad (22)$$

where superscripts "." and T denote the differentiation with respect to time and the transpose matrix in question, respectively. $(\rho A r / \cos \alpha) M_e$ and $(EI \cos \alpha / r^3) K_e$ are the mass and stiffness matrices, respectively.

3.4 Governing equations and eigenvalue analysis

Inserting Eq. (21) into Lagrange's equations of motion, we derive the governing equations of the free vibration for

the spring element. In absence of the damping and externally applied forces, these equations can be written as

$$\frac{\rho A r}{\cos \alpha} M_e \ddot{U}_e + \frac{EI \cos \alpha}{r^3} K_e U_e = F_e \quad (23)$$

where $\ddot{U}_e$ denotes the second derivative of U_e with respect to time, and F_e the nodal section force vector. Taking account of equilibrium of forces and continuity of displacements at joints of adjacent spring elements, and rearranging Eq.(23), we obtain the equation of the natural vibration for the complete spring as follows

$$\frac{\rho A r}{\cos \alpha} M \ddot{U} + \frac{EI \cos \alpha}{r^3} K U = 0 \quad (24)$$

where $(\rho A r / \cos \alpha) M$, $(EI \cos \alpha / r^3) K$, and U denote the mass matrix, the stiffness matrix, and the nodal displacement vector of the complete spring, respectively. Moreover, 0 denotes zero vector. Since free vibration is harmonic, putting $U = e^{j\omega t} U_0$, Eq. (24) leads to a generalized eigenvalue problem

$$K U_0 = \nu^2 M U_0 \quad (25)$$

where

$$\omega = \frac{\nu \cos \alpha}{r^2} \sqrt{\frac{EI}{\rho A}} \quad (26)$$

We solve Eq. (25) as an eigenvalue problem associated with the matrices K and M , and find eigenvalues ν and associated eigenvectors U_0 . The eigenvalues ν hereafter are called the dimensionless natural frequencies.

Table 1 Material and dimensions of a valve spring

Material	SWP V
Wire diameter mm	4
Mean coil diameter mm	26
Free length mm	50
Number of total coils	8.5
Number of active coils	6.5
Pitch angle	4.8°

Fig. 2 Diagram of experimental and measuring system

4. Experiment

We perform a simple experiment with single-frequency sinusoidal excitation and check the validity of the method proposed in the present report. In this experiment a valve spring, on whose wire surface a strain gage is glued at a position of 1.75 turns from its end, is used. One end of the spring is attached to the support and the other end to a vibration exciter with clamping jigs. The spring is excited along the axis of the spring, and its response is examined with the strain-meter, the synchroscope, and stroboscope. Fig.2 shows a diagram of the experimental and measuring system. Fig.3 shows an experimental result for a valve spring with the dimension given in Table 1 which is excited along its axis with acceleration of 25G.

5. Results and Discussion

5.1 Numerical example

As a numerical example, calculation was carried out for a valve spring, i.e., a typical helical spring, with fixed ends and the dimensions in Table 1. Table 2 and Fig.4 show the dimensionless natural frequencies and associated modes, respectively. In Fig.4 the abscissa represents the angle associated with coil's turns and measured from the middle of the spring while the ordinate represents the ratio

Table 2 Dimensionless natural frequencies (Fixed ends, $N=6.5$)

Order	$\nu (\alpha = 4.8^\circ)$	$\nu (\alpha = 0^\circ)$
1	0.0669	0.0663
2	0.0747	0.0755
3	0.0840	0.0993
4	0.0855	0.0997
5	0.131	0.107
6	0.139	0.108

Fig. 3 Experimental result

between each component of displacement at any point on the wire axis and the maximum of each component.

In the case of a pitch angle $\alpha=4.8^\circ$, the modes associated with the first and the fifth of v_s in Table 2 are translational ones in which the spring vibrates mainly along its axis [Fig. 4(a)], and the modes associated with the second and the sixth of v_s are torsional ones in which the spring vibrates mainly about its axis [Fig. 4(b)]. In this case, a small fluctuation with 2π period in Fig. 4(b) is due to a strong coupling of u with v and an incidental transverse vibration perpendicular to its axis. The modes associated with two very close dimensionless natural frequencies, the third and fourth of v_s , are transverse (bending) ones in which the spring vibrates perpendicularly to its axis [Fig. 4(c)]. The reason why two close natural frequencies with similar modes exist is that the lateral stiffness of spring, depending on the number of active coils and the pitch angle, is not uniform in all lateral directions and have small anisotropy.

In Table 2 are also shown the dimensionless natural frequencies in the case of $\alpha=0^\circ$ for comparison purposes. In this case, w - β vibration system, which is a vibration system along the axis of spring, and u - v vibration system, which is a vibration about and perpendicular to the axis of spring, are uncoupled, so that these systems may be treated separately. In the case of $\alpha=0^\circ$, the first, the third, and the fourth of v_s correspond to the dimensionless natural frequencies with associated modes in which the spring vibrates mainly along its axis. On the other hand, the second of v_s is a dimen-

sionless natural frequency with associated mode in which the spring vibrates mainly about its axis, and the fifth and the sixth of v_s are dimensionless natural frequencies with associated modes in which the spring vibrates perpendicularly to its axis. According to the above results, the cases of $\alpha=0^\circ$ and $\alpha=4.8^\circ$ differ remarkably from each other in the shapes of the natural vibration except the first and the second orders. In the case of a small pitch angle, the natural frequencies and associated modes so far have been estimated by setting α equal to zero, but it is seen that, in this case, the results are erroneous except the first translational and the first torsional natural frequencies.

5.2 Comparison of the experimental results with the estimated results

In Table 3 the estimated natural frequencies are compared with the natural frequencies obtained by the experiment. The experimental results adopted here for comparison are those of a spring with a ratio of 30.2 between the movable mass in exciting system and the spring mass. It was checked beforehand that the movable mass did not affect the accuracy of the experimental results, if the ratio was greater than ten. Moreover, it was observed with stroboscope that the vibration modes were the same as ones depicted in Fig. 4. The experimental results agree well with the estimated results, and so support the validity of the analytical method proposed in the present report.

Table 3 Comparison of calculated values with experimental values
(Fixed ends, $\alpha=4.8^\circ$, $N=6.5$)

Order	Calculated Values Hz	Experimental Values Hz
1	322	322
2	360	361
3	405	402
4	412	414
5	630	635
6	670	669

Table 4 Comparison of present results with other author's results
(Fixed ends, $N=6.5$)

Order	Pitch angle	Ref(1)	Ref(10)	Present results	Experimental results
1	4.8°	319 Hz		322 Hz	322 Hz
	0°	325	325 Hz	321	
2	4.8°			360	361
	0°	372		365	
3	4.8°			405	402
	0°			521

Fig. 4 Natural modes of a valve spring
(Fixed ends, $\alpha=4.8^\circ$, $N=6.5$)

5.3 Comparison of the present results with the results obtained by other authors

Table 4 compares our values of the natural frequencies with those obtained by other authors⁽¹⁾⁽¹⁰⁾. The reason the other authors' values are a little greater than our values is that the wire elongation was ignored in other authors' estimation. Inoue et al., further reported in reference (3) that they had obtained experimentally the first translational natural frequency of 53.4 Hz and the second of 105 Hz. The method proposed in the present report provides 52.3 Hz and 106 Hz for values corresponding to these experimental values, respectively. They also reported in the same reference that when they excited the spring along its axis with an exciting frequency of 42 Hz, they observed a strong transverse vibration with the same frequency. This frequency corresponds to the first transverse natural frequency of 42.3 Hz which may be evaluated by the method proposed in the present report. It requires rather cumbersome calculation to estimate analytically the natural frequencies of the helical spring taking account of the effects of the pitch angle. Correspondingly, it is difficult to estimate the natural frequencies actually except rather lower ones⁽¹⁾. However, according to the method proposed in the present report, it is possible to estimate the natural frequencies and associated modes exactly up to higher order.

5.4 Effects of the pitch angle and the number of active coils

Fig. 5 shows how the pitch angle influences the natural frequencies of the helical spring. It is seen from this figure, if the pitch angle is small, the natural frequencies are ones associated

Fig. 5 Effect of pitch angle on natural frequencies (Fixed ends, $N=6.5$)

with the translational, the torsional, and the transverse modes in order of the first, the second, and the third of ω s. But the transverse natural frequency associated with the mode in which the spring vibrates transversely to its axis, decreases rapidly with an increase in the pitch angle, and this frequency becomes the lowest one for the helical spring having the number of active coils of $N=6.5$, if the pitch angle is greater than 8.0° .

A similar result is obtained from Fig. 6 showing how the number of active coils influences the natural frequency. Each natural frequency decreases with an increase in the number of active coils, and such a tendency is remarkable in the transverse natural frequency. Thus, this natural frequency becomes the lowest one in the helical spring having the pitch angle of $\alpha=4.8^\circ$, if the number of active coils is greater than 11. As it is fundamental and important to know the transverse natural frequencies in the analysis of the dynamic stability, the above mentioned is worth taking into consideration.

6. Conclusions

In the present report, we proposed a method for estimating the natural frequencies and associated modes of the helical spring by using a finite element approach based on the analytical solutions of the equations of the helical spring in static equilibrium, and analyzed the natural vibration of the helical spring taking the effects of the pitch angle and the wire elongation into account. The results may be summarized as follows:

(1) The mass and stiffness matrices are derived for a space-curve-type

Fig. 6 Effect of number of active coils on natural frequencies (Fixed ends, $\alpha=4.8^\circ$)

structural element available for a finite element approach of the helical spring.

(2) According to a finite element approach based on the solutions of the equations of the helical spring in static equilibrium, it is easy to estimate the natural frequencies and associated modes numerically, and the estimated results agree well with the experimental results.

(3) If the effects of the pitch angle are ignored, the estimated natural frequencies and associated modes differ remarkably in higher order from the actual ones, and so an analysis taking the effects of the pitch angle into account is required.

(4) The effects of the pitch angle and the number of active coils on the natural frequencies and associated modes are clear, and it is shown that the mode associated with the fundamental natural frequency is not translational one, but transverse one, if the pitch angle or the number of active coils is greater than a certain value. This fact should be taken into account in analysis of the dynamic stability of the helical spring.

Acknowledgement

The authors wish to thank Mr. S. Nishiyama, who is the chief of the third laboratory in NHK SPRING Co., LTD, for providing samples of helical springs and also Mr. N. Kondo, who is a graduate of Yamanashi University, for his assistance with the present experiments. In addition, the authors used the large-capacity com-

puter of University of Tokyo to calculate the numerical values in this report.

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